THE MOMENTS OF THE SUM-OF-DIGITS FUNCTION IN NUMBER FIELDS

BERNHARD GITTENBERGER* AND JÖRG M. THUSWALDNER

Abstract. We consider the asymptotic behavior of the moments of the sum-of-digits function of canonical number systems in number fields. Using Delange's method we obtain the main term and smaller order terms which contain periodic fluctuations.

1. Introduction

Let $\nu_q(n)$ denote the sum of digits function of n in its q-ary representation for some integers $q \geq 2$ and $n \geq 0$. In 1975 Delange [2] considered the average of $\nu_q(n)$. He obtained

$$E_N = \frac{1}{N} \sum_{n<N} \nu_q(n) = \frac{q-1}{2} \log_q N + \gamma_1(\log_q N) \tag{1.1}$$

with a continuous, periodic fluctuation γ_1 of period 1. In [15] Kirschenhofer computed the variance of $\nu_q(n)$. His result was

$$V_N = \frac{1}{N} \sum_{n<N} \nu_q^2(n) - \frac{1}{N^2} \left(\sum_{n<N} \nu_q(n) \right)^2 = \left(\frac{q-1}{2} \right)^2 \log_q N + \gamma(\log_q N)$$

with a continuous fluctuation γ of period 1. The same result was obtained independently by Kennedy and Cooper [14]. Finally, Grabner, Kirschenhofer, Prodinger and Tichy [8] established an exact formula for the d-th moment of the binary sum of digits function:

$$\frac{1}{N} \sum_{n<N} \nu_2(n)^d = \frac{1}{2^d} (\log_2 N)^d + \sum_{i=0}^{d-1} (\log_2 N)^i \gamma_i(\log_2 N),$$

where the γ_i are again continuous fluctuations of period 1. All these results can be extended to so-called canonical number systems. We recall the definition of these number systems:

Definition 1.1. Let $\mathbf{K}$ be a number field and $\mathbf{Z_K}$ its ring of integers. A pair $(b, \mathcal{N})$ with $b \in \mathbf{Z_K}$ and $\mathcal{N} = \{0, 1, \ldots, |N(b)| - 1\}$ is called *canonical number system* if any $\gamma \in \mathbf{Z_K}$ has a representation of the form

$$\gamma = c_0 + c_1 b + \cdots + c_h b^h, \quad c_h \neq 0 \quad \text{if} \quad h \neq 0,$$

where $h \in \mathbb{N}_0$ and $c_i \in \mathcal{N}$ for $i = 0, 1, \ldots, h$. b is called base and $\mathcal{N}$ is called set of digits of $(b, \mathcal{N})$.

The sum of digits of γ with respect to the base b is defined by

$$\nu_b(\gamma) = c_0 + c_1 + \cdots + c_h.$$

Remark 1.1. Of course, the set of digits is uniquely determined by the base of a canonical number system. The bases for canonical number systems in arbitrary number fields were characterized by Kovács and Pethő [17]. Kovács [16] proved that a number field $\mathbf{K}$ contains bases of canonical number systems if and only if its ring of integers has a power basis. In this case $\mathbf{K}$ is called canonical number field. The shape of the bases of $\mathbf{K}$ is intimately related to the

Date: November 17, 1997.

1991 *Mathematics Subject Classification.* Primary: 11A63, Secondary: 11N60.

*This author's work was supported by the Austrian Science Foundation FWF, grant P10223-MAT.

integral basis of its ring of integers $\mathbf{Z_K}$. So they can be determined explicitely only for number fields with known power integral bases. This has been done in [13, 11, 12] for bases of canonical number systems in quadratic fields.

Kátai and Szabó [13] showed that the only bases in the ring of Gaussian integers $\mathbb{Z}[i]$ are given by $b = -n \pm i$, where $n \in \mathbb{N}$. Recently, Grabner, Kirschenhofer and Prodinger [7] generalized Delange's result (1.1) to Gaussian integers. They showed that $(b = -n \pm i)$

$$\frac{1}{N\pi + O(\sqrt{N})} \sum_{|z|^2<N} \nu_b(z) = 2\log_{n^2+1} N + \delta_1(\log_{n^2+1} N) + O(N^{-\frac{1}{2}} \log N), \tag{1.2}$$

where the sum is extended over all Gaussian integers z with $|z|^2 < N$ and δ_1 is a periodic fluctuation of period 1. The denominator $\pi N + O(\sqrt{N})$ denotes the number of Gaussian integers with this property. In Thuswaldner [19] this result was generalized to canonical number systems. We want to study the higher moments in the general case. To be able to state the general results we need some preliminaries (cf. [19]).

We use the natural embedding of a number field $\mathbf{K}$ of degree n and signature (s,t), $n = s+2t$, given by

$$\mathbf{v}(\gamma) = (\sigma_1(\gamma), \dots, \sigma_s(\gamma), \sigma_{s+1}(\gamma), \dots, \sigma_{s+t}(\gamma)),$$

where $\sigma_1, \dots, \sigma_s$ are the real, and $\sigma_{s+1}, \dots, \sigma_{s+t}$ are the complex isomorphisms, that map $\mathbf{K}$ onto its equivalent fields. The elements of the ring of integers $\mathbf{Z_K}$ form a lattice in this vector space (cf. for instance [1]). By the above remark we can confine ourselves to number fields whose rings of integers have power bases $\{1, \alpha, \dots, \alpha^{n-1}\}$. Hence, the vectors $\mathbf{v}(1), \mathbf{v}(\alpha), \dots, \mathbf{v}(\alpha^{n-1})$ generate the lattice formed by the elements of $\mathbf{Z_K}$. The volume of the fundamental domain of this lattice is now given by

$$I(\alpha) = \sqrt{\det(<\mathbf{v}(\alpha^i), \mathbf{v}(\alpha^j)>; i,j = 0, \dots, n-1)}.$$

In order to generalize the region of summation $|z|^2 < N$ in the above mentioned result of Grabner, Kirschenhofer and Prodinger [7] we consider regions of the shape (cf. [19])

$$\begin{array}{rcl} |z_1| & \leq & l_1, \\ \cdots & & \cdots \\ |z_s| & \leq & l_s, \\ z_{s+1}\bar{z}_{s+1} & \leq & l_{s+1}, \\ \cdots & & \cdots \\ z_{s+t}\bar{z}_{s+t} & \leq & l_{s+t}, \end{array} \tag{1.3}$$

where $z_1, \dots, z_s, z_{s+1}, \bar{z}_{s+1}, \dots, z_{s+t}, \bar{z}_{s+t}$ are the conjugates of $z \in \mathbf{K}$ and $l_1, \dots, l_{s+t}$ are certain numbers. We denote the set of all $z \in \mathbf{K}$ that fulfill (1.3) by

$$\mathcal{D}(l_1, \dots, l_s; l_{s+1}, \dots, l_{s+t}).$$

Remark 1.2. Note, that the closure of the set $\mathbf{v}(\mathcal{D}(l_1, \dots, l_s; l_{s+1}, \dots, l_{s+t}))$ is given by the set of all vectors $(z_1, \dots, z_s, z_{s+1}, \dots, z_{s+t})^T$ with $z_1, \dots, z_s \in \mathbb{R}$ and $z_{s+1}, \dots, z_{s+t} \in \mathbb{C}$ that fulfill (1.3) (cf. [1]). This set can be interpreted geometrically as a product of circles and lines.

2. Statement of Results

We are now in the position to state the main result of this paper.

Theorem 2.1. *Let $\mathbf{K}$ be a number field of degree n and signature (s,t) and $\mathbf{Z_K}$ its ring of integers with integral basis $\{1, \alpha, \dots, \alpha^{n-1}\}$. Furthermore, let b be the base of a canonical number system and denote by $b_1, \dots, b_s; b_{s+1}, \bar{b}_{s+1}, \dots, b_{s+t}, \bar{b}_{s+t}$ the conjugates of b and set*

$M = |N(b)|$. *Moreover, choose* x *such that* $1 < x < |b_1|$ *(*$1 < x < |b_1|^2$ *if* $s = 0$*, respectively) and*

$$x_1(x) = x, \ \ x_i(x) = a_i x + c_i; \quad a_i = \frac{|b_i| - 1}{|b_1| - 1}\ , \ \ c_i = \frac{|b_1| - |b_i|}{|b_1| - 1}\ (i = 2, \dots , s); \tag{2.1}$$

$$x_i(x) = a_i x + c_i; \quad a_i = \frac{|b_i|^2 - 1}{|b_1| - 1}\ , \ \ c_i = \frac{|b_1| - |b_i|^2}{|b_1| - 1}\ (i = s+1, \dots , s+t).$$

If $s = 0$*, set* $x_1 = x$ *and* $x_i = a_i x + c_i$ *with*

$$a_i = \frac{|b_i|^2 - 1}{|b_1|^2 - 1} \quad \text{and} \quad c_i = \frac{|b_1|^2 - |b_i|^2}{|b_1|^2 - 1} \quad \text{for} \quad i = 2, \dots , t.$$

Furthermore, put

$$\mathcal{D}(b, k, x) = \mathcal{D}(|b_1|^k x_1(x), \dots , |b_s|^k x_s(x); |b_{s+1}|^{2k} x_{s+1}(x), \dots , |b_{s+t}|^{2k} x_{s+t}(x)), \tag{2.2}$$

and $N = M^k x_1 \cdots x_{s+t}$. *Then we have*

$$S_d(N) = S_d(b, k, x) = \sum_{z \in \mathcal{D}(b,k,x)} (\nu_b(z))^d$$
$$= \frac{2^s \pi^t}{I(\alpha)} \Big(\frac{M-1}{2}\Big)^d N \log_M^d N + N \sum_{j=0}^{d-1} \log_M^j N \Phi_j(\log_M N) + \mathcal{O}\left(N^{\frac{n-1}{n}} \log_M^d N\right),$$

where the sum runs over all algebraic integers in $\mathcal{D}(b, k, x)$ *and* $\Phi_0, \dots , \Phi_{d-1}$ *are continuous periodic fluctuations of period* 1*.*

Remark 2.1. Theorem 2.1 has been proved for $d = 1$ (cf. [19], we used the notation of this paper in the formulation of Theorem 2.1). The only result on arbitrary moments of the sum-of-digits function known to us was established by Grabner, Kirschenhofer, Prodinger, and Tichy [8] for the binary sum-of-digits function. The main difficulty in our proof are the error terms, that occur only in the general case of canonical number systems.

Since the formulation of Theorem 2.1 is rather long and complicated, we list some special cases as corollaries. First we treat the imaginary quadratic case, then the case of number systems in $\mathbb{Z}$ with negative integers as bases.

Corollary 2.1. *Let* $D \geq 1$ *be squarefree and* $\mathbf{K} = \mathbb{Q}(\sqrt{-D})$ *an imaginary quadratic number field with ring of integers* $\mathbf{Z_K}$.

If $D \not\equiv -1(4)$ *and* $A \in \mathbb{N}$*, then we have* $b = -A \pm i\sqrt{D}$, $\bar{b} = -A \mp i\sqrt{D}$, $M = N(b)$, $N = M^k x$ *with* $1 \leq x < M$*. For the moments of the sum of digits function we get*

$$S_d(N) = S_d(b, k, x) = \sum_{\substack{|z|^2 < N \\ z \in \mathbf{Z_K}}} (\nu_b(z))^d$$
$$= \frac{\pi}{\sqrt{D}} \Big(\frac{M-1}{2}\Big)^d N \log_M^d N + N \sum_{j=0}^{d-1} \log_M^j N \Phi_j(\log_M N) + \mathcal{O}\left(N^{\frac{1}{2}} \log N\right).$$

If $D \equiv -1(4)$ *and* $B \in \mathbb{N}$, $B \equiv 1(2)$*, then we have* $b = \frac{-B + i\sqrt{D}}{2}$, $\bar{b} = \frac{-B - i\sqrt{D}}{2}$, $M = N(b) = \frac{B^2 + D}{4}$, $N = M^k x$ *with* $1 \leq x < M$*. For the sum of digits function we get*

$$S_d(N) = S_d(b, k, x) = \sum_{\substack{|z|^2 < N \\ z \in \mathbf{Z_K}}} (\nu_b(z))^d$$
$$= \frac{2\pi}{\sqrt{D}} \Big(\frac{M-1}{2}\Big)^d N \log_M^d N + N \sum_{j=0}^{d-1} \log_M^j N \Phi_j(\log_M N) + \mathcal{O}\left(N^{\frac{1}{2}} \log N\right).$$

Proof. The result follows immidiately from Theorem 2.1 by observing, that the indicated bases b are excactly the admissible bases of imaginary quadratic number fields (cf. [12]). We note, that the integral basis has the shape $\{1, i\sqrt{D}\}$ in the first case and in the second case $\{1, \frac{1+i\sqrt{D}}{2}\}$, respectively. ○

Remark 2.2. Note that this corollary also covers the case of number systems in Gaussian integers. Hence, it is a generalization of (1.2) to higher moments.

Corollary 2.2. *Let $b \in \mathbb{Z}$ and $b \leq -2$. then we get for*

$$S_d(N) = \sum_{|n|<N} (\nu_b(n))^d$$

the exact formula

$$S(N) = 2\Big(\frac{b-1}{2}\Big)^d + \sum_{j=0}^{d-1} N \log_{|b|}^j N \Phi_j \left(\frac{1}{2}\log_{|b|} N\right).$$

Proof. This formula is exact, because the fundamental region coincides with the line of integration in this case. Hence, the substitution of the sum by an integral in (3.3) in the proof of the theorem causes an additional fluctuation instead of the error term. ○

3. Proof of the Theorem

In order to prove the theorem we will use Delange's method (see [2]) in a form as used in [7] and [19].

The preparations at the beginning of the proof are exactly the same as in [19]. We therefore refer to this paper for details. Let

$$\mathcal{F} = \Big\{z | z = \sum_{\ell=1}^{\infty} \varepsilon_\ell(z) b^{-\ell}\Big\}$$

where $\varepsilon_i(z)$ is the i-th digit of z in its fractional part. As in Kátai [9, Theorem 2] one can prove that $\mathcal{F}$ is a compact set and the embedding of $\mathcal{F}$ to $\mathbb{R}^n$ is also compact (cf. [10]). The fractal geometry of $\mathcal{F}$ for complex bases has been extensively studied by Gilbert (see [3, 4, 5, 6]). Since the length of a b-adic representation is uniformly bounded (see [18]), we set $\mu = \max_z(L_b(z))$, where the maximum is taken over all $z \in \mathbf{Z_K}$ with $(z+\mathcal{F}) \cap \mathcal{D}(|b_1|, \ldots, |b_s|; |b_{s+1}|^2, \ldots, |b_{s+t}|^2) \neq \emptyset$ and $L_b(z)$ the length of the digit expansion of z with respect to b. Then we define the sets $\mathcal{F}_k$ by

$$\mathcal{F}_k = \Big\{z | z = \sum_{\ell=-\mu}^{k} \varepsilon_\ell(z) b^{-\ell}\Big\}. \tag{3.1}$$

One can easily see, that all elements of $\mathcal{D}(|b_1|, \ldots, |b_s|; |b_{s+1}|^2, \ldots, |b_{s+t}|^2)$ with at most k digits in their fractional part are contained in $\mathcal{F}_k$ by the definition of μ. Now $S_d(b,k,x)$ can be written as

$$S_d(b,k,x) = \sum_{\ell_1,\ldots,\ell_d=-\mu}^{k} \sum_{\substack{z \in \mathcal{D}(b,0,x) \\ z \in \mathcal{F}_k}} \varepsilon_{l_1}(z) \cdots \varepsilon_{l_d}(z). \tag{3.2}$$

For brevity let us write $\mathbf{z} = (z_1, \ldots z_s, z_{s+1}, \bar{z}_{s+1}, \ldots, z_{s+t}, \bar{z}_{s+t})^T$ throughout the proof. Then by applying Delange's approach the last sum may be represented by an integral and we get

$$
\begin{aligned}
S_d(b,k,x)(N) \quad = \quad & \frac{M^k}{I(\alpha)} \sum_{\ell_1,\dots,\ell_d=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} \varepsilon_{\ell_1}(\mathbf{v}^{-1}(\mathbf{z})) \cdots \varepsilon_{\ell_d}(\mathbf{v}^{-1}(\mathbf{z}))\, d\lambda_d(\mathbf{z}) \\
& + \mathcal{O}\left(k^d M^{\frac{k(n-1)}{n}}\right),
\end{aligned} \tag{3.3}
$$

where λ_d denotes the d-dimensional Lebesgue measure. Observe that the functions $\varepsilon_\ell(\mathbf{v}^{-1}(\mathbf{z}))$ ($\ell \le k$) are constant on each piece of the tiling of $\mathbf{R}^n$ induced by the translates of $\mathbf{v}((-n \pm i)^{-k}\mathcal{F})$ which follows immediately from the definition of $\mathcal{F}$. Hence the only difference between sum and integral is caused by the pieces which intersect the boundary of $\overline{\mathbf{v}(\mathcal{D}(b,0,x))}$. Since the product in the integrand is bounded by $(M-1)^d$ and we have k^d summands, the order of the error is $k^d M^{\frac{k(n-1)}{n}}$. The factor $\frac{M^k}{I(\alpha)}$ is due to the volume of the fundamental domain of the lattice induced by the elements of $\mathcal{F}_k$.

In order to prove the theorem, we have to split the integral in a way that will enable us to separate the terms contributing to the periodic fluctuations from the nonperiodic ones. As the mean value of ε_ℓ within the integration domain (with the exception of those pieces which intersect the boundary) is $\frac{M-1}{2}$ we set $L_\ell(\mathbf{z}) = \varepsilon_\ell(\mathbf{v}^{-1}(\mathbf{z})) - \frac{M-1}{2}$ and write the above integral as

$$
\begin{aligned}
S_d(b,k,x) = & \frac{M^k}{I(\alpha)} \sum_{\ell_1,\dots,\ell_d=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} \prod_{i=1}^{d} \left(L_{\ell_i}(\mathbf{z}) + \frac{M-1}{2}\right) d\lambda_d(\mathbf{z}) + \mathcal{O}\left(k^d M^{\frac{k(n-1)}{n}}\right) \\
= & \frac{M^k}{I(\alpha)} \sum_{\ell_1,\dots,\ell_d=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} \sum_{i=0}^{d} \left(\frac{M-1}{2}\right)^{d-i} \tau_i(L_{\ell_1}(\mathbf{z}),\dots,L_{\ell_d}(\mathbf{z}))\, d\lambda_d(\mathbf{z}) \\
& + \mathcal{O}\left(k^d M^{\frac{k(n-1)}{n}}\right),
\end{aligned}
$$

where τ_i denotes the i-th elementary symmetric function. Interchanging summations and integrals and keeping in mind that the integrals over $L_{j_1} \cdots L_{j_i}$ only depend on the number of factors and how many of the numbers $\ell_1, \dots, \ell_d$ are pairwise equal yields

$$
\begin{aligned}
S_d(b,k,x) = & \frac{M^k}{I(\alpha)} \left(\frac{M-1}{2}\right)^d (k+\mu+1)^d \int_{\mathbf{v}(\mathcal{D}(b,0,x))} 1\, d\lambda_d(\mathbf{z}) \\
& + \frac{M^k}{I(\alpha)} \sum_{i=1}^{d} \left(\frac{M-1}{2}\right)^{d-i} \binom{d}{i} \sum_{\ell_1,\dots,\ell_d=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} L_{\ell_1} \cdots L_{\ell_i}\, d\lambda_d(\mathbf{z}) \\
& + \mathcal{O}\left(k^d M^{\frac{k(n-1)}{n}}\right) \\
= & \frac{M^k}{I(\alpha)} 2^s \pi^t x_1 \cdots x_{s+t} \left(\frac{M-1}{2}\right)^d (k+\mu+1)^d \\
& + \frac{M^k}{I(\alpha)} \sum_{i=1}^{d} \left(\frac{M-1}{2}\right)^{d-i} \binom{d}{i} (k+\mu+1)^{d-i} \sum_{\ell_1,\dots,\ell_i=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} L_{\ell_1} \cdots L_{\ell_i}\, d\lambda_d(\mathbf{z}) \\
& + \mathcal{O}\left(k^d M^{\frac{k(n-1)}{n}}\right)
\end{aligned} \tag{3.4}
$$

Now let us examine the integrals. Note that the integrand in (3.4) has the shape $L_{\ell_1}^{m_1} \cdots L_{\ell_j}^{m_j}$ with suitably chosen $j, m_1, \dots, m_j \in \mathbb{N}$ and thus the inner sum can be rewritten as

$$
\sum_{j=1}^{i} \sum_{m_1+\cdots+m_j=i} \sum_{\ell_1,\dots,\ell_j=-\mu}^{k} \int_{\mathbf{v}(\mathcal{D}(b,0,x))} L_{\ell_1}^{m_1} \cdots L_{\ell_j}^{m_j}\, d\lambda_d(\mathbf{z}), \tag{3.5}
$$

where the inner sum runs only over all j-tuples of pairwise non-equal numbers $\ell_1, \dots, \ell_j$.

Let $Q(m)$ be the expectation of L_ℓ^m (note, that $Q(m)$ does not depend on ℓ and is zero for $m \equiv 1(2)$). Then, the integral

$$\int_{\overline{\mathbf{v}(\zeta+b^{-\eta}\mathcal{F})}} (L_{\ell_1}^{m_1} - Q(m_1)) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}) = 0 \tag{3.6}$$

for $\eta = \max_r \ell_r - 1$ and for all $\zeta \in \mathcal{F}_\eta$, since the mean of the term with index $\eta+1$ is zero while all factors are constant. Hence,

$$\int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_1}^{m_1} - Q(m_1)) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}) = \mathcal{O}\left(M^{-\frac{\max_r \ell_r}{n}}\right), \tag{3.7}$$

because due to (3.6) the only contribution to the value of the integral comes from the fundamental regions which intersect the boundary of $\overline{\mathbf{v}(\mathcal{D}(b,0,x))}$.

Now we split the integrals in (3.5) in order to get a representation in terms of the form (3.7). One step of this splitting procedure is

$$\begin{aligned}\sum_{\ell_1,\ldots,\ell_j=-\mu}^{k} \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} L_{\ell_1}^{m_1} \cdots L_{\ell_j}^{m_j}\, d\lambda_d(z) = \\ \sum_{\ell_1,\ldots,\ell_j=-\mu}^{k} \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_1}^{m_1} - Q(m_1)) \cdots L_{\ell_j}^{m_j}\, d\lambda_d(\mathbf{z}) \\ + \sum_{\ell_1,\ldots,\ell_j=-\mu}^{k} Q(m_1) \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} L_{\ell_2}^{m_2} \cdots L_{\ell_j}^{m_j}\, d\lambda_d(\mathbf{z}).\end{aligned}$$

Continuing in this way yields expressions of the form

$$\sum_{\ell_1,\ldots,\ell_j=-\mu}^{k} Q(m_1)\cdots Q(m_a) \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_{a+1}}^{m_{a+1}} - Q(m_{a+1})) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}),$$

with $1 \le a \le j \le i$ (Note that for reasons of the symmetry inherent in the summation also the first summand splits into terms of the above form). These expressions are zero if any of the numbers $m_1, \ldots, m_a$ is odd. So we have only to consider those terms where $m_1, \ldots, m_a$ are all even and, in particular, not less than 2. Since $m_1 + \cdots + m_a \le i$ we conclude that $a \le \frac{i}{2}$. Note that the summands only depend on $\ell_{a+1}, \ldots, \ell_j$ and therefore we obtain

$$(\mu+k+1)^a Q(m_1)\cdots Q(m_a) \sum_{\ell_{a+1},\ldots,\ell_j=-\mu}^{k} \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_{a+1}}^{m_{a+1}} - Q(m_{a+1})) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}). \tag{3.8}$$

If we insert now (3.8) into (3.4), the corresponding term in (3.4) turns into

$$CM^k(\mu+k+1)^{a+d-i} \sum_{\ell_{a+1},\ldots,\ell_j=-\mu}^{k} \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_{a+1}}^{m_{a+1}} - Q(m_{a+1})) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}), \tag{3.9}$$

where $a + d - i \le d - (\left[\frac{i}{2}\right] + 1) < d$ ($[\zeta]$ denotes the largest integer not greater than ζ) and C is a constant that is explicitely computable. Now we show that the integral in this expression is essentially a periodic fluctuation. By (3.7), replacing the sum in (3.9) by

$$\Psi(x) = \sum_{\ell_{a+1},\ldots,\ell_j=-\mu}^{\infty} \int_{\overline{\mathbf{v}(\mathcal{D}(b,0,x))}} (L_{\ell_{a+1}}^{m_{a+1}} - Q(m_{a+1})) \cdots (L_{\ell_j}^{m_j} - Q(m_j))\, d\lambda_d(\mathbf{z}).$$

causes an error of order $\mathcal{O}\left(k^d M^{-\frac{k}{n}}\right)$. Following [19] set $y = x_1(x)\cdots x_{s+t}(x) = M^{\{\log_M N\}}$ ($\{w\}$ denotes the fractional part of w, i.e. $\{w\} = w - [w]$). Then $y = P(x)$ is a polynomial consisting of positive and strictly monotonic factors. Hence $P(x)$ is positive and strictly monotonic in $[1, b_1]$. So the inverse function $x = P^{-1}(y)$ exists. By the definition of y we have

$$P^{-1}(M^{\{\log_M N\}}) = x.$$

Note that $\log_M N = [\log_M N] + \{\log_M N\}$, and $k = [\log_M N]$, and define a new function

$$\delta(w) = M^{\{-w\}}\Psi(P^{-1}(M^{\{w\}}))$$

which is obviously a continuous periodic function with period 1. Applying

$$\begin{aligned}(\mu+k+1)^{a+d-i} &= (\log_M N - \{\log_M N\} + \mu + 1)^{a+d+1} \\ &= \sum_{r=0}^{a+d-i} \binom{a+d-i}{r} \log_M^r N(\mu + 1 - \{\log_M N\})^{a+d-i-r}\end{aligned} \tag{3.10}$$

on (3.9) yields

$$\begin{aligned}&CN \sum_{r=0}^{a+d-i} \binom{a+d-i}{r} \log_M^r N(\mu+1-\{\log_M N\})^{a+d-i-r}\delta(\log_M N) + \mathcal{O}\left(N^{-\frac{n-1}{n}} \log_M^d N\right) \\ &= N \sum_{r=0}^{a+d-i} \log_M^r N\delta_r(\log_M N) + \mathcal{O}\left(N^{-\frac{n-1}{n}} \log_M^d\right),\end{aligned}$$

where $\delta_r(x) = C\binom{a+d-i}{r}(\mu+1-\{x\})^{a+d-i-r}\delta(x)$ $(r = 0,\dots,a+d-i)$. Noting that there are only finitely many summands of this kind we conclude that the contribution to $S_d(b,k,x)$, coming from the terms in the second line of (3.4) has the form

$$N\sum_{j=0}^{d-1} \log_M^j N\tilde{\Phi}_j(\log_M N),$$

where the $\tilde{\Phi}_j$ are finite sums of periodic fluctuations of period 1 and hence periodic fluctuations of period 1, too. It remains the investigation of the term corresponding to $i = 0$ in (3.4). Applying again (3.10) we get

$$\begin{aligned}&\frac{M^k}{I(\alpha)} 2^s\pi^t x_1 \cdots x_{x+t} \left(\frac{M-1}{2}\right)^d (k+\mu+1)^d \\ &= \frac{N}{I(\alpha)} 2^s \pi^t \left(\frac{M-1}{2}\right)^d \log_M^d N + N\sum_{j=0}^{d-1} \log_M^j N\bar{\Phi}_j(\log_M N)\end{aligned}$$

where $\bar{\Phi}_j$ are periodic fluctuations of period 1. Setting $\Phi_j(x) = \tilde{\Phi}_j(x)+\bar{\Phi}_j(x)$ for $j = 0,\dots,d-1$ we derive

$$\begin{aligned}S_d(N) =& S_d(b,k,x) = \sum_{\substack{|z|^2<N \\ z\in\mathbf{Z_K}}} (\nu_b(z))^d \\ =& \frac{2^s\pi^t}{I(\alpha)} \left(\frac{M-1}{2}\right)^d N\log_M^d N + \sum_{j=0}^{d-1} N\log_M^j N\Phi_j(\log_M N) + \mathcal{O}\left(N^{\frac{n-1}{n}} \log_M^d N\right)\end{aligned}$$

and the theorem is proved.

References

[1] S. I. Borevics and I. R. Šafarevič. *Zahlentheorie.* Birkhäuser Verlag, Basel, 1966.
[2] H. Delange. Sur la fonction sommatoire de la fonction "somme des chiffres". *Enseign. Math.*, 21(2):31–47, 1975.
[3] W. J. Gilbert. Fractal geometry derived from complex bases. *Math. Intell.*, 4:78–86, 1982.
[4] W. J. Gilbert. *Geometry of Radix Representation. The Geometric Vein. The Coxeter Festschrift. (D. Chandler, B. Grünbaum, F. A. Sharks, eds.).* Springer, Berlin, 1982.

[5] W. J. Gilbert. The fractal dimension of sets derived from complex bases. *Can. Math. Bull.*, 29:495–500, 1986.

[6] W. J. Gilbert. Complex bases and fractal similarity. *Ann. Sci. Math. Québec*, 11:65–77, 1987.

[7] P. J. Grabner, P. Kirschenhofer, and H. Prodinger. The sum-of-digits-function for complex bases. *J. London Math. Soc.* to appear.

[8] P. J. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy. On the moments of the sum-of-digits function. *Applications of Fibbonacci Numbers. G. E. Bergum, A. N. Philippou and A. F. Horadam, eds*, 5:263–273, 1993.

[9] I. Kátai. Number systems and fractal geometry. *preprint*, 1996.

[10] I. Kátai and I. Kőrnyei. On number systems in algebraic number fields. *Publ. Math. Debrecen*, 41(3–4):289–294, 1992.

[11] I. Kátai and B. Kovács. Kanonische Zahlensysteme in der Theorie der Quadratischen Zahlen. *Acta Sci. Math. (Szeged)*, 42:99–107, 1980.

[12] I. Kátai and B. Kovács. Canonical number systems in imaginary quadratic fields. *Acta Math. Hungar.*, 37:159–164, 1981.

[13] I. Kátai and J. Szabó. Canonical number systems for complex integers. *Acta Sci. Math. (Szeged)*, 37:255–260, 1975.

[14] R. E. Kennedy and C. N. Cooper. An extension of a theorem by Cheo and Yien concerning digital sums. *Fibonacci Quaterly*, 29:145–149, 1991.

[15] P. Kirschenhofer. On the variance of the sum of digits function. *Number Theoretic Analysis, Lecture Notes in Mathematics (E. Hlawka and R. F. Tichy, eds)*, 1452:112–116, 1990.

[16] B. Kovács. Canonical number systems in algebraic number fields. *Acta Math. Hungar.*, 37:405–407, 1981.

[17] B. Kovács and A. Pethő. Number systems in integral domains, especially in orders of algebraic number fields. *Acta Sci. Math. (Szeged)*, 55:286–299, 1991.

[18] B. Kovács and A. Pethő. On a representation of algebraic integers. *Studia Sci. Math. Hungar.*, 27:169–172, 1992.

[19] J. M. Thuswaldner. The sum of digits function in number fields. *Bull. London Math. Soc.*, to appear.

Bernhard Gittenberger, Department of Algebra and Discrete Mathematics, Technische Universität Wien, Wiedner Hauptstrasse 8-10/118, A-1040 Wien, Austria.

Jörg M. Thuswaldner, Department of Mathematics and Statistics, Montanuniversität Leoben, Franz-Josef-Strasse 18, A-8700 Leoben, Austria.