THE DISTRIBUTION OF NODES OF GIVEN DEGREE IN RANDOM TREES

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Abstract. Let $\mathcal{T}_n$ denote the set of unrooted unlabeled trees of size n and let $k \geq 1$ be given. By assuming that every tree of $\mathcal{T}_n$ is equally likely it is shown that the limiting distribution of the number of nodes of degree k is normal with mean value $\sim \mu_k n$ and variance $\sim \sigma_k^2 n$ with positive constants μ_k and σ_k. Besides, the asymptotic behavior of μ_k and σ_k for $k \to \infty$ as well as the corresponding multivariate distributions are derived. Furthermore, similar results can be proved for plane trees, for labeled trees, and for forests.

1. Introduction

Let $\mathcal{T}_n$ denote the set of unlabeled unrooted trees of size n and $\mathcal{T}_n^{(r)}$ the set of unlabeled rooted trees. The corresponding cardinalities will be denoted by $t_n = |\mathcal{T}_n|$ and $t_n^{(r)} = |\mathcal{T}_n^{(r)}|$. In 1937 Pólya [7] already discussed the generating function

$$t^{(r)}(x) = \sum_{n \geq 1} t_n^{(r)} x^n$$

and showed that the radius of convergence ρ satisfies $0 < \rho < 1$ and that $x = \rho$ is the only singularity on the circle of convergence $|x| = \rho$. Later Otter [6] showed that $t^{(r)}(\rho) = 1$ and used the asymptotic expansion

$$t^{(r)}(x) = 1 - b(\rho - x)^{1/2} + c(\rho - x) + d(\rho - x)^{3/2} + \cdots \tag{1.1}$$

to deduce that

$$t_n^{(r)} \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} n^{-3/2} \rho^{-n}.$$

(Note that $c = b^2/3 \approx 2.3961466$.) He also calculated $\rho \approx 0.3383219$ and $b \approx 2.6811266$. However, his main contribution was to show that

$$t(x) = \sum_{n \geq 1} t_n x^n = t^{(r)}(x) - \frac{1}{2} t^{(r)}(x)^2 + \frac{1}{2} t^{(r)}(x^2).$$

Hence $t(x)$ has a similar expansion, namely

$$t(x) = \frac{1 + t^{(r)}(\rho^2)}{2} + \frac{b^2 - \rho (t^{(r)})'(\rho^2)}{2} (\rho - x) + bc(\rho - x)^{3/2} + \cdots ,$$

and it follows that

$$t_n \sim \frac{b^3 \rho^{3/2}}{4\sqrt{\pi}} n^{-5/2} \rho^{-n}.$$

In 1975 Robinson and Schwenk [8] showed by an extension of Pólya's and Otter's method that the mean value of the number of those nodes of degree k is approximately $\mu_k n$. The asymptotic behavior of μ_k is given by

$$\mu_k \sim C \rho^k$$

where $C \approx 6.380045$ see [10]. However, the distribution has not been determined.

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The main purpose of this paper is to determine the limiting distribution of the number of nodes of degree k. Furthermore we will determine the joint limiting distribution of the number of nodes of different (given) degrees. By using the same methods similar results will be obtained for plane trees, for labeled trees, and for forests.

2. Combinatorial Background and Results

2.1. **Unlabeled, Nonplane Trees.** In addition to $\mathcal{T}_n$ and $\mathcal{T}_n^{(r)}$ we will also make use of the set $\mathcal{T}_n^{(p)}$ of planted unlabeled trees, i.e. the root of any unlabeled rooted tree is adjoined with an additional node which is not counted. This means that the degree of the root is increased by 1. Obviously $|\mathcal{T}_n^{(p)}| = |\mathcal{T}_n^{(r)}| = t_n^{(r)}$. Furthermore, let $\mathcal{T}_n^{(f)}$ denote the set of unlabeled forests of size n.

Let $\boldsymbol{k} = (k_1, \dots, k_M)$ be a given vector of M positive different integers. For every vector $\boldsymbol{m} = (m_1, \dots, m_M)$ of non-negative integers let $t_{n\boldsymbol{m}\boldsymbol{k}}^{(p)}$ denote the number of planted unlabeled rooted trees of size n with m_j nodes of degree k_j, $1 \le j \le M$, and $t_{n\boldsymbol{m}\boldsymbol{k}}^{(r)}$, $t_{n\boldsymbol{m}\boldsymbol{k}}$, and $t_{n\boldsymbol{m}\boldsymbol{k}}^{(f)}$ the corresponding numbers of unlabeled rooted trees, of unlabeled unrooted trees and of unlabeled forests, respectively. Note that we are considering arbitrary trees, i.e. we do not have any restriction to the vertex degrees (so we have $\sum_{i=1}^M m_i \le n$ and not necessarily equality). We are only focusing on those vertices having degree in $\boldsymbol{k}$ for counting. Let

$$\begin{aligned} t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) &= \sum_{n,\boldsymbol{m}} t_{n\boldsymbol{m}\boldsymbol{k}}^{(p)} x^n \boldsymbol{u}^{\boldsymbol{m}}, \\ t_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) &= \sum_{n,\boldsymbol{m}} t_{n\boldsymbol{m}\boldsymbol{k}}^{(r)} x^n \boldsymbol{u}^{\boldsymbol{m}}, \\ t_{\boldsymbol{k}}(x,\boldsymbol{u}) &= \sum_{n,\boldsymbol{m}} t_{n\boldsymbol{m}\boldsymbol{k}} x^n \boldsymbol{u}^{\boldsymbol{m}}, \\ t_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) &= \sum_{n,\boldsymbol{m}} t_{n\boldsymbol{m}\boldsymbol{k}}^{(f)} x^n \boldsymbol{u}^{\boldsymbol{m}}, \end{aligned}$$

where $\boldsymbol{u} = (u_1, \dots, u_M)$ and $\boldsymbol{u}^{\boldsymbol{m}} = u_1^{m_1} \cdots u_M^{m_M}$. Then these generating functions satisfy the following functional equations:

Lemma 2.1. *Let $Z(S_k; x_1, \dots, x_k)$ denote the cycle index of the symmetric group S_k of k elements. Then we have*

$$\begin{aligned} t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) =& x \exp\left(\sum_{i\ge1} \frac{t_{\boldsymbol{k}}^{(p)}(x^i,\boldsymbol{u}^i)}{i} \right) \qquad (2.1)\\ &+ \sum_{j=1}^M x(u_j-1) Z(S_{k_j-1}; t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}), t_{\boldsymbol{k}}^{(p)}(x^2,\boldsymbol{u}^2), \dots, t_{\boldsymbol{k}}^{(p)}(x^{k_j-1},\boldsymbol{u}^{k_j-1})), \\ t_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) =& x \exp\left(\sum_{i\ge1} \frac{t_{\boldsymbol{k}}^{(p)}(x^i,\boldsymbol{u}^i)}{i} \right) \qquad (2.2)\\ &+ \sum_{j=1}^M x(u_j-1) Z(S_{k_j}; t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}), t_{\boldsymbol{k}}^{(p)}(x^2,\boldsymbol{u}^2), \dots, t_{\boldsymbol{k}}^{(p)}(x^{k_j},\boldsymbol{u}^{k_j})), \\ t_{\boldsymbol{k}}(x,\boldsymbol{u}) =& t_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) - \frac12 t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^2 + \frac12 t_{\boldsymbol{k}}^{(p)}(x^2,\boldsymbol{u}^2), \qquad (2.3)\\ t_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) =& \exp\left(\sum_{i\ge1} \frac{t_{\boldsymbol{k}}(x^i,\boldsymbol{u}^i)}{i} \right). \qquad (2.4) \end{aligned}$$

For $M = 1$ these relations are already established in [8]. The general case only requires obvious modifications.

Note that the term

$$\exp\left(\sum_{i\ge 1}\frac{t_{\boldsymbol{k}}^{(p)}(x^i,\boldsymbol{u}^i)}{i}\right)$$

corresponds to the multiset construction of unlabeled combinatorial objects described e.g. in [11] and that

$$Z(S_l; t_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}),\dots,t_{\boldsymbol{k}}^{(p)}(x^l,\boldsymbol{u}^l)) = [v^l]\exp\left(\sum_{i\ge 1} v^i\frac{t_{\boldsymbol{k}}^{(p)}(x^i,\boldsymbol{u}^i)}{i}\right)$$

is exactly the generating function of a forest consisting of exactly l planted trees. This relation can be used to determine the partial derivatives of the cylcle index. Obviously, we have

$$\sum_{k\ge 0} Z(S_k; x_1,\dots,x_k)v^k = \exp\left(\sum_{l\ge 1}\frac{x_l}{l}v^l\right)$$

and consequently

$$\begin{aligned}\sum_{k\ge 0}\frac{\partial}{\partial x_i} Z(S_k; x_1,\dots,x_k)v^k &= \exp\left(\sum_{l\ge 1}\frac{x_l}{l}v^l\right)\frac{v^i}{i}\\ &= \sum_{k\ge 0} Z(S_k; x_1,\dots,x_k)\frac{v^{k+i}}{i}\end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial x_i} Z(S_k; x_1,\dots,x_k) = \frac{1}{i} Z(S_{k-i}; x_1,\dots,x_{k-i}) \tag{2.5}$$

Now we are ready to state our first main result:

Theorem 2.1. *Let $\boldsymbol{X}_{n\boldsymbol{k}} = (X^{(1)}_{nk_1},\dots,X^{(M)}_{nk_M})$ denote the vector of the numbers of nodes in an unrooted unlabeled random tree or forest of n nodes that have degrees $k_1,\dots,k_M$. Set*

$$\begin{aligned}\boldsymbol{\mu}_{\boldsymbol{k}} &= (\mu_{k_i})_{i=1,\dots,M} = \left(-\frac{f_i}{\rho}\right)_{i=1,\dots,M},\\ \boldsymbol{\Sigma} &= (\sigma_{ij})_{i,j=1,\dots,M} = \left(\frac{f_i f_j}{\rho^2} - \frac{f_{ij}}{\rho} - \delta_{ij}\frac{f_i}{\rho}\right)_{i,j=1,\dots,M},\end{aligned}$$

where δ_{ij} denotes the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j\\ 0 & \text{if } i \ne j\end{cases}$$

and

$$f_i = -\frac{F_i}{F_x}(\rho,\mathbf{1},1) \tag{2.6}$$

$$\begin{aligned} f_{ij} = \Bigg[&\frac{1}{F_{tt}F_x}\left(\frac{F_iF_{tx}}{F_x} - F_{ti}\right)\left(\frac{F_jF_{tx}}{F_x} - F_{tj}\right)\\ &- \frac{1}{F_x}\left(\frac{F_iF_jF_{xx}}{F_x^2} - \frac{F_iF_{xj} + F_jF_{xi}}{F_x} + F_{ij}\right)\Bigg](\rho,\mathbf{1},1)\end{aligned} \tag{2.7}$$

with

$$F(x,\boldsymbol{u},t) = xe^t \exp\left(\sum_{i\ge 2} \frac{t_{\boldsymbol{k}}(x^i,\boldsymbol{u}^i)}{i}\right)$$
$$+ \sum_{j=1}^{M} x(u_j-1)Z(S_{k_j-1}; t, t_{\boldsymbol{k}}(x^2,\boldsymbol{u}^2),\ldots,t_{\boldsymbol{k}}(x^{k_j-1},\boldsymbol{u}^{k_j-1}))$$

and $F_i = \partial/\partial u_i F(x,\boldsymbol{u},t)$.

If $\det\Sigma \ne 0$ *then* $\boldsymbol{X}_{n\boldsymbol{k}}$ *is asymptotically normally distributed with mean value* $\sim \boldsymbol{\mu}_{\boldsymbol{k}} n$ *and covariance matrix* $\sim \boldsymbol{\Sigma} n$.

Furthermore we have for large $k_1,\ldots,k_M$*:*

$$\mu_{k_i} \sim \frac{2C}{b^2\rho}\rho^{k_i} \tag{2.8}$$

$$\sigma_{ij} \sim \begin{cases} \dfrac{2C}{b^2\rho}\rho^{k_i} & \text{if } i=j \\ -\dfrac{4C^2}{b^4\rho^2}\rho^{k_i+k_j}(k_i+k_j) & \text{if } i\ne j \end{cases} \tag{2.9}$$

where

$$C = \exp\left(\frac{1}{l}\left(\frac{t_{\boldsymbol{k}}(\rho^l,\boldsymbol{1})}{\rho^l}-1\right)\right) \approx 7.7581604 \tag{2.10}$$

Remark 1. The same theorem holds if only planted or rooted unlabeled trees are considered instead of unlabeled (unrooted) trees or forests.

Remark 2. The asymptotic expansions for σ_{ij} show that $\det\Sigma$ is surely nonzero if k_i, $1\le i\le M$, are sufficiently large. There is no doubt that $\det\Sigma\ne 0$ in any case. However, at the moment we are not able to prove this rigorously.

2.2. **Plane Trees.** Let $\mathcal{P}_n^{(p)}$ denote the set of planted plane trees of size n, $\mathcal{P}_n^{(r)}$ the set of rooted plane trees of size n, $\mathcal{P}_n$ the set of (unrooted unlabeled) plane trees of size n, and $\mathcal{P}_n^{(f)}$ the set of plane forests of size n.

Let $\boldsymbol{k} = (k_1,\ldots,k_M)$ be a given vector of M positive different integers. For every vector $\boldsymbol{m} = (m_1,\ldots,m_M)$ of non-negative integers let $p_{n\boldsymbol{m}\boldsymbol{k}}^{(p)}$ denote the number of planted plane trees of size n with m_j nodes of degree k_j, $1\le j\le M$, and $p_{n\boldsymbol{m}\boldsymbol{k}}^{(r)}$, $p_{n\boldsymbol{m}\boldsymbol{k}}$, and $p_{n\boldsymbol{m}\boldsymbol{k}}^{(f)}$ the corresponding numbers of rooted plane trees, of plane trees and of plane forests, respectively. Furthermore let

$$p_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} p_{n\boldsymbol{m}\boldsymbol{k}}^{(p)} x^n \boldsymbol{u}^{\boldsymbol{m}},$$
$$p_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} p_{n\boldsymbol{m}\boldsymbol{k}}^{(r)} x^n \boldsymbol{u}^{\boldsymbol{m}},$$
$$p_{\boldsymbol{k}}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} p_{n\boldsymbol{m}\boldsymbol{k}} x^n \boldsymbol{u}^{\boldsymbol{m}},$$
$$p_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} p_{n\boldsymbol{m}\boldsymbol{k}}^{(f)} x^n \boldsymbol{u}^{\boldsymbol{m}},$$

where $\boldsymbol{u} = (u_1,\ldots,u_M)$ and $\boldsymbol{u}^{\boldsymbol{m}} = u_1^{m_1}\cdots u_M^{m_M}$. Then these generating functions satisfy the following functional equations:

Lemma 2.2. *Let* $Z(C_k; x_1, \dots, x_k) = \frac{1}{k} \sum_{d|k} \varphi(d) x_d^{k/d}$ *denote the cycle index of the cyclic group* C_k *of* k *elements. Then we have*

$$p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u}) = \frac{x}{1 - p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})} + \sum_{j=1}^{M} x(u_j - 1) p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})^{k_j - 1}, \tag{2.11}$$

$$p_{\boldsymbol{k}}^{(r)}(x, \boldsymbol{u}) = x \sum_{k \ge 1} Z(C_k; p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u}), \dots, p_{\boldsymbol{k}}^{(p)}(x^k, \boldsymbol{u}^k)) + \sum_{j=1}^{M} x(u_j - 1) Z(C_{k_j}; p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u}), \dots, p_{\boldsymbol{k}}^{(p)}(x^{k_j}, \boldsymbol{u}^{k_j})), \tag{2.12}$$

$$p_{\boldsymbol{k}}(x, \boldsymbol{u}) = p_{\boldsymbol{k}}^{(r)}(x, \boldsymbol{u}) - \frac{1}{2} p_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})^2 + \frac{1}{2} p_{\boldsymbol{k}}^{(p)}(x^2, \boldsymbol{u}^2), \tag{2.13}$$

$$p_{\boldsymbol{k}}^{(f)}(x, \boldsymbol{u}) = \exp\left(\sum_{i \ge 1} \frac{p_{\boldsymbol{k}}(x^i, \boldsymbol{u}^i)}{i} \right). \tag{2.14}$$

The case $M = 0$, i.e. we are counting $|\mathcal{P}_n^{(p)}|$, $|\mathcal{P}_n^{(r)}|$, $|\mathcal{P}_n|$, and $|\mathcal{P}_n^{(f)}|$, is already treated in [5]. Especially, we have for $p^{(p)}(x) = \sum_{n=1}^{\infty} |\mathcal{P}_n^{(p)}|$

$$p^{(p)}(x) = \frac{1}{1 - p^{(p)}(x)} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4x} = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n.$$

The general case $M > 0$ is now obvious by following the rules used for the proof of Lemma 2.1.

Our second theorem is quite similar to Theorem 2.1:

Theorem 2.2. *Let* $\boldsymbol{Y}_{n\boldsymbol{k}} = (Y_{nk_1}^{(1)}, \dots, Y_{nk_M}^{(M)})$ *denote the vector of the numbers of nodes in a plane random tree or forest of* n *nodes that have degrees* $k_1, \dots, k_M$*. Set*

$$\boldsymbol{\mu}_{\boldsymbol{k}} = (\mu_{k_i})_{i=1,\dots,M} = \left(\frac{1}{2^{k_i}} \right)_{i=1,\dots,M}$$

$$\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,\dots,M} = \left(\frac{1}{2^{k_i + k_j}} - \frac{(k_i - 2)(k_j - 2)}{2^{k_i + k_j + 1}} + \delta_{ij} \frac{1}{2^{k_i}} \right)_{i,j=1,\dots,M}$$

Then $\det \Sigma > 0$ *and* $\boldsymbol{Y}_{n\boldsymbol{k}}$ *is asymptotically normally distributed with mean value* $\sim \boldsymbol{\mu}_{\boldsymbol{k}} n$ *and covariance matrix* $\sim \boldsymbol{\Sigma} n$*.*

Remark . The same theorem holds if only planted or rooted plane trees are considered instead of plane (unrooted) trees or forests.

2.3. **Labeled Trees.** Let $\mathcal{L}_n^{(p)}$ denote the set of planted labeled trees of size n, $\mathcal{L}_n^{(r)}$ the set of rooted labeled trees of size n, $\mathcal{L}_n$ the set of (unrooted) labeled trees of size n, and $\mathcal{L}_n^{(f)}$ the set of labeled forests of size n.

Let $\boldsymbol{k} = (k_1, \dots, k_M)$ be a given vector of M positive different integers. For every vector $\boldsymbol{m} = (m_1, \dots, m_M)$ of non-negative integers let $l_{n\boldsymbol{m}\boldsymbol{k}}^{(p)}$ denote the number of planted labeled trees of size n with m_j nodes of degree k_j, $1 \le j \le M$, and $l_{n\boldsymbol{m}\boldsymbol{k}}^{(r)}$, $l_{n\boldsymbol{m}\boldsymbol{k}}$, and $l_{n\boldsymbol{m}\boldsymbol{k}}^{(f)}$ the corresponding numbers of rooted labeled trees, of labeled trees and of labeled forests, respectively. Furthermore let

$$l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} l_{n\boldsymbol{m}\boldsymbol{k}}^{(p)} \frac{x^n}{n!}\boldsymbol{u}^{\boldsymbol{m}},$$

$$l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} l_{n\boldsymbol{m}\boldsymbol{k}}^{(r)} \frac{x^n}{n!}\boldsymbol{u}^{\boldsymbol{m}},$$

$$l_{\boldsymbol{k}}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} l_{n\boldsymbol{m}\boldsymbol{k}} \frac{x^n}{n!}\boldsymbol{u}^{\boldsymbol{m}},$$

$$l_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) = \sum_{n,\boldsymbol{m}} l_{n\boldsymbol{m}\boldsymbol{k}}^{(f)} \frac{x^n}{n!}\boldsymbol{u}^{\boldsymbol{m}},$$

where $\boldsymbol{u} = (u_1, \ldots, u_M)$ and $\boldsymbol{u}^{\boldsymbol{m}} = u_1^{m_1}\cdots u_M^{m_M}$. Then these generating functions satisfy the following functional equations:

Lemma 2.3. *We have*

$$l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) = xe^{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})} + \sum_{j=1}^{M} x(u_j-1)\frac{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^{k_j-1}}{(k_j-1)!}, \tag{2.15}$$

$$l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) = xe^{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})} + \sum_{j=1}^{M} x(u_j-1)\frac{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^{k_j}}{k_j!}, \tag{2.16}$$

$$l_{\boldsymbol{k}}(x,\boldsymbol{u}) = l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) - \frac{1}{2} l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^2, \tag{2.17}$$

$$l_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) = e^{l_{\boldsymbol{k}}(x,\boldsymbol{u})}. \tag{2.18}$$

The advantage of labeled structures is that if we use exponential generating functions, i.e. x^n is replaced by $x^n/n!$, then we do not need Polya's theorem to obtain relations for the corresponding tree functions (compare with [5, 11, 9]). So Lemma 2.3 follows immediately.

Remark . It is well known that $|\mathcal{L}_n^{(p)}| = |\mathcal{L}_n^{(r)}| = n^{n-1}$ and $|\mathcal{L}_n| = n^{n-2}$. Even without knowing this it is clear that $|\mathcal{L}_n^{(r)}| = n|\mathcal{L}_n|$ since every labeled tree of size n represents exactly n rooted labeled trees. Similarly

$$|l_{n\boldsymbol{m}\boldsymbol{k}}^{(r)}| = n|l_{n\boldsymbol{m}\boldsymbol{k}}|.$$

This fact can also be used to prove (2.17). We only have to show that

$$\int_0^x \frac{l_{\boldsymbol{k}}^{(r)}(\xi,\boldsymbol{u})}{\xi}\, d\xi = l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) - \frac{1}{2} l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}). \tag{2.19}$$

By (2.15) and (2.16) we obtain

$$\begin{aligned}
\frac{\partial}{\partial x} l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) &= e^{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})} + \sum_{j=1}^{M} (u_j-1)\frac{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^{k_j}}{k_j!} \\
&\quad + xe^{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})}\frac{\partial}{\partial x} l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) + \sum_{j=1}^{M} x(u_j-1)\frac{l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})^{k_j-1}}{(k_j-1)!}\frac{\partial}{\partial x} l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) \\
&= \frac{l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u})}{x} - l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})\frac{\partial}{\partial x} l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})
\end{aligned}$$

which directly yields (2.19).

This class of trees behaves similarly to the other ones as is shown by our third theorem:

Theorem 2.3. *Let* $\boldsymbol{Z}_{n\boldsymbol{k}} = (Z^{(1)}_{nk_1}, \ldots, Z^{(M)}_{nk_M})$ *denote the vector of the numbers of nodes in a labeled random tree or forest of* n *nodes that have degrees* $k_1, \ldots, k_M$*. Set*

$$\boldsymbol{\mu}_{\boldsymbol{k}} = (\mu_{k_i})_{i=1,\ldots,M} = \left(\frac{1}{e}\frac{1}{(k_i-1)!}\right)_{i=1,\ldots,M}$$
$$\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1,\ldots,M} = \left(-\frac{1+(k_i-2)(k_j-2)}{e^2(k_i-1)!(k_j-1)!} + \delta_{ij}\frac{1}{e}\frac{1}{(k_i-1)!}\right)_{i,j=1,\ldots,M}$$

Then $\det\Sigma > 0$ *and* $\boldsymbol{Z}_{n\boldsymbol{k}}$ *is asymptotically normally distributed with mean value* $\sim \boldsymbol{\mu}_{\boldsymbol{k}} n$ *and covariance matrix* $\sim \boldsymbol{\Sigma} n$.

Remark . The same theorem holds if only planted or rooted labeled trees are considered instead of labeled (unrooted) trees or forests.

3. Analytic Background

The basic property which will be used in the sequel is the following observation (compare with [2, 3]):

Proposition 3.1. *Set* $\boldsymbol{u} = (u_1, \ldots, u_M)$ *and suppose that* $F(x, \boldsymbol{u}, y)$ *is an analytic function around* $(x_0, \boldsymbol{u}_0, y_0)$ *such that*

$$F(x_0, \boldsymbol{u}_0, y_0) = y_0,$$
$$F_y(x_0, \boldsymbol{u}_0, y_0) = 1,$$
$$F_{yy}(x_0, \boldsymbol{u}_0, y_0) \neq 0,$$
$$F_x(x_0, \boldsymbol{u}_0, y_0) \neq 0.$$

Then there exist a neighborhood U *of* $(x_0, \boldsymbol{u}_0)$*, a neighborhood* V *of* y_0 *and analytic functions* $g(x, \boldsymbol{u})$*,* $h(x, \boldsymbol{u})$ *and* $f(\boldsymbol{u})$ *which are defined on* U *such that the only solutions* $y \in V$ *with* $y = F(x, \boldsymbol{u}, y)$ *(*$(x, \boldsymbol{u}) \in U$*) are given by*

$$y = g(x, \boldsymbol{u}) \pm h(x, \boldsymbol{u})\sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.1}$$

Furthermore $g(x_0, \boldsymbol{u}_0) = y_0$ *and* $h(x_0, \boldsymbol{u}_0) = \sqrt{2f(\boldsymbol{u}_0)F_x(x_0, \boldsymbol{u}_0, y_0)/F_{yy}(x_0, \boldsymbol{u}_0, y_0)}$.

Proof. See [3]. □

3.1. **Unlabeled, Nonplane Trees.** As a first application of Proposition 3.1 we show that representation (1.1) follows just from the facts that the radius of convergence ρ satisfies $0 < \rho < 1$ and that $t^{(r)}(\rho) = \lim_{x\to\rho-} t^{(r)}(x)$ is finite. By using this representation corresponding representations for $t(x)$ and $t^{(f)}(x)$ follow immediately.

Lemma 3.1. *There exist* $\eta > 0$ *and functions* $r_1(x), r_2(x), r_3(x), r_4(x), r_5(x), r_6(x)$*, which are analytic for* $|x - \rho| < \eta$ *and satisfy* $r_i(\rho) > 0$*,* $1 \le i \le 6$*, such that*

$$\begin{aligned} t^{(r)}(x) &= t^{(p)}(x) \\ &= 1 - b\sqrt{\rho}\sqrt{1-\frac{x}{\rho}} + \frac{b^2\rho}{3}\left(1-\frac{x}{\rho}\right) + r_1(x)\left(1-\frac{x}{\rho}\right)^{3/2} + r_2(x)\left(1-\frac{x}{\rho}\right)^2, \end{aligned} \tag{3.2}$$

$$t(x) = \frac{b^3\rho^{3/2}}{3}\left(1-\frac{x}{\rho}\right)^{3/2} + r_3(x)\left(1-\frac{x}{\rho}\right)^{5/2} + r_4(x), \tag{3.3}$$

$$t^{(f)}(x) = r_5(x)\left(1-\frac{x}{\rho}\right)^{3/2} + r_6(x). \tag{3.4}$$

Furthermore, $t^{(r)}(x), t(x),$ *and* $t^{(f)}(x)$ *can be analytically continued to the region* $|x| < \rho + \frac{\eta}{2}$*,* $\arg(x - \rho) \neq 0$.

Proof. Since $\rho < 1$ and $t^{(r)}(0) = 0$ there exists $\bar\rho < \rho$ with $\bar\rho > \rho^2$ and a constant $C > 0$ with $|t^{(r)}(x)| \le C|x|$ for $|x| \le \bar\rho$. Hence the function

$$Q(x) = \exp\left(\sum_{i\ge 2} \frac{t^{(r)}(x^i)}{i}\right)$$

is analytic for $|x| \le \rho^{-1/2}$ and the functional equation $t^{(r)}(x) = x\exp\left(\sum_{i\ge1} t^{(r)}(x^i)/i\right)$ (compare with (2.1) and (2.2) for $\boldsymbol{u} = (1,1,\ldots,1)$) can be rewritten to

$$t^{(r)}(x) = xQ(x)e^{t^{(r)}(x)}$$

Set $F(x,\boldsymbol{u},y) = xQ(x)e^y$. Then all assumptions of Proposition 3.1 are satisfied for $x_0 = \rho$, $y_0 = t^{(r)}(\rho)$ and arbitrary $\boldsymbol{u}_0$. You only have to check that $F_y(x_0,\boldsymbol{u}_0,y_0) = 1$. Namely, if $F_y(x_0,\boldsymbol{u}_0,y_0) \ne 1$ then the implicit function theorem would imply that $t(x)$ is analytic at $x = \rho$. Thus we obtain (3.2). Note that by the principle of analytic continuation only one sign in (3.1) is relevant. Furthermore, from $F_y(x_0,\boldsymbol{u}_0,y_0) = F(x_0,\boldsymbol{u}_0,y_0)$ it follows that $t^{(r)}(\rho-) = 1$.

Finally, since $t_n \ne 0$ for all $n \ge 1$ it follows that $|F_y(x,\boldsymbol{u}_0,t^{(r)}(x))| < F_y(|x|,\boldsymbol{u}_0,t^{(r)}(|x|))$ for non-real x. Hence $F_y(x,\boldsymbol{u}_0,t(x)) \ne 1$ for $|x| \le \rho$, $x \ne \rho$ and by the implicit function theorem there exists an analytic continuation to R.

Since $t(x) = t^{(r)}(x) - \frac12(t^{(p)}(x))^2 + \frac12 t^{(r)}(x^2)$

$$t^{(f)}(x) = \exp\left(\sum_{i\ge1}\frac{t(x^i)}{i}\right)$$

we immediately obtain (3.3) and (3.4). We only have to observe that

$$\tilde Q(x) = \exp\left(\sum_{i\ge2}\frac{t(x^i)}{i}\right)$$

is an analytic function for $|x| < \rho + \eta$. □

In a similar manner we can prove that $t^{(p)}_{\boldsymbol{j}}(x,\boldsymbol{u})$ (which surely exists for $|x| < \rho$ and $\boldsymbol{u}$ with $|u_j| \le 1$ for $1 \le j \le M$) has a proper analytic continuation and a similar representation as $t^{(r)}(x)$ around its singularity.

Lemma 3.2. *Let $\boldsymbol{k} = (k_1,\ldots,k_M)$ be a given vector of M different positive integers. Then there exist $\eta > 0$ and functions $g_1(x,\boldsymbol{u})$, $g_2(x,\boldsymbol{u})$, $h_1(x,\boldsymbol{u})$, $h_2(x,\boldsymbol{u})$, $f(\boldsymbol{u})$ ($\boldsymbol{u} = (u_1,\ldots,u_M)$) with the following properties:*

1. *$g_1(x,\boldsymbol{u})$, $g_2(x,\boldsymbol{u})$, $h_1(x,\boldsymbol{u})$, $h_2(x,\boldsymbol{u})$, $f(\boldsymbol{u})$ are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - f(\boldsymbol{u})| < \eta$.*
2. *$g_i(\rho,1,\ldots,1) = 1$, $h_i(\rho,1,\ldots,1) = b\sqrt{\rho}$, $i = 1,2$, where b is given by (1.1), and $f(1,\ldots,1) = \rho$.*
3. *$t^{(p)}_{\boldsymbol{k}}(x,\boldsymbol{u})$ and $t^{(r)}_{\boldsymbol{k}}(x,\boldsymbol{u})$ can be analytically continued to the region*

$$R = \{(x,\boldsymbol{u}) \in \mathbf{C}^{M+1} : |u_j| \le 1 + \frac{\eta}{2}, 1 \le j \le M, |x| \le \rho + \frac{\eta}{2}, \arg(x - f(\boldsymbol{u})) \ne 0\}$$

such that

$$t^{(p)}_{\boldsymbol{k}}(x,\boldsymbol{u}) = g_1(x,\boldsymbol{u}) - h_1(x,\boldsymbol{u})\sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.5}$$

and

$$t^{(r)}_{\boldsymbol{k}}(x,\boldsymbol{u}) = g_2(x,\boldsymbol{u}) - h_2(x,\boldsymbol{u})\sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.6}$$

for $(x,\boldsymbol{u}) \in R$ and $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

Proof. The first step is to show that there exists $\eta_1 > 0$ such that $t_{\boldsymbol{k}}^{(p)}(x^i, \boldsymbol{u}^i)$ is analytic for $|x| < \rho + \eta_1$ and $|u_j| < 1 + \eta_1$, $1 \le j \le M$, if $i > 1$. Suppose that $|u_j| < 1 + \varepsilon$, $1 \le j \le M$. Then

$$|t_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})| \le p(|x|(1+\varepsilon), 1, \dots, 1) = t(|x|(1+\varepsilon)).$$

Hence, if $\eta_1 = \varepsilon > 0$ is sufficiently small it follows (as in the proof of Lemma 3.1) that $t_{\boldsymbol{k}}^{(p)}(x^i, \boldsymbol{u}^i)$ is analytic for $|x| < \rho + \eta_1$ and $|u_j| < 1 + \eta_1$, $1 \le j \le M$, and that there exists a constant $C > 0$ with

$$|t_{\boldsymbol{k}}^{(p)}(x^i, \boldsymbol{u}^i)| \le C|x(1+\eta_1)|^i.$$

Thus

$$Q(x, \boldsymbol{u}) = \exp\left(\sum_{i \ge 2} \frac{t_{\boldsymbol{k}}^{(p)}(x^i, \boldsymbol{u}^i)}{i}\right)$$

is analytic for $|x| < \rho + \eta_1$ and $|u_j| < 1 + \eta_1$, $1 \le j \le M$. So $t = t_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})$ satisfies the functional equation

$$t = F(x, \boldsymbol{u}, t)$$

with

$$F(x, \boldsymbol{u}, t) = xQ(x, \boldsymbol{u})e^t + \sum_{j=1}^{M} x(u_j - 1) Z(S_{k_j - 1}; t, t_{\boldsymbol{k}}^{(p)}(x^2, \boldsymbol{u}^2), \dots, t_{\boldsymbol{k}}^{(p)}(x^{k_j - 1}, \boldsymbol{u}^{k_j - 1})). \tag{3.7}$$

Note that $Z(S_k; t, \dots)$ is a polynomial in t with analytic coefficients. Thus we can apply Proposition 3.1 for $x_0 = \rho$, $\boldsymbol{u}_0 = (1, \dots, 1)$ and $y_0 = 1$ in order to obtain the local behavior of $t_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})$ around its singularity $x = f(\boldsymbol{u})$. Finally, it follows from $F_y(x, \boldsymbol{u}_0, t_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u}_0)) \ne 1$ (for $|x| = \rho, x \ne \rho$) that $t_{\boldsymbol{k}}^{(p)}(x, \boldsymbol{u})$ can be analytically continued to R for some $\eta > 0$. By Lemma 2.1 a similar representation is obtained for $t_{\boldsymbol{k}}^{(r)}(x, \boldsymbol{u})$. □

The representations of $t_{\boldsymbol{k}}(x, \boldsymbol{u})$ and $t_{\boldsymbol{k}}^{(f)}(x, \boldsymbol{u})$ are different from the preceding ones:

Lemma 3.3. *Let $\boldsymbol{k} = (k_1, \dots, k_M)$ be a given vector of M different positive integers. Then there exist $\eta > 0$ and functions $g_3(x, \boldsymbol{u})$, $g_4(x, \boldsymbol{u})$, $h_3(x, \boldsymbol{u})$, $h_4(x, \boldsymbol{u})$ ($\boldsymbol{u} = (u_1, \dots, u_M)$) with the following properties:*

1. *$g_3(x, \boldsymbol{u})$, $g_4(x, \boldsymbol{u})$, $h_3(x, \boldsymbol{u})$, $h_4(x, \boldsymbol{u})$, are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - f(\boldsymbol{u})| < \eta$ (with $f(\boldsymbol{u})$ from Lemma 3.2).*
2. *$g_3(\rho, 1, \dots, 1) > 0$, $g_4(\rho, 1, \dots, 1) > 0$, $h_3(\rho, 1, \dots, 1) = b^3/3 \ne 0$, where b is given by (1.1), and $h_4(\rho, 1, \dots, 1) \ne 0$.*
3. *$t_{\boldsymbol{k}}(x, \boldsymbol{u})$ and $t_{\boldsymbol{k}}^{(f)}(x, \boldsymbol{u})$ can be analytically continued to R (which is defined in Lemma 3.2) such that*

$$t_{\boldsymbol{k}}(x, \boldsymbol{u}) = g_3(x, \boldsymbol{u}) - h_3(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.8}$$

and

$$t_{\boldsymbol{k}}^{(f)}(x, \boldsymbol{u}) = g_4(x, \boldsymbol{u}) - h_4(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.9}$$

for $(x, \boldsymbol{u}) \in R$ and $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

Proof. Throughout this proof let $A_i = A_i(x, \boldsymbol{u})$, $i = 1, 2, \dots$, denote analytic functions. We have

$$t_{\boldsymbol{k}} = t_{\boldsymbol{k}}^{(r)} - \frac{1}{2}(t_{\boldsymbol{k}}^{(p)})^2 + A_1.$$

This in conjunction with (2.2) and (3.5) gives

$$\begin{aligned}
t_{\boldsymbol{k}} =& g - \bar{h}\sqrt{1-x/f(\boldsymbol{u})} \\
=& g_1 - h_1\sqrt{1-x/f(\boldsymbol{u})} + \sum_j x(u_j-1)(Z(S_{k_j}; g_1 - h_1\sqrt{1-x/f(\boldsymbol{u})}, A_2, \ldots, A_{k_j}) \\
& - Z(S_{k_j-1}; g_1 - h_1\sqrt{1-x/f(\boldsymbol{u})}, A_2, \ldots, A_{k_j-1})) - \frac{g_1^2 + h_1^2(\rho - x)}{2} + g_1 h_1 \sqrt{1-x/f(\boldsymbol{u})}
\end{aligned}$$

By means of Taylor's theorem we get

$$Z(S_k; g_1 - h_1\sqrt{1-x/f(\boldsymbol{u})}) = \sum_{i=0}^{k} Z^{(i)}(S_k; g_1) h_1^i \left(1 - x/f(\boldsymbol{u})\right)^{i/2} \frac{(-1)^i}{i!}$$

where $Z(S_k, \cdot) = Z(S_k; \cdot, A_2, \ldots, A_k)$ and $Z^{(i)}$ denotes the i-th derivative with respect to the first variable of the cycle index. Thus we obtain

$$\bar{h} = h_1 \left(1 - g_1 + \sum_j x(u_j - 1)(Z'(S_{k_j}; g_1) - Z'(S_{k_j-1}; g_1)) + \left(1 - \frac{x}{f(\boldsymbol{u})}\right) H \right)$$

with an analytic function H. and especially

$$\bar{h}(f(\boldsymbol{u}), \boldsymbol{u}) = h_1(f(\boldsymbol{u}), \boldsymbol{u}) \left(g_1(f(\boldsymbol{u}), \boldsymbol{u}) - 1 + f(\boldsymbol{u}) \sum_j (u_j - 1)(Z'(S_{k_j}; g_1) - Z'(S_{k_j-1}; g_1)) \right).$$

On the other hand note that $x = f(\boldsymbol{u})$ and $t = g_1(f(\boldsymbol{u}), \boldsymbol{u})$ are the solutions of

$$\begin{aligned}
t &= xQ(x, \boldsymbol{u}) e^t + \sum_j x(u_j - 1) Z(S_{k_j-1}; t) \\
1 &= xQ(x, \boldsymbol{u}) e^t + \sum_j x(u_j - 1) Z'(S_{k_j-1}; t)
\end{aligned}$$

which yields

$$g_1(f(\boldsymbol{u}), \boldsymbol{u}) = 1 - f(\boldsymbol{u}) \sum_j (u_j - 1)(Z'(S_{k_j-1}; g_1) - Z(S_{k_j-1}; g_1)).$$

By (2.5) this implies $\bar{h}(f(\boldsymbol{u}), \boldsymbol{u}) \equiv 0$ and setting

$$h_3(x, \boldsymbol{u}) = \bar{h}(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{-1}$$

we obtain (3.8).

Since

$$t_{\boldsymbol{k}}^{(f)} = A e^{t_{\boldsymbol{k}}},$$

where A denotes an analytic function, (3.9) follows from (2.4) and (3.8). □

3.2. **Plane Trees.** If we set $p^{(p)}(x) = \sum_{n\ge1} |\mathcal{P}_n^{(p)}| x^n$ then

$$p^{(p)}(x) = \frac{x}{1 - p^{(p)}(x)}$$

or

$$p^{(p)}(x) = \frac{1}{2} - \frac{\sqrt{1-4x}}{2} = \sum_{n\ge1} \frac{1}{n} \binom{2n-2}{n-1} x^n.$$

However, $p^{(r)}(x) = \sum_{n\ge1} |\mathcal{P}_n^{(r)}| x^n, p(x) = \sum_{n\ge1} |\mathcal{P}_n| x^n$, and $p^{(f)}(x) = \sum_{n\ge1} |\mathcal{P}_n^{(f)}| x^n$ are not as explicit as $p^{(p)}(x)$.

Lemma 3.4. *There exist $\eta > 0$ and functions $r_1(x), r_2(x), s_1(x), s_2(x)$, which are analytic for $|x - \frac{1}{4}| < \eta$ such that*

$$p^{(r)}(x) = -\frac{1}{4}\sqrt{1-4x} + \left(\frac{1}{4} + \frac{\log 2}{12}\right)(1-4x)^{3/2} + r_1(x)(1-4x)^{5/2} + s_1(x), \tag{3.10}$$

$$p(x) = \left(\frac{1}{4} + \frac{\log 2}{12}\right)(1-4x)^{3/2} + r_1(x)(1-4x)^{5/2} + s_1(x) - \frac{1}{4} + \frac{x}{2}, \tag{3.11}$$

$$p^{(f)}(x) = \left(\frac{1}{4} + \frac{\log 2}{12}\right)e^{s_1(\frac{1}{4})-\frac{1}{8}}(1-4x)^{3/2} + r_2(x)(1-4x)^{5/2} + s_2(x), \tag{3.12}$$

where

$$s_1\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{d\geq 1}\frac{\varphi(d)}{d}\log\frac{2}{1+\sqrt{1+4^{1-d}}}.$$

Furthermore, $p^{(r)}(x), p(x)$, and $p^{(f)}(x)$ can be analytically continued to the region $|x| < \frac{1}{4} + \frac{\eta}{2}$, $\arg(x - \frac{1}{4}) \neq 0$.

Proof. First we prove (3.10). Since

$$\begin{aligned} p^{(r)}(x) &= x\sum_{n\geq 1} Z(C_n, p^{(p)}(x)) \\ &= x\sum_{n\geq 1}\frac{1}{n}(p^{(r)}(x))^n + x\sum_{n\geq 1}\frac{1}{n}\sum_{d|n, d>1}\varphi(d)p^{(p)}(x^{n/d}) \\ &= S_1 + S_2 \end{aligned}$$

and

$$\begin{aligned} S_1 &= x\log\frac{1}{1-\frac{1}{2}+\frac{1}{2}\sqrt{1-4x}} \\ &= \frac{\log 2}{4} - \frac{1}{4}\sqrt{1-4x} + \left(\frac{1}{8} - \frac{\log 2}{4}\right)(1-4x) + \frac{\log 2}{12}(1-4x)^{3/2} + \cdots \end{aligned}$$

it suffices to show that S_2 is analytic around $x = \frac{1}{4}$. For this purpose we use the fact that $|p^{(p)}(x)| \leq 2|x|$ for $|x| \leq \frac{1}{4}$. Hence, if $|x| \leq \frac{1}{3}$ then $|x^d| \leq \frac{1}{9} < \frac{1}{4}$ for $d > 1$ and we obtain the estimate

$$\begin{aligned} S_2 &\leq \frac{1}{3}\sum_{n\geq 1}\frac{1}{n}\sum_{d|n, d>1}\varphi(d)2^{n/d}|x^d|^{n/d} \\ &\leq \frac{1}{3}\sum_{n\geq 1}\frac{1}{n}|2x|^n\sum_{d|n}\varphi(d) \\ &= \frac{1}{3}\frac{|2x|}{1-|2x|} \leq \frac{2}{3}. \end{aligned}$$

Thus, S_2 is even analytic for $|x| < \frac{1}{3}$ and (3.10) follows.

(3.11) and (3.12) are immediate from (3.10). □

Corollary . *We have*

$$\begin{aligned} |\mathcal{P}_n^{(p)}| &= \frac{1}{\sqrt{\pi}}4^{n-1}n^{-3/2}\left(1+O(n^{-1})\right), \\ |\mathcal{P}_n^{(r)}| &= \frac{1}{2\sqrt{\pi}}4^{n-1}n^{-3/2}\left(1+O(n^{-1})\right), \\ |\mathcal{P}_n| &= \frac{3}{\sqrt{\pi}}\left(\frac{1}{4}+\frac{\log 2}{12}\right)4^{n-1}n^{-5/2}\left(1+O(n^{-1})\right), \\ |\mathcal{P}_n^{(f)}| &= \frac{3}{\sqrt{\pi}}\left(\frac{1}{4}+\frac{\log 2}{12}\right)e^{s_1(\frac{1}{4})-\frac{1}{8}}4^{n-1}n^{-5/2}\left(1+O(n^{-1})\right). \end{aligned}$$

The next two lemmas correspond to Lemma 3.2 and Lemma 3.3.

Lemma 3.5. *Let $\boldsymbol{k} = (k_1, \dots, k_M)$ be a given vector of M different positive integers. Then there exist $\eta > 0$ and functions $g_1(x, \boldsymbol{u})$, $g_2(x, \boldsymbol{u})$, $h_1(x, \boldsymbol{u})$, $h_2(x, \boldsymbol{u})$, $f(\boldsymbol{u})$ ($\boldsymbol{u} = (u_1, \dots, u_M)$) with the following properties:*

1. *$g_1(x, \boldsymbol{u})$, $g_2(x, \boldsymbol{u})$, $h_1(x, \boldsymbol{u})$, $h_2(x, \boldsymbol{u})$, $f(\boldsymbol{u})$ are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$, and $|x - f(\boldsymbol{u})| < \eta$.*
2. *$g_1(\frac14, 1, \dots, 1) = \frac12$, $g_2(\frac14, 1, \dots, 1) > 0$, $h_1(\frac14, 1, \dots, 1) = \frac12$, $h_2(\frac14, 1, \dots, 1) = \frac14$, and $f(1, \dots, 1) = \frac14$.*
3. *$p^{(p)}_{\boldsymbol{k}}(x, \boldsymbol{u})$ and $p^{(r)}_{\boldsymbol{k}}(x, \boldsymbol{u})$ can be analytically continued to the region*
$$R = \{(x, \boldsymbol{u}) \in \mathbf{C}^{M+1} : |u_j| \le 1 + \frac{\eta}{2}, 1 \le j \le M, |x| \le \frac14 + \frac{\eta}{2}, \arg(x - f(\boldsymbol{u})) \ne 0\}$$
such that
$$p^{(p)}_{\boldsymbol{k}}(x, \boldsymbol{u}) = g_1(x, \boldsymbol{u}) - h_1(x, \boldsymbol{u}) \sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.13}$$
and
$$p^{(r)}_{\boldsymbol{k}}(x, \boldsymbol{u}) = g_2(x, \boldsymbol{u}) - h_2(x, \boldsymbol{u}) \sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.14}$$
for $(x, \boldsymbol{u}) \in R$ and $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

Proof. First, (3.13) follows from Lemma 2.2 and Proposition 3.1.

Next, it follows as in the proof of Lemma 3.2 that there exists $\eta_1 > 0$ such that $p^{(p)}_{\boldsymbol{k}}(x^i, \boldsymbol{u}^i)$ is analytic for $|x| < \frac14 + \eta_1$ and $|u_j| < 1 + \eta_1$, $1 \le j \le M$, and that
$$|p^{(p)}_{\boldsymbol{k}}(x^i, \boldsymbol{u}^i)| \le 2|x(1 + \eta_1)|^i.$$
Hence, it follows that
$$A = \sum_{n \ge 1} \frac{1}{n} \sum_{d|n, d>1} \varphi(d) \left(p^{(p)}_{\boldsymbol{k}}(x^d, \boldsymbol{u}^d) \right)^{n/d}$$
is analytic for $|x| < \frac14 + \eta_1$ and $|u_j| < 1 + \eta_1$, $1 \le j \le M$. Thus
$$p^{(r)}_{\boldsymbol{k}}(x, \boldsymbol{u}) = x \log \frac{1}{1 - p^{(p)}_{\boldsymbol{k}}(x, \boldsymbol{u})} + xA + \sum_{j=1}^{M} x(u_j - 1) Z(C_{k_j - 1}, p^{(p)}_{\boldsymbol{k}}(x, \boldsymbol{u}), p^{(p)}_{\boldsymbol{k}}(x^2, \boldsymbol{u}^2), \dots)$$
has a representation of the form (3.14). □

Lemma 3.6. *Let $\boldsymbol{k} = (k_1, \dots, k_M)$ be a given vector of M different positive integers. Then there exist $\eta > 0$ and functions $g_3(x, \boldsymbol{u})$, $g_4(x, \boldsymbol{u})$, $h_3(x, \boldsymbol{u})$, $h_4(x, \boldsymbol{u})$ ($\boldsymbol{u} = (u_1, \dots, u_M)$) with the following properties:*

1. *$g_3(x, \boldsymbol{u})$, $g_4(x, \boldsymbol{u})$, $h_3(x, \boldsymbol{u})$, $h_4(x, \boldsymbol{u})$, are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - f(\boldsymbol{u})| < \eta$ (with $f(\boldsymbol{u})$ from Lemma 3.5).*
2. *$g_3(\rho, 1, \dots, 1) > 0$, $g_4(\rho, 1, \dots, 1) > 0$, $h_3(\rho, 1, \dots, 1) = \left(\frac14 + \frac{\log 2}{12}\right)$, and $h_4(\rho, 1, \dots, 1) \ne 0$.*
3. *$t_{\boldsymbol{k}}(x, \boldsymbol{u})$ and $t^{(f)}_{\boldsymbol{k}}(x, \boldsymbol{u})$ can be analytically continued to R (which is defined in Lemma 3.5) such that*
$$p_{\boldsymbol{k}}(x, \boldsymbol{u}) = g_3(x, \boldsymbol{u}) - h_3(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.15}$$

and

$$p_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) = g_4(x,\boldsymbol{u}) - h_4(x,\boldsymbol{u})\left(1-\frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.16}$$

for $(x,\boldsymbol{u}) \in R$ *and* $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

Proof. Since, $p_{\boldsymbol{k}} = p_{\boldsymbol{k}}^{(r)} - \frac{1}{2}(p_{\boldsymbol{k}}^{(p)})^2 + A$ (where A denotes an analytic function) it is clear that $p_{\boldsymbol{k}}$ has a representation of the form

$$p_{\boldsymbol{k}} = g_3 - \overline{h}\sqrt{1-\frac{x}{f}},$$

with $\overline{h}(\frac{1}{4},1,\dots,1) = 0$. Therefore, we only have to show that $\overline{h}(f(\boldsymbol{u}),\boldsymbol{u}) \equiv 0$ for $|u_j-1| < \eta$, $1 \le j \le M$. For this purpose we represent $p_{\boldsymbol{k}}$ as

$$p_{\boldsymbol{k}} = h_1\left(g_1 - x\frac{h_1}{1-g_1} - \sum_{j=1}^{M} x(u_j-1)g_1^{k_j-1}\right)\sqrt{1-\frac{x}{f}} + \left(1-\frac{x}{f}\right)H,$$

where H denotes a bounded function. (We only have to use (3.13), Lemma 2.2, and Taylor's theorem.) Since $g_1(f(\boldsymbol{u}),\boldsymbol{u}) = p_{\boldsymbol{k}}^{(p)}(f(\boldsymbol{u}),\boldsymbol{u})$ it follows from (2.11) that

$$g_1 - x\frac{h_1}{1-g_1} - \sum_{j=1}^{M} x(u_j-1)g_1^{k_j-1} \equiv 0$$

for $x = f(\boldsymbol{u})$. Thus, $\overline{h}(f(\boldsymbol{u}),\boldsymbol{u}) \equiv 0$ and (3.15) follows.

Finally, (3.16) follows from (2.14), (3.16), and Taylor's theorem. □

3.3. **Labeled Trees.** As already mentioned the solution of $l^{(p)} = xe^{l^{(p)}}$ is given by $l^{(p)}(x) = \sum_{n\ge1} n^{n-1}x^n/n!$, i.e. $|\mathcal{L}_n^{(p)}| = |\mathcal{L}_n^{(r)}| = n^{n-1}$. Furthermore, $|\mathcal{L}_n| = n^{n-2}$ and (as we will see in a moment) $|\mathcal{L}_n^{(f)}| \sim \sqrt{e}n^{n-2}$.

Lemma 3.7. *There exists* $\eta > 0$ *and functions* $r_1(x), r_2(x), r_3(x), r_4(x), r_5(x)$, *which are analytic for* $|x - \frac{1}{4}| < \eta$ *such that*

$$\begin{aligned} l^{(p)}(x) &= l^{(r)}(x) \\ &= 1 - \sqrt{2}\sqrt{1-ex} + \frac{2}{3}(1-ex) + r_1(x)(1-ex)^{3/2} + r_2(x)(1-ex)^2, \end{aligned} \tag{3.17}$$

$$l(x) = \frac{1}{2} + \frac{2\sqrt{2}}{3}(1-ex)^{3/2} + r_3(x)(1-ex)^2 + r_4(x)(1-ex)^{5/2}, \tag{3.18}$$

$$p^{(f)}(x) = \sqrt{e} + \frac{2\sqrt{2e}}{3}(1-ex)^{3/2} + r_4(x)(1-ex)^2 + r_5(x)(1-ex)^{5/2}. \tag{3.19}$$

Furthermore, $l^{(r)}(x), l(x)$, *and* $l^{(f)}(x)$ *can be analytically continued to the region* $|x| < \frac{1}{4} + \frac{\eta}{2}$, $\arg(x - \frac{1}{4}) \ne 0$.

Proof. (3.17) follows directly from Proposition 3.1. The coefficient of $(1-ex)$ can easily be determined by inserting $l^{(p)}$ into the Taylor series expansion of $F(l,x) = xe^l - l$ around $x_0 = \frac{1}{e}$, $l_0 = 1$.

Since $l(x) = l^{(r)}(x) - \frac{1}{2}(l^{(p)}(x))^2$ and $l^{(f)}(x) = e^{l(x)}$ (3.18) and (3.19) follow immediately from (3.17). □

The next two lemmas correspond to Lemma 3.2 and Lemma 3.3 resp. to Lemma 3.5 and Lemma 3.6. We state them without proof.

Lemma 3.8. *Let $\boldsymbol{k} = (k_1, \ldots, k_M)$ be a given vector of M different positive integers. Then there exists $\eta > 0$ and functions $g_1(x,\boldsymbol{u})$, $g_2(x,\boldsymbol{u})$, $h_1(x,\boldsymbol{u})$, $h_2(x,\boldsymbol{u})$, $f(\boldsymbol{u})$ ($\boldsymbol{u} = (u_1, \ldots, u_M)$) with the following properties:*

1. *$g_1(x,\boldsymbol{u})$, $g_2(x,\boldsymbol{u})$, $h_1(x,\boldsymbol{u})$, $h_2(x,\boldsymbol{u})$, $f(\boldsymbol{u})$ are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - f(\boldsymbol{u})| < \eta$.*
2. *$g_i(\frac{1}{4}, 1, \ldots, 1) = 1$, $h_i(\frac{1}{4}, 1, \ldots, 1) = \sqrt{2}$, $i = 1, 2$, and $f(1, \ldots, 1) = \frac{1}{e}$.*
3. *$l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u})$ and $l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u})$ can be analytically continued to the region*
$$R = \{(x,\boldsymbol{u}) \in \mathbf{C}^{M+1} : |u_j| \le 1 + \frac{\eta}{2}, 1 \le j \le M, |x| \le \frac{1}{e} + \frac{\eta}{2}, \arg(x - f(\boldsymbol{u})) \ne 0\}$$
such that
$$l_{\boldsymbol{k}}^{(p)}(x,\boldsymbol{u}) = g_1(x,\boldsymbol{u}) - h_1(x,\boldsymbol{u}) \sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.20}$$
and
$$l_{\boldsymbol{k}}^{(r)}(x,\boldsymbol{u}) = g_2(x,\boldsymbol{u}) - h_2(x,\boldsymbol{u}) \sqrt{1 - \frac{x}{f(\boldsymbol{u})}} \tag{3.21}$$
for $(x,\boldsymbol{u}) \in R$ and $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

Lemma 3.9. *Let $\boldsymbol{k} = (k_1, \ldots, k_M)$ be a given vector of M different positive integers. Then there exist $\eta > 0$ and functions $g_3(x,\boldsymbol{u})$, $g_4(x,\boldsymbol{u})$, $h_3(x,\boldsymbol{u})$, $h_4(x,\boldsymbol{u})$ ($\boldsymbol{u} = (u_1, \ldots, u_M)$) with the following properties:*

1. *$g_3(x,\boldsymbol{u})$, $g_4(x,\boldsymbol{u})$, $h_3(x,\boldsymbol{u})$, $h_4(x,\boldsymbol{u})$, are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - f(\boldsymbol{u})| < \eta$ (with $f(\boldsymbol{u})$ from Lemma 3.8).*
2. *$g_3(\rho, 1, \ldots, 1) = \frac{1}{2}$, $g_4(\rho, 1, \ldots, 1) = \sqrt{e}$, $h_3(\rho, 1, \ldots, 1) = 2\sqrt{2}/3$, and $h_4(\rho, 1, \ldots, 1) = 2\sqrt{2e}/3$.*
3. *$l_{\boldsymbol{k}}(x,\boldsymbol{u})$ and $l_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u})$ can be analytically continued to R (which is defined in Lemma 3.8) such that*
$$l_{\boldsymbol{k}}(x,\boldsymbol{u}) = g_3(x,\boldsymbol{u}) - h_3(x,\boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.22}$$
and
$$l_{\boldsymbol{k}}^{(f)}(x,\boldsymbol{u}) = g_4(x,\boldsymbol{u}) - h_4(x,\boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2} \tag{3.23}$$
for $(x,\boldsymbol{u}) \in R$ and $|u_j - 1| < \eta$, $1 \le j \le M$, $|x - f(\boldsymbol{u})| < \eta$.

4. Limiting Distributions

With help of Lemmas 3.2, 3.3, 3.5, 3.6, 3.8, and 3.9 it is rather easy to obtain the proposed limiting distributions.

Proposition 4.1. *Suppose that $y(x,\boldsymbol{u}) = \sum y_{n\boldsymbol{m}} x^n \boldsymbol{u}^{\boldsymbol{m}}$ ($\boldsymbol{u} = (u_1, \ldots, u_M), \boldsymbol{m} = (m_1, \ldots, m_M)$) is an analytic function with $y_{n\boldsymbol{m}} \ge 0$ for all $n, \boldsymbol{m}$ and that there exists $\eta > 0$ and functions $g(x,\boldsymbol{u})$, $h(x,\boldsymbol{u})$, $f(\boldsymbol{u})$ which are analytic for $|u_j - 1| < \eta$, $1 \le j \le M$ and $|x - \rho| < \eta$, where $\rho > 0$ is the radius of convergence of $y(x, 1, \ldots, 1)$ such that $y(x,\boldsymbol{u})$ can be analytically continued to R and that*
$$y(x,\boldsymbol{u}) = g(x,\boldsymbol{u}) - h(x,\boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{1/2}$$
for $(x,\boldsymbol{u}) \in R$, $|u_j - 1| < \eta$, $1 \le j \le M$, and $|x - f(\boldsymbol{u})| < \eta$. Then $y_n(\boldsymbol{u}) = \sum_{\boldsymbol{m}} y_{n\boldsymbol{m}} \boldsymbol{u}^{\boldsymbol{m}} = [x^n] y(x,\boldsymbol{u})$ is asymptotically given by
$$y_n(\boldsymbol{u}) = \frac{h(f(\boldsymbol{u}),\boldsymbol{u})}{2\sqrt{\pi} n^{3/2}} f(\boldsymbol{u})^{-n+1} + \mathcal{O}\left(\frac{f(\boldsymbol{u})^{-n}}{n^{5/2}}\right) \tag{4.1}$$

uniformly for $|u_j - 1| < \eta$, $1 \le j \le M$.

Similarly, if

$$y(x, \boldsymbol{u}) = g(x, \boldsymbol{u}) + h(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2}$$

for $(x, \boldsymbol{u}) \in R$, $|u_j - 1| < \eta$, $1 \le j \le M$, *and* $|x - f(\boldsymbol{u})| < \eta$. *Then* $y_n(\boldsymbol{u}) = \sum_{\boldsymbol{m}} y_{n\boldsymbol{m}} \boldsymbol{u}^{\boldsymbol{m}} = [x^n] y(x, \boldsymbol{u})$ *is asymptotically given by*

$$y_n(\boldsymbol{u}) = \frac{3h(f(\boldsymbol{u}), \boldsymbol{u})}{4\sqrt{\pi} n^{5/2}} f(\boldsymbol{u})^{-n+1} + \mathcal{O}\left(\frac{f(\boldsymbol{u})^{-n}}{n^{7/2}}\right) \tag{4.2}$$

uniformly for $|u_j - 1| < \eta$, $1 \le j \le M$.

Proof. By Taylor's theorem

$$h(x, \boldsymbol{u}) = \sum_{l \ge 0} h_l(\boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^l,$$

where

$$h_l(\boldsymbol{u}) = \frac{(-f(\boldsymbol{u}))^l}{l!} \left[\frac{\partial^l}{\partial x^l} h(x, \boldsymbol{u})\right]_{x = f(\boldsymbol{u})}.$$

Thus

$$h(x, \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{1/2} = h(f(\boldsymbol{u}), \boldsymbol{u}) \left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{1/2} + \mathcal{O}\left(\left(1 - \frac{x}{f(\boldsymbol{u})}\right)^{3/2}\right)$$

uniformly for $|u_j - 1| \le \frac{3\eta}{4}$, $1 \le j \le M$, and $|x - f(\boldsymbol{u})| \le \frac{3\eta}{4}$. By the analytic continuation property and continuity it follows that (4) also holds uniformly for $(x, \boldsymbol{u}) \in R$. Hence [4, Theorem 3A] implies (4.1).

The proof of (4.2) is the same. □

Now suppose that $y_{n\boldsymbol{m}}$ ($\boldsymbol{m} = (m_1, \ldots, m_M)$) are non-negative numbers such that

$$y_n = \sum_{\boldsymbol{m}} y_{n\boldsymbol{m}}$$

is finite for all $n > 0$. Then it is possible to study random vectors $\boldsymbol{X}_n = (X_{n1}, \ldots, X_{nM})$ with

$$\mathbf{P}\{\boldsymbol{X}_n = \boldsymbol{m}\} = \frac{y_{n\boldsymbol{m}}}{y_n}.$$

For example, if $y_n(\boldsymbol{u}) = \sum_{\boldsymbol{m}} y_{n\boldsymbol{m}} \boldsymbol{u}^{\boldsymbol{m}}$ behaves like (4.1) with $h(\rho, 1, \ldots, 1) \ne 0$ then $\boldsymbol{X}_n$ has a proper limiting distribution.

Proposition 4.2 ([1])**.** *Suppose that* $y_{n\boldsymbol{m}} \ge 0$ *and that there exist functions* $H(\boldsymbol{u}), f(\boldsymbol{u})$ *defined for* $u_j = e^{it_j}, 1 \le j \le M$, $|t_j| < \varepsilon$, t_j *real, such that* $H(1, \ldots, 1) \ne 0$ *and* $H(\boldsymbol{u})$ *is uniformly continuous and that* $f(1, \ldots, 1) = \rho > 0$ *and* $f(e^{it_1}, \ldots, e^{it_M})$ *has continuous third derivatives with*

$$y_n(\boldsymbol{u}) = \sum_{\boldsymbol{m}} y_{n\boldsymbol{m}} \boldsymbol{u}^{\boldsymbol{m}} \sim a_n H(\boldsymbol{u}) f(\boldsymbol{u})^{-n} \tag{4.3}$$

uniformly for $|t_j| < \varepsilon$, $1 \le j \le M$, *for some sequence* $a_n > 0$. *Furthermore set* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ *and* $\boldsymbol{\Sigma} = (\sigma_{ij})$ *where*

$$\mu_j = i \frac{\partial}{\partial t_j} \log f(e^{it_1}, \ldots, e^{it_M}) \Big|_{t_1 = \cdots = t_M = 0} \qquad 1 \le j \le M$$

$$\sigma_{ij} = -\frac{\partial^2}{\partial t_i \partial t_j} \log f(e^{it_1}, \ldots, e^{it_M}) \Big|_{t_1 = \cdots = t_M = 0} \qquad 1 \le i, j \le M$$

and assume that $\det \mathbf{\Sigma} \neq 0$. *Then*

$$\frac{\boldsymbol{X}_n - n\boldsymbol{\mu}}{\sqrt{n}} \to \mathcal{N}(0, \mathbf{\Sigma}),$$

i.e. $\boldsymbol{X}_n$ *is asymptotically normal with mean value* $\sim n\boldsymbol{\mu}$ *and covariance matrix* $\sim n\mathbf{\Sigma}$.

Remark 3. If we have more informations about $y_n(\boldsymbol{u})$ than (4.3), e.g. an asymptotic expansion of the form (4.1) with $h(f(1,\dots,1),1,\dots,1) \neq 0$ then we can also obtain local limit theorems and asymptotic expansions for mean value and covariance matrix:

$$\begin{aligned} \mathbf{E}\boldsymbol{X}_n &= \boldsymbol{\mu} n + \mathcal{O}(1) \\ \mathbf{Cov}\boldsymbol{X}_n &= \mathbf{\Sigma} n + \mathcal{O}(1) \end{aligned}$$

See [1, 2, 3] for more details.

Remark 4. In Proposition 3.1 it assumed that $y = y(x, \boldsymbol{u})$ satisfies a functional equation $y = F(x, \boldsymbol{u}, y)$. By varying $\boldsymbol{u}_0$ (in Proposition 3.1, compare also with [3]) it follows that $y = y(f(\boldsymbol{u}), \boldsymbol{u}), x = f(\boldsymbol{u})$ are the solutions of the system of equations

$$y = F(x, \boldsymbol{u}, y), \tag{4.4}$$

$$1 = F_y(x, \boldsymbol{u}, y). \tag{4.5}$$

Furthermore, the parameters of interest $\boldsymbol{\mu} = (\mu_1, \dots, \mu_M)$ and $\mathbf{\Sigma} = (\sigma_{ij})$ are given by

$$\mu_i = -\frac{f_i(\mathbf{1})}{f(\mathbf{1})}$$

and by

$$\sigma_{ij} = \frac{f_i(\mathbf{1})f_j(\mathbf{1}) - f_{ij}(\mathbf{1})f(\mathbf{1})}{f^2(\mathbf{1})} - \delta_{ij}\frac{f_i(\mathbf{1})}{f(\mathbf{1})},$$

where the subscripts i, j denote the partial derivative with respect to u_i resp. u_j, e.g. $f_j = \partial/\partial u_j \; f(\boldsymbol{u})$. Implicit differentiation of (4.4) yields

$$y_i = F_x f_i + F_i + F_y y_i.$$

By means (4.5) we get $F_x f_i + F_i \equiv 0$ which implies

$$\mu_i = \frac{F_i(x_0, \mathbf{1}, y_0)}{x_0 F_x(x_0, \mathbf{1}, y_0)},$$

where $x_0 = f(\mathbf{1})$ and $y_0 = y(x_0, \mathbf{1})$. Again implicit differentiation of this equation and the preceding one gives

$$\begin{aligned} f_{ij} = &\frac{1}{F_{yy}F_x}\left(\frac{F_i F_{yx}}{F_x} - F_{yi}\right)\left(\frac{F_j F_{yx}}{F_x} - F_{yj}\right) \\ &- \frac{1}{F_x}\left(\frac{F_i F_j F_{xx}}{F_x^2} - \frac{F_i F_{xj} + F_j F_{xi}}{F_x} + F_{ij}\right). \end{aligned} \tag{4.6}$$

Hence we can also evaluate $\mathbf{\Sigma}$ (compare also with (2.7)).

4.1. **Unlabeled, Nonplane Trees.** Now we will use Proposition 4.2 to prove Theorem 2.1. By applying the preceding remark it follows that the parameters of interest $\boldsymbol{\mu} = (\mu_{k_1}, \dots, \mu_{k_M})$ and $\mathbf{\Sigma} = (\sigma_{ij})$ are correct.

In the introduction we mentioned that the constants μ_{k_i} determining the asymptotic behavior of the mean value of the number of nodes of degree k_i decrease geometrically in k_i. Let us now examine how the entries of $\mathbf{\Sigma}$ behave for large $\boldsymbol{k}$, i.e. we have to determine the partial derivatives of F asymptotically. Using the functional equations (4.4) and (4.5) and evaluating at $(x, \boldsymbol{u}, t) = (\rho, \mathbf{1}, 1)$ yields $Q = 1/\rho e$ and

$$\begin{aligned}
F_x =& F_{tx} = Qe + \rho Q_x e = \frac{1}{\rho}\left(1+\sum_{l\ge 2} t_x(\rho^l,\mathbf{1})\rho^l\right)\\
F_t =& F_{tt} = \rho Q e = 1\\
F_i =& \rho Q_i e + \rho Z(S_{k_i-1};t) = \sum_{l\ge 2} t_i(\rho^l,\mathbf{1}) + \rho Z(S_{k_i-1};t)\\
F_{ti} =& \rho Q_i e + \rho Z(S_{k_i-2};t) = \sum_{l\ge 2} t_i(\rho^l,\mathbf{1}) + \rho Z(S_{k_i-1};t)\\
F_{ij} =& \rho Q_{ii} e + \rho\frac{\partial}{\partial u_i} Z(S_{k_j-1};t) + \rho\frac{\partial}{\partial u_j} Z(S_{k_i-1};t)\\
=& \left(\sum_{l\ge 2} t_i(\rho^l,\mathbf{1})\right)\left(\sum_{l\ge 2} t_j(\rho^l,\mathbf{1})\right) + \sum_{l\ge 2} l t_{ij}(\rho^l,\mathbf{1}) + \delta_{ij}\sum_{l\ge 2} l(l-1)t_i(\rho^l,\mathbf{1})\\
&+ \rho\frac{\partial}{\partial u_i} Z(S_{k_j-1};t) + \rho\frac{\partial}{\partial u_j} Z(S_{k_i-1};t)\\
F_{xi} =& \rho Q_{xi} e + Q_i e + Z(S_{k_i-1};t) + \rho\frac{\partial}{\partial x} Z(S_{k_i-1};t)\\
=& \frac{1}{\rho}\left(1+\sum_{l\ge 2} t_x(\rho^l,\mathbf{1})\rho^l\right)\sum_{l\ge 2} t_i(\rho^l,\mathbf{1}) + \sum_{l\ge 2} l t_{xi}(\rho^l,\mathbf{1})\rho^{l-1}\\
&+ Z(S_{k_j-1};t) + \rho\frac{\partial}{\partial x} Z(S_{k_i-1};t)\\
F_{xx} =& 2Q_x e + \rho Q_{xx} e
\end{aligned}$$

Schwenk [10] showed that

$$\sum_{l\ge 2} t_i(\rho^l,\mathbf{1}) = o\left(\rho^{k_i}\right). \tag{4.7}$$

and $Z(S_k;t) \sim C\rho^k$ where C is given by (2.10). Modifying the proof of [10, Corollary 4.1] properly gives

$$\sum_{l\ge 2} l t_{xi}(\rho^l,\mathbf{1})\rho^{l-1} = o\left(\rho^{k_i}\right)$$
$$\sum_{l\ge 2} l t_{ij}(\rho^l,\mathbf{1}) = o\left(\rho^{k_i+k_j}\right)$$

Now let us turn to the terms containing derivatives of the cycle index. We have

$$\begin{aligned}
\frac{\partial}{\partial u_i} Z(S_k;t) &= \sum_{l\ge 2} \frac{\partial}{\partial t_l} Z(S_k;t_1,\dots,t_k)\bigg|_{t_m=t(\rho^m,\mathbf{1}),m=1,\dots,k} l t_i(\rho^l,\mathbf{1})\\
&= \sum_{l\ge 2} Z(S_{k-l};t)t_i(\rho^l,\mathbf{1})
\end{aligned}$$

Note that $Z(S_{k-l};t) \sim C\rho^{k-l}$ and (4.7) was established by showing that $t_i(\rho^l) \le (2\rho^l)^{k_i}$ (see [10]). Since $2\rho^2 < \rho$ this implies $t_i(\rho^l) < (2\rho^2)^k \rho^{(l-2)k} = o\left(\rho^{(l-1)k}\right)$. Moreover we have $k \ge 1$ and $l \ge 2$ and thus $k(l-1) \ge k+l-2$ which yields $t_i(\rho^l) = o\left(\rho^{k+l}\right)$ and consequently

$$\frac{\partial}{\partial u_i} Z(S_k;t) = o\left(\rho^{k+k_i}\right).$$

The second term of this kind satisfies

$$\frac{\partial}{\partial x} Z(S_k;t) = \sum_{l\ge 2} Z(S_{k-l};t) t_x(\rho^l,\mathbf{1})\rho^{l-1}.$$

We have $Z(S_{k-l};t) = C\rho^{k-l} + o\left(\rho^{k-l}\right)$. Furthermore $t_x(y,\mathbf{1})$ is analytic at $y=0$ and thus $t_x(y,\mathbf{1}) = 1 + \mathcal{O}(y)$. Thus

$$\frac{\partial}{\partial x} Z(S_k;t) = \frac{C}{\rho} k\rho^k + o\left(\rho^k\right).$$

Hence in case of $i \ne j$ the dominating term in (4.6) is

$$\frac{F_i F_{xj} + F_j F_{xi}}{F_x^2} \sim \frac{C^2}{\rho F_x^2} \rho^{k_i+k_j}(k_i+k_j).$$

In order to compute F_x observe that $t(x,\mathbf{1}) = xQe^{t(x,\mathbf{1})}$ and by differentiation and using the fact that $t(x,\mathbf{1}) = t^{(r)}(x)$ in conjunction with (1.1) we obtain

$$1 + \sum_{l\ge 2} t_x(\rho^l,\mathbf{1})\rho^l = \lim_{x\to\rho} \frac{x t_x(x,1)(1-t(x,1))}{t(x,1)} = \frac{b^2\rho}{2}$$

and hence we get (2.8) and (2.9) and the proof of Theorem 2.1 is complete.

4.2. **Plane Trees.** It is clear that Theorem 2.2 follows from Lemma 3.6 via Propositions 4.1 and 4.2 provided that $\det \mathbf{\Sigma} > 0$.

First, it is an easy exercise to determine

$$\mathbf{\Sigma} = \left(-\frac{1}{2^{k_i+k_j}} - \frac{(k_i-2)(k_j-2)}{2^{k_i+k_j+1}} + \delta_{ij}\frac{1}{2^{k_i}} \right)_{i,j=1,\dots,M}. \tag{4.8}$$

Then the next lemma completes the proof of Theorem 2.2.

Lemma 4.1. *Suppose that $\mathbf{\Sigma}$ is a $M\times M$-matrix given by (4.8), where $k_1,k_2,\dots,k_M$ are different positive integers. Then*

$$\det \mathbf{\Sigma} = 2^{-k_1-\cdots-k_M} \left(1 - \sum_{i=1}^M \frac{1}{2^{k_i}} - \sum_{i=1}^M \frac{(k_i-2)^2}{2^{k_i+1}} + \sum_{i,j=1}^M \frac{(k_i-k_j)^2}{2^{k_i+k_j+2}} \right) > 0. \tag{4.9}$$

Proof. Set

$$\begin{aligned}
\boldsymbol{e}_1 &= (1,0,0,\dots,0),\\
\boldsymbol{e}_2 &= (0,1,0,\dots,0),\\
&\vdots\\
\boldsymbol{e}_M &= (0,0,\dots,0,1),\\
\boldsymbol{a} &= (2^{-k_1},\dots,2^{-k_M}),\\
\boldsymbol{b} &= ((k_1-2)2^{-k_1-1},\dots,(k_M-2)2^{-k_M-1}).
\end{aligned}$$

Then

$$\begin{aligned}
2^{k_1+\cdots+k_M}\det\boldsymbol{\Sigma} &= \det(\boldsymbol{e}_1-\boldsymbol{a}-(k_1-2)\boldsymbol{b},\dots,\boldsymbol{e}_M-\boldsymbol{a}-(k_M-2)\boldsymbol{b})\\
&= \det(\boldsymbol{e}_1,\dots,\boldsymbol{e}_M)\\
&\quad-\sum_{j=1}^{M}\det(\boldsymbol{e}_1,\dots,\boldsymbol{e}_{j-1},\boldsymbol{a},\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_M)\\
&\quad-\sum_{j=1}^{M}\det(\boldsymbol{e}_1,\dots,\boldsymbol{e}_{j-1},(k_j-2)\boldsymbol{b},\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_M)\\
&\quad+\sum_{1\le i<j\le M}\det\big(\boldsymbol{e}_1,\dots,\boldsymbol{e}_{i-1},\boldsymbol{a}+(k_i-2)\boldsymbol{b},\boldsymbol{e}_{i+1},\dots,\\
&\qquad\qquad\boldsymbol{e}_{j-1},\boldsymbol{a}+(k_j-2)\boldsymbol{b},\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_M\big)\\
&= 1-\sum_{i=1}^{M}\frac{1}{2^{k_i}}-\sum_{i=1}^{M}\frac{(k_i-2)^2}{2^{k_i+1}}+\sum_{1\le i<j\le M}\frac{(k_i-k_j)^2}{2^{k_i+k_j+1}},
\end{aligned}$$

which proves (4.9).

Next set $K=\{k_1,k_2,\dots,k_M\}$ and $L=\mathbf{Z}^+\setminus K$. Furthermore, let $s_k=1$ for $k\in K$ and $s_k=0$ for $k\in L$. Then we have

$$\begin{aligned}
&\frac{1}{2}+\sum_{k,m\in K}\frac{(k-m)^2}{2^{k+m+3}}+\sum_{k,m\in L}\frac{(k-m)^2}{2^{k+m+3}}-2\sum_{k\in K,m\in L}\frac{(k-m)^2}{2^{k+m+3}}\\
&=\frac{1}{2}+\sum_{k,m\in K}\frac{(k-m)^2(-1)^{s_k+s_m}}{2^{k+m+3}}\\
&=\frac{1}{2}+\frac{1}{4}\sum_{k\ge1}\frac{k^2(-1)^{s_k}}{2^k}\sum_{k\ge1}\frac{(-1)^{s_k}}{2^k}-\frac{1}{4}\left(\sum_{k\ge1}\frac{k(-1)^{s_k}}{2^k}\right)^2\\
&=\frac{1}{2}+\frac{1}{4}\left(6-2\sum_{k\in K}\frac{k^2}{2^k}\right)\left(1-2\sum_{k\in K}\frac{1}{2^k}\right)-\frac{1}{4}\left(2-2\sum_{k\in K}\frac{k}{2^k}\right)^2\\
&=1-\frac{1}{2}\sum_{k\in K}\frac{k^2}{2^k}+2\sum_{k\in K}\frac{k}{2^k}-3\sum_{k\in K}\frac{1}{2^k}+\sum_{k,m\in K}\frac{k^2}{2^{k+m}}-\sum_{k,m\in K}\frac{km}{2^{k+m}}\\
&=1-\sum_{k\in K}\frac{1}{2^k}-\sum_{k\in K}\frac{(k-2)^2}{2^{k+1}}+2\sum_{k,m\in K}\frac{(k-m)^2}{2^{k+m+2}}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2}&=\sum_{k,m\ge1}\frac{(k-m)^2}{2^{k+m+3}}\\
&=\sum_{k,m\in K}\frac{(k-m)^2}{2^{k+m+3}}+\sum_{k,m\in L}\frac{(k-m)^2}{2^{k+m+3}}+2\sum_{k\in K,m\in L}\frac{(k-m)^2}{2^{k+m+3}}
\end{aligned}$$

we immediately obtain

$$2^{k_1+\cdots+k_M}\det\boldsymbol{\Sigma}=\sum_{k,m\in L}\frac{(k-m)^2}{2^{k+m+2}}>0,$$

which completes the proof of Lemma 4.1. □

4.3. **Labeled Trees.** As in the case of plane trees Theorem 2.3 follows from Lemma 3.9 via Propositions 4.1 and 4.2 provided that $\det \mathbf{\Sigma} > 0$.

Again it is easy to calculate

$$\mathbf{\Sigma} = \left(-\frac{1+(k_i-2)(k_j-2)}{e^2(k_i-1)!(k_j-1)!} + \delta_{ij}\frac{1}{e}\frac{1}{(k_i-1)!} \right)_{i,j=1,\ldots,M}. \tag{4.10}$$

Finally the following lemma completes the proof of Theorem 2.3.

Lemma 4.2. *Suppose that $\mathbf{\Sigma}$ is a $M \times M$-matrix given by (4.10), where $k_1, k_2, \ldots, k_M$ are different positive integers. Then*

$$\det \mathbf{\Sigma} = \left(\prod_{j=1}^{M} \frac{1}{(k_j-1)!} \right) \times \tag{4.11}$$
$$\left(1 - \frac{1}{e}\sum_{i=1}^{M} \frac{1}{(k_j-1)!} - \frac{1}{e}\sum_{i=1}^{M} \frac{(k_i-2)^2}{(k_j-1)!} + \frac{1}{2e^2}\sum_{i,j=1}^{M} \frac{(k_i-k_j)^2}{(k_i-1)!(k_j-1)!} \right) > 0$$

Proof. The proof of Lemma 4.2 is almost the same as that of Lemma 4.1. Especially (4.11) follows as above.

If we set $K = \{k_1, k_2, \ldots, k_M\}$, $L = \mathbf{Z}^+ \setminus K$, and $s_k = 1$ for $k \in K$ and $s_k = 0$ for $k \in L$ then we obtain

$$\frac{1}{2} + \frac{1}{4e^2} \sum_{k,m\in K} \frac{(k-m)^2(-1)^{s_k+s_m}}{(k-1)!(m-1)!}$$
$$= 1 - \frac{1}{e}\sum_{k\in K}\frac{1}{(k-1)!} - \frac{1}{e}\sum_{k\in K}\frac{(k-2)^2}{(k-1)!} + \frac{1}{e^2}\sum_{k,m\in K}\frac{(k-m)^2}{(k-1)!(m-1)!},$$

which implies

$$\left(\prod_{j=1}^{M} (k_j-1)! \right) \det \mathbf{\Sigma} = \frac{1}{e^2} \sum_{k,m\in L} \frac{(k-m)^2}{(k-1)!(m-1)!} > 0.$$

□

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