HAYMAN ADMISSIBLE FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. An alternative generalisation of Hayman's admissible functions ([17]) to functions in several variables is developed and a multivariate asymptotic expansion for the coefficients is proved. In contrast to existing generalisations of Hayman admissibility ([7]), most of the closure properties which are satisfied by Hayman's admissible functions can be shown to hold for this class of functions as well.

1. Introduction

1.1. **General Remarks and History.** Hayman [17] defined a class of analytic functions $\sum y_n x^n$ for which their coefficients y_n can be computed asymptotically by applying the saddle point method in a rather uniform fashion. Moreover those functions satisfy nice algebraic closure properties which makes checking a function for admissibility amenable to a computer.

Many extensions of this concept can be found in the literature. E.g., Harris and Schoenfeld [16] introduced an admissibility imposing much stronger technical requirements on the functions. The consequence is that they obtain a full asymptotic expansion for the coefficients and not only the main term. The disadvantage is the loss of the closure properties. Moreover, it can be shown that if $y(x)$ is H-admissible, then $e^{y(x)}$ is HS-admissible and the error term is bounded. There are numerous applications of H-admissible or HS-admissible functions in various fields, see for instance [1, 2, 3, 8, 9, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Roughly speaking, the coefficients of an H-admissible function satisfy a normal limit law (cf. Theorem 1 in the next section). This has been generalised by Mutafchiev [25] to different limit laws.

Other investigations of limit laws for coefficients of power series can be found in [4, 5, 13, 11, 12].

1.2. **Generalisation to Functions in Several Variables.** Of course, it is a natural problem to generalise Hayman's concept to the multivariate case. Two definitions have been presented by Bender and Richmond [6, 7] which we do not state in this paper due to their complexity. The advantage of BR-admissibility and the even more general BR-superadmissibility is a wide applicability. There is an impressing list of examples in [7]. However, one loses some of the closure properties of the univariate case. Moreover, the closure properties fulfilled by BR-admissible and BR-superadmissible functions do not seem to be well suitable for an automatic treatment by a computer (in contrary to Hayman's closure properties, see e.g. [34] for H-admissibility or [10] for a generalisation).

The intention of this paper is to define an alternative generalisation of Hayman's admissibility which preserves (most of) the closure properties of the univariate case. The importance of the closure properties is that they enable us to construct new classes of H-admissible functions by applying algebraic rules on a basic class of functions known to be H-admissible. Conversely, it is possible to try to decompose a given function into H-admissible *atoms* and use such a decomposition for an admissibility check which can be done automatically by a computer. A first investigation in this direction was done recently in [10] for bivariate functions whose coefficients are related to combinatorial random variables. The univariate case was treated in [34].

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In order to achieve our goal we will stay as close as possible to Hayman's definition. This allows us to prove multivariate generalisations of most of his technical auxiliary results for the multivariate case. Then we can use essentially Hayman's proof to show the closure properties. We will require some technical side conditions which Hayman did not need. However, verifying these needs asymptotic evaluation of functions which can be done automatically using the tools developped by Salvy et al. (see [33, 35, 36]).

1.3. **Comparison with BR-admissibility.**

Advantages. The advantage of H-admissibility is that the closure properties are more similar to those of univariate H-admissibility which are more amenable to computer algebra systems. Indeed, for H-admissible functions as well as a special class of multivariate function admissibility check have successfully been implemented in Maple (see [10, 34] and remarks above).

Drawbacks. H-admissibility seems to be a narrower concept than BR-admissibility. For an important closure property, the product, we have to be more restrictive than Bender and Richmond [7]. And the only (nonobvious) combinatorial example known not to be BR-admissible which was presented by Bender and Richmond themselves is neither H-admissible.

Other remarks. If we consider functions in only one variable, then our concept of multivariate H-admissible functions coincides with Hayman's. This is not true for BR-admissible functions: Any (univariate) H-admissible function is BR-admissible as well, but the converse is not true.

1.4. **Plan of the paper.** In the next section we recall Hayman's admissibility. Then we present the definition and some basic properties of H-admissible functions in several variables. Afterwards, asymptotic properties for H-admissible functions and their derivatives are shown. In Section 5, we characterise the polynomials $P(z_1, \ldots, z_d)$ in d variables with real coefficients such that e^P is an H-admissible function. This provides a basic class of H-admissible functions as a starting point. The closure properties are shown in Section 6. The final section lists some combinatorial applications.

2. Univariate Admissible Functions

Our starting point is Hayman's [17] definition of functions whose coefficients can be computed by application of the saddle point method in a rather uniform fashion.

Definition 1. A function
$$y(x) = \sum_{n\geq 0} y_n x^n \tag{1}$$
is called admissible in the sense of Hayman (H-admissible) if it is analytic in $|x| < R$ where $0 < R \leq \infty$ and positive for $R_0 < x < R$ with some $R_0 < R$ and satisfies the following conditions:

(1) There exists a function $\delta(z) : (R_0, R) \to (0, \pi)$ such that for $R_0 < r < R$ we have
$$y\left(re^{i\theta}\right) \sim y(r) \exp\left(i\theta a(r) - \frac{\theta^2}{2} b(r)\right), \quad \text{as } r \to R,$$
uniformly for $|\theta| \leq \delta(r)$, where
$$a(r) = r\frac{y'(r)}{y(r)}$$
and
$$b(r) = ra'(r) = r\frac{y'(r)}{y(r)} + r^2\frac{y''(r)}{y(r)} - r^2\left(\frac{y'(r)}{y(r)}\right)^2.$$

(2) For $R_0 < r < R$ we have
$$y\left(re^{i\theta}\right) = o\left(\frac{y(r)}{\sqrt{b(r)}}\right), \quad \text{as } r \to R,$$
uniformly for $\delta(r) \leq |\theta| \leq \pi$.

(3) $b(r) \to \infty$ as $r \to R$.

For H-admissible functions Hayman [17] proved the following basic result:

Theorem 1. *Let $y(x)$ be a function defined in (1) which is H-admissible. Then as $r \to R$ we have*

$$y_n = \frac{y(r)}{r^n\sqrt{2\pi b(r)}}\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right)+o(1)\right), \quad \text{as } n \to \infty,$$

uniformly in n.

Corollary 1. *The function $a(r)$ is positive and increasing for sufficiently large r, and*

$$b(r) = o(a(r)^2), \quad \text{as } r \to R.$$

If we choose $r = \rho_n$ to be the (uniquely determined) solution of $a(\rho_n) = n$, then we get a simpler estimate:

Corollary 2. *Let $y(x)$ be an H-admissible function. Then we have as $n \to \infty$*

$$y_n \sim \frac{y(\rho_n)}{\rho_n^n\sqrt{2\pi b(\rho_n)}},$$

where ρ_n is uniquely defined for sufficiently large n.

The proof of the theorem is an application of the saddle point method.

By means of several technical lemmas, which we do not state here, Hayman [17] proved H-admissibility for some basic function classes. One of them is given in the following theorem.

Theorem 2. *Suppose that $p(x)$ is a polynomial with real coefficients and that all but finitely many coefficients in the power series expansion of $e^{p(x)}$ are positive, then $e^{p(x)}$ is H-admissible in the whole complex plane.*

Furthermore he showed some simple closure properties which are satisfied by H-admissible functions:

Theorem 3. (1) *If $y(x)$ is H-admissible, then $e^{y(x)}$ is H-admissible, too.*

(2) *If $y_1(x), y_2(x)$ are H-admissible, then so is $y_1(x)y_2(x)$.*

(3) *If $y(x)$ is H-admissible in $|x| < R$ and $p(x)$ is a polynomial with real coefficients and $p(R) > 0$ if $R < \infty$ and positive leading coefficient if $R = \infty$, then $y(x)p(x)$ is H-admissible in $|x| < R$.*

(4) *Let $y(x)$ be H-admissible in $|x| < R$ and $f(x)$ an analytic function in this region. Assume that $f(x)$ is real if x is real and that there exists a $\delta > 0$ such that*

$$\max_{|x|=r} |f(x)| = O\left(y(r)^{1-\delta}\right), \text{ as } r \to R.$$

Then $y(x) + f(x)$ is H-admissible in $|x| < R$.

(5) *If $y(x)$ is H-admissible in $|x| < R$ and $p(x)$ is a polynomial with real coefficients, then $y(x) + p(x)$ is H-admissible in $|x| < R$. If $p(x)$ has a positive leading coefficient, then $p(y(x))$ is also H-admissible.*

3. Multivariate Admissible Functions: Definition and Behaviour of Coefficients

In this section we will extend Hayman's results to functions in several variables. In particular, we will consider functions which are admissible in some range $\mathcal{R} \subset \mathbb{R}^d$. We will for technical simplicity assume that $\mathcal{R}$ is a simply connected set which contains the origin and has $(\infty, \dots, \infty)$ as a boundary point.

3.1. **Notations used throughout the paper.** In the sequel we will use bold letters $\mathbf{x} = (x_1, \dots, x_d)$ to denote vector valued variables (d-dimensional row vectors) and the notation $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$. Moreover, inequalities $\mathbf{x} < \mathbf{y}$ between vectors are to be understood componentwise, i.e., $\mathbf{x} < \mathbf{y} \iff x_i < y_i$ for $i = 1, \dots, d$. $\mathbf{r} \to \infty$ means that all components of $\mathbf{r}$ tend to infinity in such a way that $\mathbf{r} \in \mathcal{R}$. $\mathbf{x}^t$ denotes the transpose of a vector or matrix $\mathbf{x}$. Subscripts x_j, etc. denote partial derivatives w.r.t. x_j, etc.

For a function $y(\mathbf{x})$, $\mathbf{x} \in \mathbf{C}^d$, $\mathbf{a}(\mathbf{x}) = (a_j(\mathbf{x}))_{j=1,\dots,d}$ denotes the vector of the logarithmic (partial) derivatives of $y(\mathbf{x})$, i.e.,

$$a_j(\mathbf{x}) = \frac{x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})},$$

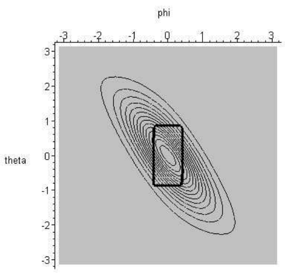
and $B(\mathbf{x}) = (B_{jk}(\mathbf{x}))_{j,k=1,\dots,d}$ denotes the matrix of the second logarithmic (partial) derivatives of $y(\mathbf{x})$, i.e.,

$$B_{jk}(\mathbf{x}) = \frac{x_j x_k y_{x_j x_k}(\mathbf{x}) + \delta_{jk} x_j y_{x_j}(\mathbf{x})}{y(\mathbf{x})} - \frac{x_j x_k y_{x_j}(\mathbf{x}) y_{x_k}(\mathbf{x})}{y(\mathbf{x})^2},$$

where δ_{jk} denotes Kronecker's δ defined by

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

3.2. **Definition and basic results.** Like in the univariate case where we required asymptotic relations depending on whether $\theta \in \Delta(r) = (-\delta(r), \delta(r))^d$ we will need a suitable domain $\mathbf{\Delta}(\mathbf{r})$ for distinguishing the behaviour of the function locally around the $\mathcal{R}$ (that means all arguments close to a real number) from the behaviour far away from $\mathcal{R}$. The geometry of multivariate functions



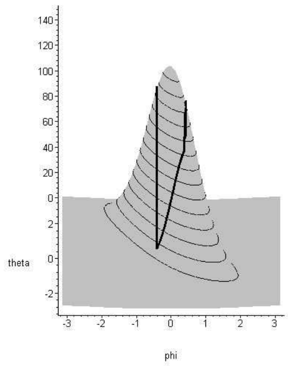


FIGURE 1. Typical shape of $|y(re^{i\varphi}, se^{i\theta})|$

is much more complicated than that of univariate ones. For instance, for $d = 2$ dimensions the typical shape of $|y(re^{i\varphi}, se^{i\theta})|$ for admissible functions is depicted in Figure 1. As the figure shows, choosing straightforwardly $\mathbf{\Delta}(\mathbf{r}) = (-\delta(r), \delta(r))^d$ will in general lead to technical difficulties, for instance if $\max_{\boldsymbol{\theta} \in \partial \mathbf{\Delta}(\mathbf{r})} \left| y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)\right|$ has to be estimated. So in order to avoid this, we have to adapt $\mathbf{\Delta}(\mathbf{r})$ to the geometry of the function. This leads to the following definition.

Definition 2. We will call a function

$$y(\mathbf{x}) = \sum_{n_1,\dots,n_d \geq 0} y_{n_1 \cdots n_d} x_1^{n_1} \cdots x_d^{n_d} \tag{2}$$

with real coefficients $y_{n_1 \cdots n_d}$ H-admissible in $\mathcal{R}$ if $y(\mathbf{x})$ is entire and positive for $\mathbf{x} \in \mathcal{R}$ and $x_j \geq R_0$ for all $j = 1, \dots, d$ (for some fixed $R_0 > 0$) and has the following properties:

(I) $B(\mathbf{r})$ is positive definite and for an orthonormal basis $\mathbf{v}_1(\mathbf{r}), \dots, \mathbf{v}_d(\mathbf{r})$ of eigenvectors of $B(\mathbf{r})$, there exists a function $\boldsymbol{\delta} : \mathbb{R}^d \to [-\pi, \pi]^d$ such that

$$y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right) \sim y(\mathbf{r}) \exp\left(i\boldsymbol{\theta}\mathbf{a}(\mathbf{r})^t - \frac{\boldsymbol{\theta} B(\mathbf{r}) \boldsymbol{\theta}^t}{2}\right), \text{ as } \mathbf{r} \to \infty, \tag{3}$$

uniformly for $\boldsymbol{\theta} \in \mathbf{\Delta}(\mathbf{r}) := \{\sum_{j=1}^d \mu_j \mathbf{v}_j(\mathbf{r})$ such that $|\mu_j| \leq \delta_j(\mathbf{r})$, for $j = 1, \dots, d\}$. That means the asymptotic formula holds uniformly for $\boldsymbol{\theta}$ inside a cuboid spanned by the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ of B, the size of which is determined by $\boldsymbol{\delta}$.

(II) The asymptotic relation

$$y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right) = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right), \text{ as } \mathbf{r} \to \infty, \tag{4}$$

holds uniformly for $\boldsymbol{\theta} \notin \boldsymbol{\Delta}(\mathbf{r})$.

(III) The eigenvalues $\lambda_1(\mathbf{r}), \ldots, \lambda_d(\mathbf{r})$ of $B(\mathbf{r})$ satisfy

$$\lambda_i(\mathbf{r}) \to \infty, \text{ as } \mathbf{r} \to \infty, \text{ for all } i = 1, \ldots, d.$$

(IV) We have $B_{ii}(\mathbf{r}) = o\left(a_i(\mathbf{r})^2\right)$, as $\mathbf{r} \to \infty$.

(V) For $\mathbf{r}$ sufficiently large and $\boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\}$ we have

$$|y(\mathbf{r}e^{i\boldsymbol{\theta}})| < y(\mathbf{r}).$$

Remark 1. Condition (IV) of the definition is a multivariate analog of Corollary 1. We want to mention that without requiring condition (IV), one can prove a weaker analog of Corollary 1, namely $\|B(\mathbf{r})\| = o(\|\mathbf{a}(\mathbf{r})\|^2)$, as $\mathbf{r} \to \infty$, where $\|\cdot\|$ denotes the spectral norm on the left-hand side and the Euclidean norm on the right-hand side. It turns out that this condition is too weak for our purposes.

Remark 2. Note that for $d = 1$ (V) follows from the other conditions. We conjecture that this is true for $d > 1$, too. However, we are only able to show that in the domains $\|\boldsymbol{\theta}\| = o\left(\sqrt{\lambda_{\min}}/\|a(\mathbf{r})\|^2\right)$ and $1/\|\boldsymbol{\theta}\| = O\left(\sqrt{\lambda_{\min}}\right)$ the inequality (V) is certainly true[1]. But since $\sqrt{\lambda_{\min}}/\|a(\mathbf{r})\|^2 = o\left(1/\sqrt{\lambda_{\min}}\right)$ there is a gap which we are not able to close.

Note that since B is a positive definite and symmetric matrix, there exists an orthogonal matrix A and a regular diagonal matrix D such that

$$B = A^t D A. \tag{5}$$

We will refer to these matrices several times throughout the paper.

Lemma 1. *Let $y(\mathbf{x})$ be a function as defined in (2) which is H-admissible. Then, as $\mathbf{r} \to \infty$, $\delta_j(\mathbf{r})^2 \lambda_j(\mathbf{r}) \to \infty$ for $j = 1, \ldots, d$.*

Proof. If we take $\boldsymbol{\theta} = \delta_j(\mathbf{r})\mathbf{v}_j(\mathbf{r})$ then we are at a point where (3) and (4) are both valid. Taking absolute values in (3) we get

$$\left|y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)\right| \sim y(\mathbf{r}) \exp\left(-\frac{\delta_j(\mathbf{r})^2 \lambda_j(\mathbf{r})}{2}\right).$$

On the other hand (4) gives

$$y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right) = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right).$$

Since $\det B(\mathbf{r}) = \prod_{j=1}^{d} \lambda_j(\mathbf{r}) \to \infty$ we must have $\delta_j(\mathbf{r})^2 \lambda_j(\mathbf{r}) \to \infty$. □

Remark 3. The above lemma shows that $\boldsymbol{\delta}$ cannot be too small. On the other hand, since the third order terms in (I) vanish asymptotically, $\|\boldsymbol{\delta}\|$ must tend to zero.

Theorem 4. *Let $y(\mathbf{x})$ be a function as defined in (2) which is H-admissible. Then as $\mathbf{r} \to \infty$ we have*

$$y_{\mathbf{n}} = \frac{y(\mathbf{r})}{\mathbf{r}^{\mathbf{n}}(2\pi)^{d/2}\sqrt{\det B(\mathbf{r})}} \left(\exp\left(-\frac{1}{2}(\mathbf{a}(\mathbf{r}) - \mathbf{n})B(\mathbf{r})^{-1}(\mathbf{a}(\mathbf{r}) - \mathbf{n})^t\right) + o(1)\right), \tag{6}$$

uniformly for all $\mathbf{n} \in \mathbf{Z}^d$.

[1] $\lambda_{\min}$ denotes the smallest eigenvalue of $B(\mathbf{r})$

Proof. Let $\mathcal{E} = \left\{\sum_j \mu_j \mathbf{v}_j \mid |\mu_j| \le \delta_j\right\}$. Then we have $y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} = I_1 + I_2$ with

$$I_1 = \frac{1}{(2\pi)^d} \underset{\mathcal{E}}{\int \cdots \int} \frac{y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)}{e^{i\mathbf{n}\boldsymbol{\theta}^t}}\, d\theta_1 \cdots d\theta_d$$

and

$$I_2 = \frac{1}{(2\pi)^d} \underset{[-\pi,\pi]^d\setminus\mathcal{E}}{\int \cdots \int} \frac{y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)}{e^{i\mathbf{n}\boldsymbol{\theta}^t}}\, d\theta_1 \cdots d\theta_d = o\left(\frac{y(\mathbf{r})}{\sqrt{\det B(\mathbf{r})}}\right)$$

as can be easily seen from the definition of H-admissibility (cf. (4)).

By (3) and the substitution $\mathbf{z} = \boldsymbol{\theta}\sqrt{(\det B(\mathbf{r}))/2}$ we have

$$\begin{aligned} I_1 &\sim \frac{y(\mathbf{r})}{(2\pi)^d} \underset{\mathcal{E}}{\int \cdots \int} \exp\left(i(\mathbf{a}(\mathbf{r})-\mathbf{n})\boldsymbol{\theta}^t - \frac{1}{2}\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t\right) d\theta_1 \cdots d\theta_d \\ &= \frac{y(\mathbf{r})}{(\pi\sqrt{2\cdot\det B(\mathbf{r})})^d} \underset{\sqrt{\frac{\det B}{2}}\cdot\mathcal{E}}{\int \cdots \int} \exp\left(i\mathbf{c}\mathbf{z}^t - \frac{\mathbf{z}B(\mathbf{r})\mathbf{z}^t}{\det B(\mathbf{r})}\right) dz_1 \cdots dz_d, \end{aligned}$$

where $\mathbf{c} = (\mathbf{a}-\mathbf{n})\sqrt{2/\det B}$. Let A and D be the matrices of (5) Substituting $\mathbf{z} = \mathbf{w}A$ and extending the integration domain to infinity (which causes an exponentially small error by Lemma 1) gives

$$I_1 \sim \frac{y(\mathbf{r})}{(\pi\sqrt{2\cdot\det B(\mathbf{r})})^d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(i\mathbf{c}A^t\mathbf{w}^t - \frac{1}{\det B(\mathbf{r})}\sum_{j=1}^d \lambda_j w_j^2\right) dw_1 \cdots dw_d,$$

where λ_j are of course the diagonal elements of D. Now observe that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\lambda_j w_j^2}{\det B(\mathbf{r})} + i(\mathbf{c}A^t)_j w_j\right) dw_j = \frac{\sqrt{\pi \det B(\mathbf{r})}}{\sqrt{\lambda_j}} \exp\left(\frac{(\mathbf{c}A^t)_j^2 \det B(\mathbf{r})}{4\lambda_j}\right)$$

and $\lambda_1 \cdots \lambda_d = \det B$ and thus

$$I_1 \sim \frac{y(\mathbf{r})}{(2\pi)^{d/2}\sqrt{\det B(\mathbf{r})}} \exp\left(-\frac{1}{4}\sum_{k=1}^d \frac{(\det B(\mathbf{r}))\cdot(\mathbf{c}A^t)_k^2}{\lambda_k}\right).$$

With

$$(\mathbf{c}A^t)_k^2 = \frac{2}{\det B(\mathbf{r})}\left(\sum_{j=1}^d (a_j(\mathbf{r}) - n_j)A_{kj}\right)^2$$

we get

$$\begin{aligned} \frac{1}{4}\sum_{k=1}^d \frac{(\det B(\mathbf{r}))\cdot(\mathbf{c}A^t)_k^2}{\lambda_k} &= \frac{1}{2}\sum_{k=1}^d \left(\frac{1}{\sqrt{\lambda_k}}\sum_{j=1}^d (a_j(\mathbf{r}) - n_j)A_{kj}\right)^2 \\ &= \frac{(\mathbf{a}(\mathbf{r})-\mathbf{n})A^tD^{-1}A(\mathbf{a}(\mathbf{r})-\mathbf{n})^t}{2} = \frac{(\mathbf{a}(\mathbf{r})-\mathbf{n})B(\mathbf{r})^{-1}(\mathbf{a}(\mathbf{r})-\mathbf{n})^t}{2} \end{aligned}$$

as desired. □

If we choose $\mathbf{r} = \boldsymbol{\rho}_{\mathbf{n}}$ to be the solution of $\mathbf{a}(\boldsymbol{\rho}_{\mathbf{n}}) = \mathbf{n}$, then we get a simpler estimate:

Corollary 3. *Let $y(x)$ be an H-admissible function. If $n_1, \ldots, n_d \to \infty$ in such a way that all components of the solution $\boldsymbol{\rho}_{\mathbf{n}}$ of $\mathbf{a}(\boldsymbol{\rho}_{\mathbf{n}}) = \mathbf{n}$ likewise tend to infinity, then we have*

$$y_{\mathbf{n}} \sim \frac{y(\boldsymbol{\rho}_{\mathbf{n}})}{\boldsymbol{\rho}_{\mathbf{n}}^{\mathbf{n}}\sqrt{(2\pi)^d \det B(\boldsymbol{\rho}_{\mathbf{n}})}},$$

where $\boldsymbol{\rho}_n$ is uniquely defined for sufficiently large $\mathbf{n}$, i.e., $\min_j n_j > N_0$ for some $N_0 > 0$.

Remark 4. Note that in contrary to the univariate case, the equation $\mathbf{a}(\boldsymbol{\rho}_\mathbf{n}) = \mathbf{n}$ has not necessarily a solution. There may occur dependencies between the variables which force all coefficients to be zero if the index $\mathbf{n}$ lies outside a cone. Thus for those $\mathbf{n}$ the expression on the right-hand side of (6) must, however, tend to zero and $\mathbf{a}(\boldsymbol{\rho}_\mathbf{n}) = \mathbf{n}$ cannot have a solution.

Even if there is a solution, some components may remain bounded.

4. Properties of H-admissible functions and their derivatives

Lemma 2. *H-admissible functions $y(\mathbf{x})$ satisfy*

$$\mathbf{a}\left(\mathbf{r}e^{\mathbf{h}}\right) \sim \mathbf{a}(\mathbf{r}), \ \ as\ \mathbf{r} \to \infty,$$

uniformly for $|h_j| = O\left(1/a_j(\mathbf{r})\right)$.

Proof. Without loss of generality assume that $d = 2$. Since B is positive definite, we have

$$B_{11}B_{22} - B_{12}^2 \geq 0 \text{ and thus } |B_{12}| \leq \sqrt{B_{11}B_{22}} = o(a_1(\mathbf{r})a_2(\mathbf{r}))$$

by condition (IV) of the definition. Note that for positive definite matrices, every 2×2-subdeterminant is nonnegative. Therefore considering only $d = 2$ is really no restriction.

Now define $\varphi_1(x_1, x_2) = a_1\left(e^{x_1}, e^{x_2}\right)$ and $\varphi_2(x_1, x_2) = a_2\left(e^{x_1}, e^{x_2}\right)$. Obviously $\frac{\partial}{\partial x_1}\varphi_1(\mathbf{x}) = B_{11}(\mathbf{x}) = o(a_1(\mathbf{x})^2)$ and $\frac{\partial}{\partial x_2}\varphi_1(\mathbf{x}) = B_{12}(\mathbf{x}) = o(a_1(\mathbf{x})a_2(\mathbf{x}))$. Analogously, we have $\frac{\partial}{\partial x_1}\varphi_2(\mathbf{x}) = o(a_1(\mathbf{x})a_2(\mathbf{x}))$ and $\frac{\partial}{\partial x_1}\varphi_1(\mathbf{x}) = o(a_2(\mathbf{x})^2)$. Let $|x_1'-x_1''| = O\left(1/a_1(\mathbf{x}')\right)$ and $|x_2'-x_2''| = O\left(1/a_2(\mathbf{x}')\right)$. Then

$$\frac{1}{\varphi_2(x_1', x_2')} - \frac{1}{\varphi_2(x_1', x_2'')} = \int_{x_2'}^{x_2''} \frac{\frac{\partial}{\partial x_2}\varphi_2(x_1', x)}{\varphi_2(x_1', x)^2}\, dx = o\left(x_2' - x_2''\right) = o\left(\frac{1}{\varphi_2(x_1', x_2')}\right), \text{ as } x_1', x_2' \to \infty,$$

which implies $\varphi_2(x_1', x_2') \sim \varphi_2(x_1', x_2'')$ or, equivalently,

$$a_2(x_1', x_2') \sim a_2(x_1', x_2'') \text{ as } x_1', x_2' \to \infty. \tag{7}$$

Now assume $x_2'' > x_2'$ and note that by Corollary 3 almost all coefficients $y_\mathbf{n}$ of $y(\mathbf{x})$ for which $\min_j n_j$ is sufficiently large are nonnegative. Hence $a_1(\mathbf{x})$ and $a_2(\mathbf{x})$ must be monotone in both variables for sufficiently large x_1, x_2. Therefore we get

$$\begin{aligned}\frac{1}{\varphi_1(\mathbf{x}')} - \frac{1}{\varphi_1(\mathbf{x}'')} &= \int_{x_2'}^{x_2''} \frac{\frac{\partial}{\partial x_2}\varphi_1(x_1', x)}{\varphi_1(x_1', x)^2}\, dx + \int_{x_1'}^{x_1''} \frac{\frac{\partial}{\partial x_1}\varphi_1(x, x_2'')}{\varphi_1(x, x_2'')^2}\, dx \\ &= o\left(\frac{a_2(x_1', x_2'')}{a_1(x_1', x_2')a_2(x_1', x_2')}\right) + o\left(x_1' - x_1''\right)\end{aligned}$$

Using (7) we finally obtain

$$\frac{1}{\varphi_1(\mathbf{x}')} - \frac{1}{\varphi_1(\mathbf{x}'')} = o\left(\frac{1}{a_1(x_1', x_2')}\right) = o\left(\frac{1}{\varphi_1(\mathbf{x}')}\right)$$

which implies $a_1(\mathbf{x}') \sim a_1(\mathbf{x}'')$. The asymptotic relation for a_2 is proved analogously and completes the proof. □

Lemma 3. *If $y(\mathbf{x})$ is an H-admissible function then for $n_j > 0$, $j = 1, \dots, d$, we have*

$$\frac{y(\mathbf{r})}{\mathbf{r}^\mathbf{n}} \to \infty \ \ as\ \mathbf{r} \to \infty.$$

Moreover, for any given $\varepsilon > 0$ we have

$$\|\mathbf{a}(\mathbf{r})\| = O\left(y(\mathbf{r})^\varepsilon\right) \ \ and\ \|B(\mathbf{r})\| = O\left(y(\mathbf{r})^\varepsilon\right)$$

as $\mathbf{r} \to \infty$.

Proof. The first relation is a trivial consequence of Theorem 4. So let us turn to the other equations. Assume that there exists $\bar{\mathbf{R}}$ such that for all $\mathbf{r} \geq \bar{\mathbf{R}}$ we have

$$\|\mathbf{a}(\mathbf{r})\|_{\max} \geq y(\mathbf{r})^{\varepsilon}.$$

This implies that for arbitrary $\mathbf{h} \in \mathbb{R}^d$ with only nonzero components, we have

$$\sum_j a_j(\bar{\mathbf{R}} + t\mathbf{h}) = \sum_j \frac{y_j(\bar{\mathbf{R}} + t\mathbf{h})}{y(\bar{\mathbf{R}} + t\mathbf{h})}(\bar{R}_j + th_j) \geq y(\bar{\mathbf{R}} + t\mathbf{h})^{\varepsilon} \cdot K$$

for $t \geq 0$ and hence

$$\sum_j \frac{y_j(\bar{\mathbf{R}} + t\mathbf{h})h_j \left(\frac{\bar{R}_j}{h_j} + t\right)}{y(\bar{\mathbf{R}} + t\mathbf{h})^{1+\varepsilon}} \geq K.$$

Let k such that

$$\max_j \frac{\bar{R}_j + th_j}{h_j} = \frac{\bar{R}_k}{h_k} + t.$$

Then

$$\sum_j \frac{y_j(\bar{\mathbf{R}} + t\mathbf{h})h_j}{y(\bar{\mathbf{R}} + t\mathbf{h})^{1+\varepsilon}} \geq \frac{K}{\frac{\bar{R}_k}{h_k} + t}.$$

Set $g(t) = y(\bar{\mathbf{R}} + t\mathbf{h})$. Therefore we have

$$\frac{g'(t)}{g(t)^{1+\varepsilon}} \geq \frac{K}{\frac{\bar{R}_k}{h_k} + t}$$

and thus

$$\int_0^{\rho} \frac{g'(t)}{g(t)^{1+\varepsilon}}\, dt \geq K \left(\log\left(\frac{\bar{R}_k}{h_k} + \rho\right) - \log\frac{\bar{R}_k}{h_k}\right) = K \log \frac{\bar{R}_k + \rho h_k}{\bar{R}_k} \tag{8}$$

Now let $\rho \to \infty$ and note that (8) is unbounded. On the other hand, the above integral evaluates to

$$\int_0^{\rho} \frac{g'(t)}{g(t)^{1+\varepsilon}}\, dt = \frac{y(\bar{\mathbf{R}})^{-\varepsilon} - y(\bar{\mathbf{R}} + \rho\mathbf{h})^{-\varepsilon}}{\varepsilon} \tag{9}$$

which is bounded for $\rho \to \infty$ and we arrive at a contradiction. □

Corollary 4. *For any $\varepsilon > 0$ we have, as $\mathbf{r} \to \infty$, $\det B(\mathbf{r}) = O\left(y(\mathbf{r})^{\varepsilon}\right)$.*

Proof. Since $\|B\|$ is the largest eigenvalue of B, we have $\det B \leq \|B\|^d$. Hence the assertion immediately follows from Lemma 3. □

Lemma 4. *Let k be fixed. Then an H-admissible function $y(\mathbf{x})$ satisfies*

$$y\left(r_1 + \frac{kr_1}{a_1(\mathbf{r})}, \dots, r_d + \frac{kr_d}{a_d(\mathbf{r})}\right) \sim e^{kd} y(r_1, \dots, r_d)$$

for $r_1, \dots, r_d \to \infty$ ($\mathbf{r} \to \infty$)

Proof. For given $h_1, \dots, h_d$ we have for some $0 < \theta < 1$

$$\log y(r_1 + h_1, \dots, r_d + h_d) - \log y(r_1, \dots, r_d) = \sum_{j=1}^{d} \frac{y_{z_j}(r_1 + \theta h_1, \dots, r_d + \theta h_d)h_j}{y(r_1 + \theta h_1, \dots, r_d + \theta h_d)}$$

$$= \sum_{j=1}^{d} \frac{h_j}{r_j + \theta h_j} a_j(r_1 + \theta h_1, \dots, r_d + \theta h_d)$$

$$= \sum_{j=1}^{d} \frac{k a_j(r_1 + \theta h_1, \dots, r_d + \theta h_d)\left(1 + O\left(\frac{1}{a_j(\mathbf{r})}\right)\right)}{a_j(r_1 + \theta h_1, \dots, r_d + \theta h_d)} \sim kd$$

where we substituted $h_j = kr_j/a_j(\mathbf{r})$ and $r_j/(r_j+\theta h_j) = 1 + O\left(1/a_j(\mathbf{r})\right)$ in the penultimate step and used Lemma 2 in the last step. $\square$

The next theorem shows that the coefficients of H-admissible functions satisfy a multivariate normal limit law.

Theorem 5. *Let $y(\mathbf{x}) = \sum_{\mathbf{n}\geq 0} y_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ be an H-admissible function. Moreover, let $\tilde{\mathbf{n}} = \mathbf{n}A^t$, where A is the orthogonal matrix defined in (5), and let $\tilde{\mathbf{a}}(\mathbf{r}) = (\tilde{a}_1(\mathbf{r}),\dots,\tilde{a}_d(\mathbf{r})) = \mathbf{a}\cdot A^t$ be the vector of the logarithmic derivatives of $y(\mathbf{x})$ w.r.t. the orthonormal eigenbasis of $B(\mathbf{r})$ given in the definition. Then we have, as $\mathbf{r}\to\infty$,*

$$\sum_{\mathbf{n}\ s.\ t.\ \forall j:\, \tilde{n}_j \leq \tilde{a}_j(\mathbf{r}) + \omega_j\sqrt{\lambda_j(\mathbf{r})}} y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} \sim \frac{y(\mathbf{r})}{(2\pi)^{d/2}} \int_{-\infty}^{\omega_d}\cdots\int_{-\infty}^{\omega_1} \exp\left(-\frac{1}{2}\sum_{j=1}^d t_j^2\right) dt_1\cdots dt_d$$

Proof. Define $N_j = \lfloor \tilde{a}_j(\mathbf{r})\rfloor$, and

$$\underline{N}_j = \left\lfloor \tilde{a}_j(\mathbf{r}) + \underline{\omega}_j\sqrt{2\det B(\mathbf{r})}\right\rfloor, \quad \overline{N}_j = \left\lfloor \tilde{a}_j(\mathbf{r}) + \overline{\omega}_j\sqrt{2\det B(\mathbf{r})}\right\rfloor$$

for some $\underline{\omega}_j < 0 < \overline{\omega}_j$. Let furthermore $N_j + 2 \leq n_j \leq \overline{N}_j$ and D be the diagonal matrix of (5). Then

$$\begin{aligned}
&\int_{n_1}^{n_1+1}\cdots\int_{n_d}^{n_d+1} \exp\left(-\frac{(\mathbf{x}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{x}-\tilde{\mathbf{a}})^t}{2}\right) dx_1\cdots dx_d\\
&\quad\leq \exp\left(-\frac{(\mathbf{n}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{n}-\tilde{\mathbf{a}})^t}{2}\right)\\
&\quad\leq \int_{\mathbf{n}-\mathbf{1}}^{\mathbf{n}} \exp\left(-\frac{(\mathbf{x}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{x}-\tilde{\mathbf{a}})^t}{2}\right) dx_1\cdots dx_d
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{N_1+2}^{\overline{N}_1+1}\cdots\int_{N_d+2}^{\overline{N}_d+1} \exp\left(-\frac{(\mathbf{x}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{x}-\tilde{\mathbf{a}})^t}{2}\right) dx_1\cdots dx_d\\
&\quad\leq \sum_{n_1=N_1+2}^{\overline{N}_1+1}\cdots\sum_{n_d=N_d+2}^{\overline{N}_d+1} \exp\left(-\frac{(\mathbf{n}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{n}-\tilde{\mathbf{a}})^t}{2}\right)\\
&\quad\leq \int_{N_1+1}^{\overline{N}_1}\cdots\int_{N_d+1}^{\overline{N}_d} \exp\left(-\frac{(\mathbf{x}-\tilde{\mathbf{a}})D(\mathbf{r})^{-1}(\mathbf{x}-\tilde{\mathbf{a}})^t}{2}\right) dx_1\cdots dx_d
\end{aligned}$$

By substituting $x_j = \tilde{a}_j(\mathbf{r}) + t_j\sqrt{\lambda_j(\mathbf{r})}$, $d\mathbf{x} = \sqrt{\det B(\mathbf{r})}\,d\mathbf{t}$, the integral becomes

$$\sqrt{\det B(\mathbf{r})}\int_{\underline{t}_1}^{\overline{t}_1}\cdots\int_{\underline{t}_d}^{\overline{t}_d} \exp\left(-\frac{1}{2}\sum_{j=1}^d t_j^2\right) dt_1\cdots dt_d$$

with $\underline{t}_j \to 0$ and $\overline{t}_j \to \omega_j$.

Now set $\tilde{N} := \left\{\mathbf{n} \in \mathbf{N}^d \text{ such that for all } j \text{ we have } \underline{N}_j \le \tilde{n}_j \le \overline{N}_j\right\}$. Then an application of Theorem 4 gives

$$\begin{aligned}\sum_{\mathbf{n}\in\tilde{N}} y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} &\sim \frac{y(\mathbf{r})}{(2\pi)^{d/2}\sqrt{\det B}} \sum_{\mathbf{n}\in\tilde{N}} \exp\left(-\frac{(\mathbf{n}-\mathbf{a})B^{-1}(\mathbf{n}-\mathbf{a})^t}{2}\right)\\ &= \frac{y(\mathbf{r})}{(2\pi)^{d/2}\sqrt{\det B}} \sum_{\tilde{\mathbf{n}}=\underline{\mathbf{N}}}^{\overline{\mathbf{N}}} \exp\left(-\frac{(\tilde{\mathbf{n}}-\tilde{\mathbf{a}})D^{-1}(\tilde{\mathbf{n}}-\tilde{\mathbf{a}})^t}{2}\right)\\ &\sim \frac{1}{(2\pi)^{d/2}} \int_{\underline{\omega}_1}^{\overline{\omega}_1} \cdots \int_{\underline{\omega}_d}^{\overline{\omega}_d} \exp\left(-\frac{1}{2}\sum_{j=1}^d t_j^2\right) dt_1\cdots dt_d\end{aligned}$$

where in the last step the considerations above were applied. On the other hand the sum $\sum_{\exists j: n_j<\underline{N}_j} y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} < \varepsilon y(\mathbf{r})$ if all $\underline{\omega}_j$ are small enough. □

Theorem 6. *Let $\mathbf{k} \in \mathbb{R}^d$ be fixed. Then, as $\mathbf{r} \to \infty$,*

$$\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} y(\mathbf{r}) \sim y(\mathbf{r}) \left(\frac{a_1(\mathbf{r})}{r_1}\right)^{k_1} \cdots \left(\frac{a_d(\mathbf{r})}{r_d}\right)^{k_d}$$

Proof. Set $\bar{R}_j = r_j\left(1+\frac{1}{a_j(\mathbf{r})}\right)$. Then, if $|z_j| < \bar{R}_j$ for all j, we have by Lemma 4

$$|y(\mathbf{z})| = \left|\sum_{\mathbf{n}} y_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}\right| \le \sum y_{\mathbf{n}}\bar{\mathbf{R}}^{\mathbf{n}} = y(\bar{\mathbf{R}}) = O\left(y(\mathbf{r})\right).$$

Let $\mathbf{h} = \bar{\mathbf{R}} - \mathbf{r} = \left(\frac{r_1}{a_1(\mathbf{r})}, \dots, \frac{r_d}{a_d(\mathbf{r})}\right)$. Then we have

$$y(\mathbf{z}) = \sum \frac{1}{k_1!\cdots k_d!} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} y(\mathbf{r})(\mathbf{z}-\mathbf{r})^{\mathbf{k}}$$

and hence by Cauchy's inequality we get

$$\begin{aligned}\left|\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} y(\mathbf{r})\right| &\le \frac{k_1!\cdots k_d!}{h_1^{k_1}\cdots h_d^{k_d}} y(\bar{\mathbf{R}})\\ & O\left(y(\mathbf{r})\left(\frac{a_1(\mathbf{r})}{r_1}\right)^{k_1} \cdots \left(\frac{a_d(\mathbf{r})}{r_d}\right)^{k_d}\right)\end{aligned}$$

Now define $(n)_k := n(n-1)\cdots(n-k+1)$ and observe that

$$\begin{aligned}r_1^{k_1}\cdots r_d^{k_d} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial x_d^{k_d}} y(\mathbf{r}) &= \sum_{\mathbf{n}} (n_1)_{k_1}\cdots(n_d)_{k_d} y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}\\ &= \sum\nolimits_1 + \sum\nolimits_2\end{aligned}$$

with

$$\sum\nolimits_1 = \sum_{\mathbf{n} \text{ such that } \forall j: |a_j(\mathbf{r})-n_j| \le \omega\sqrt{B_{jj}(\mathbf{r})}} (n_1)_{k_1}\cdots(n_d)_{k_d} y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}$$

and $\sum_2 = \sum - \sum_1$. In the range of summation we have $(n_1)_{k_1}\cdots(n_d)_{k_d} \sim \mathbf{a}(\mathbf{r})^{\mathbf{k}}$. Let $\tilde{\mathbf{n}}$ as in Theorem 5 and set $s_j = n_j - a_j$ and $\tilde{s}_j = \tilde{n}_j - \tilde{a}_j$. Since A is orthogonal, we have

$$\|\tilde{\mathbf{s}}\|^2 = \|\mathbf{s}\|^2 = \omega^2 \sum_{j=1}^d B_{jj}$$

Hence the range of summation covers the set $\{\mathbf{n} : \forall j : |\tilde{a}_j(\mathbf{r}) - \tilde{n}_j| \le \omega\sqrt{\lambda_j(\mathbf{r})}\}$. Therefore we obtain by means of Theorem 5 $\sum_1 \sim C(\omega)y(\mathbf{r})\mathbf{a}(\mathbf{r})^{\mathbf{k}}$ with

$$\frac{1}{\pi^{d/2}}\int_{-\omega}^{\omega}\cdots\int_{-\omega}^{\omega}\exp\left(-\frac{1}{2}\sum_{j=1}^{d}t_j^2\right)\,dt_1\cdots dt_d < C(\omega) < 1.$$

On the other hand define

$$\sum{}' := \sum_{\mathbf{n}:\exists j:|a_j-n_j|>\omega\sqrt{B_{jj}(\mathbf{r})}}.$$

Then we have

$$\begin{aligned}\left|\sum{}_2\right| &\le \sum{}'(n_1)_{k_1}\cdots(n_d)_{k_d}y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}} \le \sum{}'\mathbf{n}^{\mathbf{k}}y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}\\ &\le \left(\sum{}'\mathbf{n}^{2\mathbf{k}}y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}\right)^{1/2}\left(\sum{}'y_{\mathbf{n}}\mathbf{r}^{\mathbf{n}}\right)^{1/2}\\ &= O\left(\left(\mathbf{r}^{2\mathbf{k}}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_d}}{\partial x_d^{2k_d}}y(\mathbf{r})\int\cdots\int_E\exp\left(-\frac{1}{2}\sum_{j=1}^{d}t_j^2\right)\,dt_1\cdots dt_d\right)^{1/2}\right),\end{aligned}$$

with the integration domain $E = (\mathbb{R}^+)^d \setminus [0,\omega]^d$. Therefore, since

$$\mathbf{r}^{2\mathbf{k}}\frac{\partial^{2k_1}}{\partial x_1^{2k_1}}\cdots\frac{\partial^{2k_d}}{\partial x_d^{2k_d}}y(\mathbf{r}) = O\left(y(\mathbf{r})\mathbf{a}(\mathbf{r})^{2\mathbf{k}}\right),$$

we have for sufficiently large ω

$$\left|\sum{}_1 + \sum{}_2 - y(\mathbf{r})\mathbf{a}(\mathbf{r})^{\mathbf{k}}\right| < \varepsilon y(\mathbf{r})\mathbf{a}(\mathbf{r})^{\mathbf{k}}$$

which completes the proof. □

Lemma 5. *Assume that there exist constants $\eta > 0$ and $C > 0$ such that for $|z_j - r_j| < \eta r_j$ $(j = 1,\dots,d)$ the matrix B satisfies $|\mathbf{h}B(\mathbf{z})\mathbf{h}^t| \le C\mathbf{h}B(\mathbf{r})\mathbf{h}^t$ for all $\mathbf{h} \in \mathbb{R}^d$. Furthermore, assume regularity of $y(\mathbf{z})$ in this region and that $y(\mathbf{z}) \ne 0$. Then*

$$\log y\left(r_1e^{i\theta_1},\dots,r_de^{i\theta_d}\right) = \log y(\mathbf{r}) + i\boldsymbol{\theta}\mathbf{a}(\mathbf{r})^t - \frac{1}{2}\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t + \varepsilon(\mathbf{r},\boldsymbol{\theta})$$

where

$$|\varepsilon(\mathbf{r},\boldsymbol{\theta})| \le \frac{C\|\boldsymbol{\theta}\|\cdot\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t}{\eta}. \tag{10}$$

Proof. Set $g(t) = \log y\left(e^{x_1+ith_1},\dots,e^{x_d+ith_d}\right)$ for $|t| \le \eta$ and some h with $\|h\| = 1$. Then

$$g''(t) = \mathbf{h}B\left(e^{x_1+ith_1},\dots,e^{x_d+ith_d}\right)\mathbf{h}^t = \sum_{n\ge 0}c_nt^n$$

with

$$|c_n| \le \frac{C'g''(|t|)}{\eta^n} \le \frac{Cg''(0)}{\eta^n},$$

with a positive constant C'. Since

$$g'(0) = i\sum_j\frac{y_{z_j}(\mathbf{r})r_jh_j}{y(\mathbf{r})} = \mathbf{a}(\mathbf{r})\mathbf{h}^t,$$

we obtain by setting $t\mathbf{h} = \boldsymbol{\theta}$ the expansion

$$\log y\left(r_1e^{i\theta_1},\dots,r_de^{i\theta_d}\right) = g(t) = g(0) + itg'(0) - \frac{t^2}{2}g''(0) + \varepsilon(r,\boldsymbol{\theta})$$

which is of the required shape. Finally, observe that

$$\varepsilon(r,\boldsymbol{\theta}) = \sum\frac{c_n}{(n+1)(n+2)}t^{n+2}$$

and

$$|c_n|\cdot|t|^{n+2}\leq\frac{Cg''(0)}{\eta^n}|t|^{n+2}\leq\frac{Cg''(0)|t|^3}{\eta}=\frac{C\|\boldsymbol{\theta}\|\cdot\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t}{\eta}$$

which immediately implies (10). □

Lemma 6. *An H-admissible function $y(\mathbf{x})$ satisfies*

$$y\left(r_1e^{i\theta_1},\dots,r_de^{i\theta_d}\right)=y(\mathbf{r})+i\boldsymbol{\theta}\tilde{\mathbf{a}}(\mathbf{r})^t-\frac{1}{2}\boldsymbol{\theta}\tilde{B}(\mathbf{r})\boldsymbol{\theta}^t+O\left(y(\mathbf{r})\cdot\|\boldsymbol{\theta}\|^3\cdot\|\mathbf{a}(\mathbf{r})\|^3\right)$$

uniformly for $|\theta_j|\leq 1/a_j(\mathbf{r})$, for $j=1,\dots,d$, where

$$\tilde{\mathbf{a}}(\mathbf{r})=\nabla y\left(e^{s_1},\dots,e^{s_d}\right)\big|_{s_1=\log r_1,\dots,s_d=\log r_d}=\left(r_jy_{x_j}(\mathbf{r})\right)_{j=1,\dots,d}$$

$$\tilde{B}(\mathbf{r})=\left(\left.\frac{\partial^2y\left(e^{s_1},\dots,e^{s_d}\right)}{\partial s_j\partial s_k}\right|_{s_1=\log r_1,\dots,s_d=\log r_d}\right)_{j,k=1,\dots,d}$$

Proof. We have $\tilde{B}(\mathbf{z})=\left(y_{z_jz_k}(\mathbf{z})z_jz_k+\delta_{jk}y_{z_j}(\mathbf{z})z_j\right)_{j,k=1,\dots,d}$. Theorem 6 yields $y_{z_jz_k}(\mathbf{r})r_jr_k\sim y(\mathbf{r})a_j(\mathbf{r})a_k(\mathbf{r})$ which implies $\|\tilde{B}(\mathbf{r})\|=O\left(y(\mathbf{r})\|\mathbf{a}(\mathbf{r})\|^2\right)$. Setting $\eta_j=1/a_j(\mathbf{r})$, $j=1,\dots,d$, and applying Theorem 6 again and Lemmas 2 and 4 afterwards yields the following asymptotic equivalence for the entries of $\tilde{B}$.

$$\begin{aligned}&\tilde{B}_{jk}(r_1(1+\eta_1),\dots,r_d(1+\eta_d))\\&\quad\sim y(r_1(1+\eta_1),\dots,r_d(1+\eta_d))a_j(r_1(1+\eta_1),\dots,r_d(1+\eta_d))a_k(r_1(1+\eta_1),\dots,r_d(1+\eta_d))\\&\quad\sim e^dy(\mathbf{r})a_j(\mathbf{r})a_k(\mathbf{r}).\end{aligned}\tag{11}$$

Furthermore, observe that all entries of $\tilde{B}(z)$ are analytic functions and thus we have

$$\tilde{B}(z)=\sum_{\mathbf{n}}B_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}=\sum_{\mathbf{n}}y_{\mathbf{n}}\cdot(n_in_j)_{i,j=1,\dots,d}\mathbf{z}^{\mathbf{n}}$$

Clearly, all matrices $(n_in_j)_{i,j=1,\dots,d}$ are positive definite and hence by (V) we get

$$\max_{|z_j|=r_j,j=1,\dots,d}|\mathbf{h}\tilde{B}(\mathbf{z})\mathbf{h}^t|\leq\mathbf{h}\tilde{B}(\mathbf{r})\mathbf{h}^t.$$

Hence (11) implies that we have $|\mathbf{h}\tilde{B}(\mathbf{z})\mathbf{h}^t|\leq C\mathbf{h}\tilde{B}(\mathbf{r})\mathbf{h}^t$ for $|z_j-r_j|\leq\eta_jr_j$, $j=1,\dots,d$. Consequently, we can apply Lemma 5 to $e^{y(\mathbf{z})}$ and get

$$y\left(r_1e^{i\theta_1},\dots,r_de^{i\theta_d}\right)=y(\mathbf{r})+i\boldsymbol{\theta}\tilde{\mathbf{a}}(\mathbf{r})^t-\frac{1}{2}\boldsymbol{\theta}\tilde{B}(\mathbf{r})\boldsymbol{\theta}^t+\varepsilon(\mathbf{r},\boldsymbol{\theta})$$

with

$$|\varepsilon(\mathbf{r},\boldsymbol{\theta})|\leq\frac{C\|\tilde{B}(\mathbf{r})\|\cdot\|\boldsymbol{\theta}\|^3}{2\min_j\eta_j}\leq\frac{C\|\tilde{B}(\mathbf{r})\|\cdot\|\boldsymbol{\theta}\|^3\cdot\|\mathbf{a}(\mathbf{r})\|}{2}=O\left(y(\mathbf{r})\cdot\|\boldsymbol{\theta}\|^3\cdot\|\mathbf{a}(\mathbf{r})\|^3\right)$$

as desired. □

Likewise we will need a more precise estimate for "large" $\boldsymbol{\theta}$.

Lemma 7. *Let $\varepsilon>0$. If $y(\mathbf{x})$ is H-admissible and $\|\boldsymbol{\theta}\|_{\max}\geq y(\mathbf{r})^{-1/2+\varepsilon}$ then*

$$\left|y\left(r_1e^{i\theta_1},\dots,r_de^{i\theta_d}\right)\right|\leq y(\mathbf{r})-y(\mathbf{r})^\eta.$$

with some constant $0<\eta<2\varepsilon$.

Proof. Assume $\theta_\ell\geq y(\mathbf{r})^{-2/5-\varepsilon}$. Set $k_j=\lfloor a_j(\mathbf{r})\rfloor$ and $\boldsymbol{\ell}=(k_1+1,k_2+1,\dots,k_\ell+1,k_{\ell+1},k_{\ell+2},\dots,k_d)$. Then define $\upsilon_\ell:=y_{\boldsymbol{\ell}}\mathbf{z}^{\boldsymbol{\ell}}$ and $\alpha_\ell:=|\upsilon_\ell|$ In the same manner as in [17, Lemma 6] one proves

$$|\upsilon_{\ell-1}+\upsilon_\ell|\leq\alpha_{\ell-1}+\alpha_\ell-\frac{1}{10}\frac{y(\mathbf{r})^{2\varepsilon}}{\sqrt{(2\pi)^d\det B(\mathbf{r})}}.$$

Then Corollary 4 implies $|\upsilon_{\ell-1}+\upsilon_\ell|\leq\alpha_{\ell-1}+\alpha_\ell-y(\mathbf{r})^\eta$ with $0<\eta<2\varepsilon$. Hence

$$\left|y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)\right|\leq|\tilde{y}(\mathbf{z})|+|\upsilon_{\ell-1}+\upsilon_\ell|\leq\tilde{y}(\mathbf{r})+\alpha_{\ell-1}+\alpha_\ell-y(\mathbf{r})^\eta=y(\mathbf{r})-y(\mathbf{r})^\eta$$

where $\tilde{y}(\mathbf{z})=y(\mathbf{z})-\upsilon_{\ell-1}(\mathbf{z})-\upsilon_\ell(\mathbf{z})$ The inequality follows from (V). □

5. A Class of H-admissible Functions

In this section we want to present conditions under which exponentials of multivariate polynomials are H-admissible. Let $\sigma > 1$ be some constant and set

$$\mathcal{R}_\sigma := \left\{ \mathbf{r} \in \left(\mathbb{R}^+\right)^d : (r_{\min})^\sigma > r_{\max} \right\}.$$

Furthermore let $E_\sigma := \{\mathbf{e} \in \mathbb{R}^d : e_j \in [1, \sigma),$ for $1 \le j \le d$, and there is an $1 \le i \le d$ such that $e_i = 1\}$. Thus $\mathbf{r} \in \mathcal{R}_\sigma$ is equivalent to the existence of some $\tau \ge 1$ and some $\mathbf{e} \in E_\sigma$ such that $\mathbf{r} = \tau^{\mathbf{e}} := (\tau^{e_1}, \dots, \tau^{e_d})$. Obviously, $\mathbf{r} \to \infty$ in $\mathcal{R}_\sigma$ is equivalent to $r_m in \to \infty$ for $\mathbf{r} \in \mathcal{R}_\sigma$ as well as to $t \to \infty$ for $\mathbf{r} = \tau^{\mathbf{e}}$ with $\mathbf{e} \in E_\sigma$. We start with some basic auxiliary results on multivariate polynomials.

Lemma 8. *Let $P(\mathbf{r}) = \sum_{\mathbf{p}} \beta_{\mathbf{p}} \mathbf{r}^{\mathbf{p}}$ and $Q(\mathbf{r}) = \sum_{\mathbf{p}} \beta_{\mathbf{p}} \mathbf{r}^{\mathbf{p}}$ be polynomials in $\mathbf{r}$ satisfying*

$$\frac{P(\mathbf{r})}{Q(\mathbf{r})} \to \infty, \text{ for } r_{\min} \to \infty \text{ (with } \mathbf{r} \in \mathcal{R}_\sigma).$$

Then there exists $e > 0$ such that

$$\frac{P(\mathbf{r})}{Q(\mathbf{r})} > r_{\min}^e, \text{ for sufficiently large } r_{\min} \text{ (with } \mathbf{r} \in \mathcal{R}_\sigma).$$

Proof. Let $\mathbf{e} \in E_\sigma$ and $\mathbf{r} = \tau^{\mathbf{e}}$. Then there exist positive numbers $c_P(\mathbf{e}), c_Q(\mathbf{e}), d_P(\mathbf{e})$, and $d_Q(\mathbf{e})$ such that

$$\frac{P(\tau^{\mathbf{e}})}{Q(\tau^{\mathbf{e}})} = \frac{\sum_{\mathbf{p}} \beta_{\mathbf{p}} \tau^{\mathbf{p}\cdot\mathbf{e}^t}}{\sum_{\mathbf{p}} \beta_{\mathbf{p}} \tau^{\mathbf{p}\cdot\mathbf{e}^t}} \sim \frac{c_P(\mathbf{e}) \tau^{d_P(\mathbf{e})}}{c_Q(\mathbf{e}) \tau^{d_Q(\mathbf{e})}} = \frac{c_P(\mathbf{e})}{c_Q(\mathbf{e})} \cdot \tau^{d_P(\mathbf{e}) - d_Q(\mathbf{e})} \to \infty, \text{ fr } \tau \to \infty.$$

Thus $d_P(\mathbf{e}) > d_Q(\mathbf{e})$. If we set $e := \min_{\mathbf{e} \in E_\sigma} \frac{d_P(\mathbf{e}) - d_Q(\mathbf{e})}{2}$, then for all $\mathbf{e} \in \mathbf{E}_\sigma$ we obtain

$$\frac{P(\tau^{\mathbf{e}})}{Q(\tau^{\mathbf{e}})} > r_{\min}^e, \text{ for sufficiently large } r_{\min} \ (\mathbf{r} \in \mathcal{R}_\sigma),$$

as desired. □

Corollary. *Let $P(\mathbf{r}) = \sum_{\mathbf{p}} \beta_{\mathbf{p}} \mathbf{r}^{\mathbf{p}}$ be a polynomial satisfying $P(\mathbf{r}) \to \infty$ as $r_{\min} \to \infty$. Then for sufficiently large $r_{\min}$ we have $P(\mathbf{r}) > \sqrt{r_{\min}}$.*

Now we are able to characterize the admissible functions which are exponentials of a polynomial.

Theorem 7. *Let $P(\mathbf{z}) = \sum_{\mathbf{m} \in M} b_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$ be a polynomial with real coefficients $b_{\mathbf{m}} \neq 0$ for $\mathbf{m} \in M$. Moreover, let $y(\mathbf{z}) = e^{P(\mathbf{z})}$. Then the following statements are equivalent.*

(i) $\forall \boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\} : \left| y(\mathbf{r} e^{i\boldsymbol{\theta}}) \right| < y(\mathbf{r})$ *if $\mathbf{r} \in \mathcal{R}_\sigma$ sufficiently large*

(ii) $\forall \boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\} : y(\mathbf{r} e^{i\boldsymbol{\theta}}) = o\left(y(\mathbf{r})\right),$ *as $\mathbf{r} \to \infty$ in $\mathcal{R}_\sigma$*

(iii) $\forall \boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\} : y(\mathbf{r} e^{i\boldsymbol{\theta}}) = o\left(\frac{y(\mathbf{r})}{\sqrt{\det(B(\mathbf{r}))}}\right),$ *as $\mathbf{r} \to \infty$ in $\mathcal{R}_\sigma$*

(iv) *$y(\mathbf{z})$ is H-admissible in $\mathcal{R}_\sigma$.*

Proof. Let L_j denote the highest exponent of z_j appearing in $P(\mathbf{z})$ and $L = \max_{1 \le j \le d} L_j$.

(i) $\Longrightarrow$ (ii): By assumption we have for sufficiently large $\mathbf{r} \in \mathcal{R}_\sigma$ and some $\boldsymbol{\theta} \in [-\pi, \pi]^d \setminus \{\mathbf{0}\}$

$$\frac{\left| e^{P(\mathbf{r} e^{i\boldsymbol{\theta}})} \right|}{e^{P(\mathbf{r})}} = e^{\Re(P(\mathbf{r} e^{i\boldsymbol{\theta}})) - P(\mathbf{r})} < 1$$

and hence

$$\begin{aligned} Q(\mathbf{r}) &:= \Re(P(\mathbf{r} e^{i\boldsymbol{\theta}})) - P(\mathbf{r}) \\ &= \Re\left(\sum_{\mathbf{m} \in M} b_{\mathbf{m}} \mathbf{r}^{\mathbf{m}} e^{i\mathbf{m}\boldsymbol{\theta}^t} \right) - P(\mathbf{r}) \\ &= \sum_{\mathbf{m} \in M} b_{\mathbf{m}} \mathbf{r}^{\mathbf{m}} \left(\cos\left(\mathbf{m}\boldsymbol{\theta}^t\right) - 1 \right) < \log(1) = 0. \end{aligned}$$

Since $Q(\mathbf{r})$ is a polynomial attaining only negative values for $\mathbf{r}\in\mathcal{R}_\sigma$. Thus $\lim_{\mathbf{r}\to\infty}Q(\mathbf{r})=-\infty$ and this is equivalent to (ii).

(ii) $\Longrightarrow$ (iii): The assumption implies by Corollary $Q(\mathbf{r})=\Re(P(\mathbf{r}e^{i\boldsymbol{\theta}}))-P(\mathbf{r})<-\sqrt{r_{\min}}$ for sufficiently large $\mathbf{r}\in\mathcal{R}_\sigma$. The entries of $B(\mathbf{r})$ are $B_{jk}(\mathbf{r}):=x_jx_k\frac{\partial^2}{\partial x_j\partial x_k}P(\mathbf{x})$ and therefore obviously

$$\log(\det(B(\mathbf{r})))=\log\left(\lambda_1(\mathbf{r})\cdots\lambda_d(\mathbf{r})\right)=O\left(\log\left(B_{11}(\mathbf{r})\cdots B_{dd}(\mathbf{r})\right)\right).$$

Since the largest exponent of $P(\mathbf{x})$ is L, we obtain $B_{jj}(\mathbf{r})=O\left(r_{\max}^{dL+1}\right)$ and therefore

$$\log(\det(B(\mathbf{r})))=O\left(\log\left(r_{\max}^{d(dL+1)}\right)\right)=O\left(\log\left(r_{\min}^{\sigma d(dL+1)}\right)\right)=O\left(\log r_{\min}\right)$$

and this implies

$$\begin{aligned}\log\left(\frac{\left|y(\mathbf{r}e^{i\boldsymbol{\theta}})\right|}{y(\mathbf{r})}\sqrt{\det(B(\mathbf{r}))}\right)&=\Re(P(\mathbf{r}e^{i\boldsymbol{\theta}}))-P(\mathbf{r})+\frac{1}{2}\log(\det(B(\mathbf{r}))\\&=-\sqrt{r_{\min}}+O\left(\log r_{\min}\right)\to-\infty\end{aligned}$$

which shows (iii).

(iii) $\Longrightarrow$ (i): This implication is trivial.

(iii) $\Longrightarrow$ (iv): We have to show the conditions (I)–(V) of the definition. (IV) and (V) are obvious. In the sequel we will first show (III), then (I) and (II) at the end. Let $\lambda_1\le\ldots\le\lambda_d$ denote the eigenvalues of B.

(III): The assumption implies that $B(\mathbf{r})$ must be positive definite. Therefore, for any fixed $\mathbf{h}\in\mathbb{R}^d$ the function $Q(\mathbf{r}):=\mathbf{h}B(\mathbf{r})\mathbf{h}^t$ is a polynomial which is positive on $\mathcal{R}_\sigma$ and hence $\lim_{\mathbf{r}\to\infty}Q(\mathbf{r})=\infty$. Now choose $\mathbf{h}=\mathbf{v}_j$, an eigenvector of $B(\mathbf{r})$ with eigenvalue λ_j, and (III) follows.

(I): Consider $B^{-1}(\mathbf{r})$. The eigenvalues are $\frac{1}{\lambda_d}\le\cdots\le\frac{1}{\lambda_1}$ and their sum, i.e., the trace of $B^{-1}(\mathbf{r})$ can be expressed in terms of the cofactors of $B(\mathbf{r})$. We have

$$\frac{1}{\lambda_1}\le\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_d}=\frac{\hat{B}_{11}(\mathbf{r})+\cdots+\hat{B}_{dd}(\mathbf{r})}{\det(B(\mathbf{r}))}\to0.$$

Thus

$$\lambda_1\ge\frac{\det(B(\mathbf{r}))}{\hat{B}_{11}(\mathbf{r})+\cdots+\hat{B}_{dd}(\mathbf{r})}\to\infty\text{ as }\mathbf{r}\to\infty$$

The determinant as well as the cofactors are polynomials in $\mathbf{r}$. Thus applying Lemma 8 we obtain

$$\lambda_1(\mathbf{r})\ge r_{\min}^e,\ \text{for } r_{\min}\text{ sufficiently large}$$

and suitable e.

Now let $\delta_j:=\lambda_j^{-\frac{1}{2}+\frac{\varepsilon}{2}}$ with $\varepsilon<\min\left\{\frac{e}{6\sigma d(Ld+1)},\frac{1}{3}\right\}$. Then for $\boldsymbol{\theta}\in\boldsymbol{\Delta}(\mathbf{r})=\left\{\sum_{j=1}^d\mu_j\mathbf{v}_j(\mathbf{r}):|\mu_j|\le\delta_j(\mathbf{r}),1\le j\le d\right\}$ we get

$$\|\boldsymbol{\theta}\|\le\sqrt{\lambda_1^{-1+\varepsilon}+\cdots+\lambda_d^{-1+\varepsilon}}\le\sqrt{d}\lambda_1^{-\frac{1}{2}+\frac{\varepsilon}{2}}\le\sqrt{d}r_{\min}^{e\left(-\frac{1}{2}+\frac{\varepsilon}{2}\right)}<r_{\min}^{-\frac{\varepsilon}{3}}$$

for $\mathbf{r}$ sufficiently large.

Set $Q(\mathbf{z}):=\mathbf{h}B(\mathbf{z})\mathbf{h}^t$. Since $Q(\mathbf{z})$ is a polynomial we have for $\mathbf{e}\in E_\sigma$ $Q(\tau^{\mathbf{e}})\sim\tilde{c}(\mathbf{e})\cdot\tau^\Lambda$ for a suitable constant Λ as well as $Q\left(\tau^{\mathbf{e}}(1+2\eta)\right)\le C\cdot Q(\tau^{\mathbf{e}})$ for sufficiently large τ. Therefore the conditions of Lemma 5 are fulfilled and we get for the third order term $\varepsilon(\mathbf{r},\boldsymbol{\theta})$ in the Taylor expansion of $P(\mathbf{z})$ the estimate

$$\begin{aligned}\max_{\boldsymbol{\theta}\in\boldsymbol{\Delta}(\mathbf{r})}|\varepsilon(\mathbf{r},\boldsymbol{\theta})|&=\max_{\boldsymbol{\theta}\in\boldsymbol{\Delta}(\mathbf{r})}\frac{\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t\cdot\|\boldsymbol{\theta}\|}{2\eta}\\&=O\left(\frac{(\lambda_1^\varepsilon+\cdots+\lambda_d^\varepsilon)\cdot\lambda_1^{-\frac{1}{2}+\frac{\varepsilon}{2}}}{\eta}\right)=O\left(\frac{\lambda_d^\varepsilon\cdot\lambda_1^{-\frac{1}{2}+\frac{\varepsilon}{2}}}{\eta}\right).\end{aligned}$$

Since $\lambda_d^{\varepsilon}\lambda_1^{\frac{\varepsilon}{2}} \leq (\lambda_1 \cdots \lambda_d)^{\varepsilon} = \det B(\mathbf{r})^{\varepsilon}$, we obtain $\det B(\mathbf{r}) = O\left(r_{\min}^{\sigma d(dL+1)}\right)$. On setting $\eta = r_{\min}^{-\frac{\varepsilon}{3}}$ this implies

$$\max_{\boldsymbol{\theta}\in\boldsymbol{\Delta}(\mathbf{r})} |\varepsilon(\mathbf{r},\boldsymbol{\theta})| = O\left(\frac{r_{\min}^{\sigma d(Ld+1)\varepsilon}\cdot r_{\min}^{-\frac{\varepsilon}{2}}}{r_{\min}^{-\frac{\varepsilon}{3}}}\right) \to 0 \text{ fr } r_{\min} \to \infty$$

because of $\varepsilon < \frac{e}{6\sigma d(Ld+1)}$.

(II): We have for $\mathbf{r}$ large enough

$$\sqrt{\det\left(B(\mathbf{r})\right)} \leq (r_{\min})^{\frac{\sigma d(Ld+1)}{2}} \leq \exp\left(\frac{1}{2}\left(r_{\min}^{e}\right)^{\varepsilon}\right) \leq \exp\left(\frac{1}{2}\lambda_1^{\varepsilon}\right)$$

and therefore on the boundary of $\boldsymbol{\Delta}(\mathbf{r})$

$$\begin{aligned}\max_{\boldsymbol{\theta}\in\partial\boldsymbol{\Delta}(\mathbf{r})} \frac{\left|y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)\right|}{y(\mathbf{r})} &\sim \max_{\boldsymbol{\theta}\in\partial\boldsymbol{\Delta}(\mathbf{r})} \exp\left(-\frac{1}{2}\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t\right)\\ &= \exp\left(-\frac{1}{2}\delta_1^2(\mathbf{r})\lambda_1(\mathbf{r})\right) = \exp\left(-\frac{1}{2}\lambda_1^{\epsilon}\right)\\ &= O\left(\frac{1}{\sqrt{\det\left(B(\mathbf{r})\right)}}\right). \end{aligned}\tag{12}$$

The estimate $|\varepsilon(\mathbf{r},\boldsymbol{\theta})| \leq \boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t \cdot \|\boldsymbol{\theta}\|/2\eta$ from above is valid for fixed η. This combined with assumption (i) guarantees that (12) is valid outside $\boldsymbol{\Delta}(\mathbf{r})$ as well.

(iv) $\Longrightarrow$ (i): This is an obvious consequence of admissibility. □

For polynomials with positive coefficients a – from a computational viewpoint – much simpler criterion can be stated. This criterion is also satisfied by admissible functions in the sense of [6].

Corollary. *Let $P(\mathbf{z}) = \sum_{j=1}^{L} a_j\mathbf{z}_j^{\mathbf{k}}$ be a multivariate polynomial with positive coefficients $a_j > 0$ and $\sigma > 0$ an arbitrary constant. Then a necessary and sufficient condition for $e^{P(\mathbf{z})}$ to be H-admissible is that the system of the equations*

$$\mathbf{k}_j\boldsymbol{\theta}^t \equiv 0 \mod 2\pi,\ j = 1\ldots,L, \tag{13}$$

has only the trivial solution $\boldsymbol{\theta} \equiv \mathbf{0} \mod 2\pi$. Equivalently, this means that the span of the vectors $\mathbf{k}_j$ over $\mathbb{Z}$ equals $\mathbb{Z}^d$.

Proof. This is an immediate consequence of the previous theorem. We have to show (i). Observe

$$\begin{aligned} y\left(r_1e^{i\theta_1},\ldots,r_de^{i\theta_d}\right) &= \exp\left(P\left(r_1e^{i\theta_1},\ldots,r_de^{i\theta_d}\right)\right)\\ &= y(\mathbf{r})\exp\left(-2\sum_{\ell=1}^{L} a_\ell r_1^{k_{1\ell}}\cdots r_d^{k_{d\ell}}\sin^2\left(\sum_{j=1}^{d}\frac{k_{j\ell}\theta_j}{2}\right)\right) \end{aligned}\tag{14}$$

Condition (i) is satisfied if and only if the exponent in (14) vanishes only for $\theta_1 = \cdots = \theta_d = 0$. But this is obviously equivalent to (13). □

6. Closure Properties

Theorem 8. *If $y(\mathbf{x})$ is H-admissible in $\mathcal{R}$, then $e^{y(\mathbf{x})}$ is H-admissible in $\mathcal{R}$, too.*

Proof. Let $\boldsymbol{\delta}(\mathbf{r}) = (y(\mathbf{r})^{-2/5},\ldots,y(\mathbf{r})^{-2/5})$ and $Y(\mathbf{x}) = e^{y(\mathbf{x})}$. Let $\bar{\mathbf{a}}$ and $\bar{B}$ denote the the vector of the first and the matrix of the second logarithmic derivatives of $e^{y(\mathbf{x})}$, respectively. Then by Lemma 6

$$\log Y\left(r_1e^{i\theta_1},\ldots,r_de^{i\theta_d}\right) = \log Y(\mathbf{r}) + i\boldsymbol{\theta}\bar{\mathbf{a}}(\mathbf{r})^t - \frac{1}{2}\boldsymbol{\theta}\bar{B}(\mathbf{r})\boldsymbol{\theta}^t + O\left(y(\mathbf{r})^{-1/5}\|\mathbf{a}(\mathbf{r})\|^3\right)$$

for $\|\boldsymbol{\theta}\| < \delta(\mathbf{r})$. Hence we have $y(\mathbf{r})^{-1/5}\|\mathbf{a}(\mathbf{r})\|^3 \to 0$ as $\mathbf{r} \to \infty$ which guarantees (I) for $\boldsymbol{\theta}$ inside the cube $\mathcal{K}$ defined by our choice of $\boldsymbol{\delta}$. Hence (I) is also true for the cube $\mathcal{E}$ spanned by the eigenvectors of $B(\mathbf{r})$ and inscribed in $\mathcal{K}$.

If $\|\boldsymbol{\theta}\|_{\max} > y(\mathbf{r})^{-2/5-\varepsilon}$, which is (for sufficiently large $\mathbf{r}$) equivalent to $\boldsymbol{\theta} \notin \mathcal{K}' = y(\mathbf{r})^{-\varepsilon}\mathcal{K}$, then Lemma 7 in conjunction with $\bar{B}_{jk} \sim y(\mathbf{r})a_j(\mathbf{r})a_k(\mathbf{r})$ yields

$$|Y\left(r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}\right)| \leq Y(\mathbf{r}) \exp\left(-y(\mathbf{r})^{-1/7}\right) \leq Y(\mathbf{r}) \exp\left(-\left(\det \tilde{B}(\mathbf{r})\right)^{1/(7d)}\right).$$

This implies (II) outside $\mathcal{K}'$ and therefore in particular outside $\mathcal{E}$.

Condition (V) is obvious. Therefore it remains to show that the eigenvalues of $\bar{B}(\mathbf{r})$ tend to infinity and condition (IV). Note that $\bar{B} = y \cdot (B + \mathbf{a}^t\mathbf{a})$ and that $\mathbf{a}^t\mathbf{a}$ is a positive semidefinite matrix of rank 1 with eigenvalues 0 and $\|\mathbf{a}\|^2$. Then the smallest eigenvalue $\lambda_{\min}\left(\bar{B}\right)$ of $\bar{B}$ satisfies

$$\lambda_{\min}\left(\bar{B}\right) = \min_{\mathbf{x}:\|\mathbf{x}\|=1} \mathbf{x}\bar{B}\mathbf{x}^t \geq \min_{\mathbf{x}:\|\mathbf{x}\|=1} \mathbf{x}B\mathbf{x}^t + \min_{\mathbf{x}:\|\mathbf{x}\|=1} \mathbf{x}\mathbf{a}^t\mathbf{a}\mathbf{x}^t \geq \min_{\mathbf{x}:\|\mathbf{x}\|=1} \mathbf{x}B\mathbf{x}^t = \lambda_{\min}(B) \to \infty.$$

and (III) follows. In order to show (IV) observe that

$$\bar{B}_{jj} = y \cdot (B_{jj} + a_j^2) \sim y \cdot a_j^2 = o\left(y^2 a_j^2\right) = o\left(\bar{a}_j^2\right)$$

as desired. □

Theorem 9. *If $y_1(\mathbf{x})$ and $y_2(\mathbf{x})$ are H-admissible in $\mathcal{R}$ and there exists a constant and C such that $\det(B_1 + B_2) \leq C \min\left(\det B_1, \det B_2\right)$. Assume furthermore that the eigenvectors of B_1 and B_2 are the same. Then $y_1(\mathbf{x})y_2(\mathbf{x})$ is H-admissible in $\mathcal{R}$, too.*

Proof. The logarithmic derivatives of $y_1(\mathbf{x})y_2(\mathbf{x})$ are $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ and $B = B_1 + B_2$, respectively. This immediately implies (III) and (IV). (V) is obvious.

Note furthermore that, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are the cuboids inside of which (I) is valid for $y_1(\mathbf{x})$ and $y_2(\mathbf{x})$, respectively, then inside the domain $\mathcal{C}_1 \cap \mathcal{C}_2$ the function $y_1(\mathbf{x})y_2(\mathbf{x})$ obviously satisfies (I). The condition on the determinant of $B = B_1 + B_2$ implies that outside this domain (II) holds. □

Remark 5. Note that powers of H-admissible functions are always H-admissible, since the assumptions of the theorem are obviously true in the case $y_1(\mathbf{x}) = y_2(\mathbf{x})$.

Theorem 10. *Let $y(\mathbf{x})$ be H-admissible in $\mathcal{R}$ and $p(\mathbf{x}) = \sum_{\mathbf{n}\in M} p_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$ be a polynomial with real coefficients. Assume that for each coefficient $p_{\mathbf{n}}$ with $p_{\mathbf{n}} < 0$ there exists an $\mathbf{m} \in M$ with $\mathbf{n} \lneqq \mathbf{m}$ and $p_{\mathbf{m}} > 0$. Then $y(\mathbf{x})p(\mathbf{x})$ is H-admissible in $\mathcal{R}$.*

Proof. Let $\bar{\mathbf{a}}$ and $\bar{B}$ denote the vector of the first and the matrix of the second logarithmic derivatives of $y(\mathbf{x})p(\mathbf{x})$, respectively. Then

$$\begin{aligned}
\bar{a}_j(\mathbf{r}) &= a_j(r) + r_j \frac{p_{x_j}(\mathbf{r})}{p(\mathbf{r})}, \\
\bar{B}_{jj}(\mathbf{r}) &= B_{jj}(\mathbf{r}) + r_j \frac{p_{x_j}(\mathbf{r})}{p(\mathbf{r})} + r_j^2 \left(\frac{p_{x_j x_j}(\mathbf{r})}{p(\mathbf{r})} - \frac{p_{x_j}(\mathbf{r})^2}{p(\mathbf{r})^2}\right), \\
\bar{B}_{jk}(\mathbf{r}) &= B_{jk}(\mathbf{r}) + r_j r_k \left(\frac{p_{x_j x_k}(\mathbf{r})}{p(\mathbf{r})} - \frac{p_{x_j}(\mathbf{r}) p_{x_k}(\mathbf{r})}{p(\mathbf{r})^2}\right).
\end{aligned}$$

Clearly, the contributions coming from the polynomial remain bounded when $\mathbf{r} \to \infty$. Moreover,

$$\frac{p\left(r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}\right)}{p(\mathbf{r})} = O\left(1\right).$$

Furthermore, note the condition on the eigenvalues of $B(\mathbf{r})$ ensures that we can choose $\boldsymbol{\delta}$ such that $\|\boldsymbol{\delta}(\mathbf{r})\| \to 0$, because in this case $c(\mathbf{r}) := \sqrt{2\log(\det B(\mathbf{r}))/\lambda_{\min}(\mathbf{r})} \to 0$. If $\boldsymbol{\theta}$ fulfils $\|\boldsymbol{\theta}\| > c(\mathbf{r})$ then (II) holds, since

$$\left|\frac{y\left(\mathbf{r}e^{i\boldsymbol{\theta}}\right)}{y(\mathbf{r})}\right| \sim \exp\left(-\frac{\boldsymbol{\theta}B(\mathbf{r})\boldsymbol{\theta}^t}{2}\right) \leq \exp\left(-\frac{\lambda_{min}}{2}\|\boldsymbol{\theta}\|^2\right) < \frac{1}{\det B(\mathbf{r})} = o\left(\frac{1}{\sqrt{\det B(\mathbf{r})}}\right).$$

Therefore it is an easy exercise to show (I)–(V). □

Theorem 11. *Let $y(\mathbf{x})$ be H-admissible in $\mathcal{R}$ and $f(\mathbf{x})$ an analytic function in this region. Assume that $f(\mathbf{x})$ is real if $\mathbf{x} \in \mathbb{R}^d$ and that there exists a $\delta > 0$ such that*

$$\max_{x_i=r_i, i=1,\dots,d} |f(\mathbf{x})| = O\left(y(\mathbf{r})^{1-\delta}\right), \text{ as } \mathbf{r} \to \infty.$$

Then $y(\mathbf{x}) + f(\mathbf{x})$ is H-admissible in $\mathcal{R}$.

Proof. Let again $\bar{\mathbf{a}}$ and $\bar{B}$ denote the vector of the first and the matrix of the second logarithmic derivatives of $y(\mathbf{x}) + f(\mathbf{x})$, respectively. Then obviously, $\bar{a}_j(\mathbf{r}) \sim a_j(\mathbf{r})$ and $\bar{B}_{jk}(\mathbf{r}) \sim B_{jk}(\mathbf{r})$ and with these relations H-admissibility of $y(\mathbf{x}) + f(\mathbf{x})$ is easily proved. □

Corollary. *If $y(\mathbf{x})$ is H-admissible in $\mathcal{R}$ and $p(\mathbf{x})$ is a polynomial with real coefficients, then $y(\mathbf{x}) + p(\mathbf{x})$ is H-admissible in $\mathcal{R}$. If $p(x)$ is a polynomial in one variable with real coefficients and a positive leading coefficient, then $p(y(\mathbf{x}))$ is also H-admissible.*

Proof. This is an immediate consequence of Theorems 9 and 11 (cf. remark after Theorem 9). □

Theorem 12. *If $y(z)$ is univariate H-admissible, then $Y(x,z) = e^{xy(z)}$ is H-admissible in $\{(r,s) : y(s)^{\varepsilon-1} \le r \le y(s)^c\}$ where ε, c are arbitrary positive constants.*

Remark 6. This closure property is true for BR-admissible functions as well.

Remark 7. We think that the same holds also for multivariate H-admissible functions, but we did not succeed in proving that all eigenvalues tend to infinity (condition (III) of the definition).

Proof. The first logarithmic derivatives of Y are given by $a_1(x,z) = xY_x/Y = xy(z)$ and $a_2(x,z) = zY_z/Y = xzy'(z)$. The matrix of the second logarithmic derivatives is

$$B(x,z) = \begin{pmatrix} xy(z) & xzy'(z) \\ xzy'(z) & xz^2y''(z) + xzy'(z) \end{pmatrix}.$$

If a_y and b_y denote the first and second logarithmic derivative of $y(x)$, respectively, then a straightforward computation shows $\det B(x,z) = x^2y(z)^2b_y(z) \to \infty$. The smaller eigenvalue is

$$\frac{xy(z) + xz^2y''(z) + xzy'(z)}{2}\left(1 - \sqrt{1 - \frac{4\det B}{(xy(z) + xz^2y''(z) + xzy'(z))^2}}\right)$$
$$\sim \frac{\det B}{xy(z) + xz^2y''(z) + xzy'(z)} \sim \frac{x^2y(z)^2b_y}{xy(z)a_y^2} \to \infty$$

which proves (III). (IV) and (V) are obvious.

Now we turn to (II). Let $x = re^{i\theta}$ and $z = se^{i\varphi}$. Then we have $|Y(x,z)| = |\exp(\Re re^{i\theta}y(se^{i\varphi}))|$. We know from [7, Lemma 2] that for $|\varphi| \ge f(s)^{-\nu}$ ($\nu > 0$ then there is a positive constant κ with $|y(se^{i\varphi})| \le y(s) - y(s)^{1-\kappa}$. Thus for $|\varphi| \le (ry(s))^{-1/3-\varepsilon}$ and $r \le y(s)^{\varepsilon-1}$ this implies

$$\left|re^{i\theta}y\left(se^{i\varphi}\right)\right| = o\left(\frac{ry(s)}{\sqrt{b_y(s)}}\right)$$

and this yields

$$\left|\exp\left(re^{i\theta}y\left(se^{i\varphi}\right)\right)\right| \le e^{ry(s)/2} = o\left(\frac{e^{2ry(s)/3}}{\sqrt{b_y(s)}}\right) = o\left(\frac{e^{ry(s)}}{\sqrt{\det B(r,s)}}\right)$$

and we get (II) for this case.

Now let $|\theta| \ge (ry(s))^{-1/3-\varepsilon/2}$ and $|\varphi| \le (ry(s))^{-1/3-\varepsilon}$. Then [17, Lemma 5] implies

$$\Re re^{i\theta}y(se^{i\varphi}) \le r\left(1 - \frac{\theta^2}{5}\right)\left(y(s) - \frac{\varphi^2}{2}(sy'(s) + s^2y''(s)\right) - r\sin\theta\cdot\varphi sy'(s) = o(ry(s))$$

where the last equation follows from applying the constraint on φ and θ as well as $r \le y(s)^c$. This shows (II).

If $|\theta| \le (ry(s))^{-1/3-\varepsilon/4}$ and $|\varphi| \le (ry(s))^{-1/3-\varepsilon/2}$ then a routine calculation shows the estimate of (I) in this range. Thus we can inscribe a cuboid $\mathbf{\Delta}(r,s)$ spanned by an orthonormal basis of eigenvectors of $B(r,s)$ into this domain and have (I) inside $\mathbf{\Delta}(r,s)$ whereas outside we are in the range where showed above the validity of (II). $\square$

Theorem 13. *If $y(\mathbf{z})$ is H-admissible in $\mathcal{R}$. Let $\lambda_{\min}$ $\lambda_{\max}$ denote the smallest and the largest eigenvalue of the matrix $B(\mathbf{r})$ of the second logarithmic derivatives of $e^{y(\mathbf{z}_1)y(\mathbf{z}_2)}$. Then $e^{y(\mathbf{z}_1)y(\mathbf{z}_2)}$ is H-admissible in $\mathcal{S} = \{(\mathbf{r}_1,\mathbf{r}_2) \in \mathcal{R}\times\mathcal{R} \mid \log\lambda_{\max} = o\left((y(\mathbf{r}_1)y(\mathbf{r}_2))^{-2/3+\rho}\lambda_{\min}\right)\}$ for any $\rho > 0$.*

Proof. Using Lemma 6 it is easy to show (I) for $\mathbf{\Delta} = [-A,A]^{2d}$ with $A = (y(\mathbf{r}_1)y(\mathbf{r}_2))^{-1/3+\rho/2}$. Moreover, we have on the boundary of $\mathbf{\Delta}$

$$\Re\left(\log y(\mathbf{r}e^{i\boldsymbol{\theta}}) - \log y(\mathbf{r}) + \frac{1}{2}\log\det B(\mathbf{r})\right) \sim -\frac{1}{2}\lambda_{\min} + \frac{1}{2}\log\det B(\mathbf{r})$$

which tends to $-\infty$ in $\mathcal{S}$ and thus proves (II).

To show (III) let $\mathbf{a}_y$ and B_y denote the first and second logarithmic derivatives of y, respectively. Note that B can be written in block matrix form

$$B(\mathbf{r}) = y(\mathbf{r}_1)y(\mathbf{r}_2)\begin{pmatrix} B_y(\mathbf{r}_1) + \mathbf{a}(\mathbf{r}_1)^t\mathbf{a}(\mathbf{r}_1) & \mathbf{a}(\mathbf{r}_1)^t\mathbf{a}(\mathbf{r}_2) \\ \mathbf{a}(\mathbf{r}_1)^t\mathbf{a}(\mathbf{r}_2) & B_y(\mathbf{r}_2) + \mathbf{a}(\mathbf{r}_2)^t\mathbf{a}(\mathbf{r}_2)\end{pmatrix}.$$

This allows a decomposition into a sum of a positive definite and a positive semidefinite matrix. So arguing as in the proof of Theorem 8 we obtain (III). (IV) and (V) are obvious. $\square$

7. Examples of H-admissible functions

7.1. **Stirling numbers of the second kind.** The generating function of the Stirling numbers of the second kind is $y(z,u) = e^{u(e^z-1)}$ and satisfies the conditions of Theorem 12. Therefore the coefficients satisfy the assertion of Theorem 4 which was already proved in [7]. It follows that the number of blocks in a random partition of size n is asymptotically normally distributed, as $n \to \infty$. This is a classical result of Harper (see [15]).

7.2. **Permutations with bounded cycle length.** Consider the set of permutations with no cycle longer than ℓ $\Longrightarrow$ counted by length and number of cycles. The generating function is then

$$y(z,u) = \exp\left(u\sum_{i=1}^{\ell}\frac{z^i}{i}\right).$$

The exponent is a polynomial satisfying the conditions of Corollary and is therefore H-admissible. So the assertion of Theorem 4 for the coefficients follows. This slightly generalises a result in [10], where only the asymptotic normal distribution of the number of cycles (this means, roughly speaking, that the marginal distribution is asymptotically normal) was established for $\ell \ge 3$.

7.3. **Partitions of a set of partitions.** The generating function of the set of partitions of the set of block of a given partition counted by number of blocks (v counting the blocks of the *inner* and u counting blocks of the *outer* partition) is $y(z,u) = \exp\left(u\left(e^{v(e^z-1)} - 1\right)\right)$. A bivariate normal distribution of these two block numbers follows now from Theorem 4. For mean and variance of the number of blocks of the outer partition were computed by Salvy and Shackell [36].

7.4. **Set partitions with bounded block size.** Set partitions with bounded block size can be counted by the generating function

$$y(z,u) = \exp\left(u\sum_{i=1}^{\ell}\frac{z^i}{i!}\right).$$

This generalises a result in [10] where under the assumption $\ell \ge 3$ it was shown that the number of blocks in such

7.5. **Covering complete bipartite graphs.** The number of coverings of a complete bipartite graph with complete bipartite graphs with at least one vertex in each part of the bipartition (see [14, Example 3.3.8]) can be described by the generating function $y(z,u) = \exp\left((e^z-1)(e^u-1)\right)$ which is H-admissible in $\mathbb{R}^2$ by Theorem 13.

7.6. **Set partitions with coloured elements.** Consider partitions of a set where each element is assigned to one of d colours. Moreover let $S \subseteq \mathbb{Z}^d$ be a finite set and let for each block $\mathcal{B}$ of the partition b_j denote the number of elements of $\mathcal{B}$ having colour j. Then partitions such that for each block we have $(b_1, \ldots, b_d) \in S$ can be counted by the generating function

$$y(\mathbf{z}) = \exp\left(\sum_{\mathbf{n}\in S} \frac{\mathbf{z}^{\mathbf{n}}}{n_1! \cdots n_d!}\right)$$

which is H-admissible.

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References

[1] Luis Báez-Duarte. Hardy-Ramanujan's asymptotic formula for partitions and the central limit theorem. *Adv. Math.*, 125(1):114–120, 1997.
[2] Jason P. Bell, Edward A. Bender, Peter J. Cameron, and L. Bruce Richmond. Asymptotics for the probability of connectedness and the distribution of number of components. *Electron. J. Combin.*, 7(1):Research Paper 33, 22 pp. (electronic), 2000.
[3] E. A. Bender, P. J. Cameron, A. M. Odlyzko, and L. B. Richmond. Connectedness, classes and cycle index. *Combin. Probab. Comput.*, 8(1-2):31–43, 1999. Recent trends in combinatorics (Mátraháza, 1995).
[4] Edward A. Bender. Central and local limit theorems applied to asymptotic enumeration. *J. Combinatorial Theory Ser. A*, 15:91–111, 1973.
[5] Edward A. Bender and L. Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration. II. Multivariate generating functions. *J. Combin. Theory Ser. A*, 34(3):255–265, 1983.
[6] Edward A. Bender and L. Bruce Richmond. A generalisation of Canfield's formula. *J. Combin. Theory Ser. A*, 41(1):50–60, 1986.
[7] Edward A. Bender and L. Bruce Richmond. Admissible functions and asymptotics for labelled structures by number of components. *Electron. J. Combin.*, 3(1):Research Paper 34, approx. 23 pp. (electronic), 1996.
[8] David Burshtein and Gadi Miller. Asymptotic enumeration methods for analyzing LDPC codes. *IEEE Trans. Inform. Theory*, 50(6):1115–1131, 2004.
[9] Rod Canfield, Sylvie Corteel, and Pawel Hitczenko. Random partitions with non negative rth differences. In *LATIN 2002: Theoretical Informatics (Cancun)*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 131–140. Springer, Berlin, 2002.
[10] Michael Drmota, Bernhard Gittenberger, and Thomas Klausner. Extended admissible functions and Gaussian limiting distributions. *Mathematics of Computation*, 74:1953–1966, 2005.
[11] Philippe Flajolet and Michèle Soria. Gaussian limiting distributions for the number of components in combinatorial structures. *J. Combin. Theory Ser. A*, 53(2):165–182, 1990.
[12] Philippe Flajolet and Michèle Soria. General combinatorial schemas: Gaussian limit distributions and exponential tails. *Discrete Math.*, 114(1-3):159–180, 1993.
[13] Zhicheng Gao and L. Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration. IV. Multivariate generating functions. *J. Comput. Appl. Math.*, 41(1-2):177–186, 1992.
[14] I. P. Goulden and D. M. Jackson. *Combinatorial enumeration.* A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1983. With a foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics.
[15] L. H. Harper. Stirling behavior is asymptotically normal. *Ann. Math. Statist.*, 38:410–414, 1967.
[16] Bernard Harris and Lowell Schoenfeld. Asymptotic expansions for the coefficients of analytic functions. *Illinois J. Math.*, 12:264–277, 1968.
[17] W. K. Hayman. A generalisation of Stirling's formula. *J. Reine Angew. Math.*, 196:67–95, 1956.
[18] Finbarr Holland and Richard Rochberg. Bergman kernel asymptotics for generalized Fock spaces. *J. Anal. Math.*, 83:207–242, 2001.
[19] John Knopfmacher and Richard Warlimont. Arithmetical semigroups related to trees and polyhedra. II. Maps on surfaces. *Math. Nachr.*, 235:59–81, 2002.
[20] C. Krattenthaler and T. W. Müller. Equations in finite semigroups: explicit enumeration and asymptotics of solution numbers. *J. Combin. Theory Ser. A*, 105(2):291–334, 2004.
[21] Gérard Letac, Dhafer Malouche, and Stefan Maurer. The real powers of the convolution of a negative binomial distribution and a Bernoulli distribution. *Proc. Amer. Math. Soc.*, 130(7):2107–2114 (electronic), 2002.
[22] Martin W. Liebeck and Aner Shalev. Fuchsian groups, coverings of Riemann surfaces, subgroup growth, random quotients and random walks. *J. Algebra*, 276(2):552–601, 2004.
[23] Wafik Boulos Lotfallah. Strong 0-1 laws in finite model theory. *J. Symbolic Logic*, 65(4):1686–1704, 2000.
[24] Thomas Müller. Finite group actions and asymptotic expansion of $e^{P(z)}$. *Combinatorica*, 17(4):523–554, 1997.
[25] Ljuben Mutafchiev. Limiting distributions for the number of distinct component sizes in relational structures. *J. Combin. Theory Ser. A*, 79(1):1–35, 1997.

[26] Ljuben Mutafchiev. Erratum to: "Limiting distributions for the number of distinct component sizes in relational structures" [J. Combin. Theory Ser. A **79** (1997), no. 1, 1–35. *J. Combin. Theory Ser. A*, 102(2):447–449, 2003.
[27] Ljuben R. Mutafchiev. On the size of the Durfee square of a random integer partition. *J. Comput. Appl. Math.*, 142(1):173–184, 2002.
[28] Ljuben R. Mutafchiev. The typical growth of the kth excess in a random integer partition. *Monatsh. Math.*, 136(4):313–325, 2002.
[29] Ljuben R. Mutafchiev. On the maximal multiplicity of parts in a random integer partition. *Ramanujan J.*, 9(3):305–316, 2005.
[30] A. M. Odlyzko. Asymptotic enumeration methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1063–1229. Elsevier, Amsterdam, 1995.
[31] A. M. Odlyzko and L. B. Richmond. Asymptotic expansions for the coefficients of analytic generating functions. *Aequationes Math.*, 28(1-2):50–63, 1985.
[32] Alon Orlitsky, Narayana P. Santhanam, and Junan Zhang. Always good Turing: asymptotically optimal probability estimation. *Science*, 302(5644):427–431, 2003.
[33] Dan Richardson, Bruno Salvy, John Shackell, and Joris Van der Hoeven. Asymptotic expansions of exp-log functions. In Y. N. Lakshman, editor, *ISSAC'96*, pages 309–313. ACM Press, 1996. Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation. July 24–26, 1996. Zürich, Switzerland.
[34] Bruno Salvy. Examples of automatic asymptotic expansions. *SIGSAM Bulletin*, 25(2):4–17, April 1991.
[35] Bruno Salvy. Fast computation of some asymptotic functional inverses. *J. Symbolic Comput.*, 17(3):227–236, 1994.
[36] Bruno Salvy and John Shackell. Symbolic asymptotics: multiseries of inverse functions. *J. Symbolic Comput.*, 27(6):543–563, 1999.