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Main Theorem on Planar Parallel Manipulators with Cylindrical Singularity Surface

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Abstract: In this article we prove that there do not exist non-architecturally singular Stewart Gough Platforms with planar base and platform and no four anchor points collinear, whose singularity set for any orientation of the platform is a cylindrical surface with rulings parallel to a given fixed direction p in the space of translations.

Key Words: *Stewart Gough Platform, planar parallel manipulator, cylindrical singularity surface, architecture singular manipulators*

1 Introduction

The geometry of the parallel manipulator is given by the six base anchor points $\mathbf{M}_i := (A_i, B_i, C_i)^T$ in the fixed space and by the six platform anchor points $\mathbf{m}_i := (a_i, b_i, c_i)^T$ in the moving space. By using Euler Parameters (e_0, e_1, e_2, e_3) for the parametrization of the spherical motion group the coordinates $\mathbf{m}_i'$ of the platform anchor points with respect to the fixed space can be written as $\mathbf{m}_i' = K^{-1}\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t}$ with

$$\mathbf{R} := (r_{ij}) = \begin{bmatrix} e_0^2+e_1^2-e_2^2-e_3^2 & 2(e_1e_2+e_0e_3) & 2(e_1e_3-e_0e_2) \\ 2(e_1e_2-e_0e_3) & e_0^2-e_1^2+e_2^2-e_3^2 & 2(e_2e_3+e_0e_1) \\ 2(e_1e_3+e_0e_2) & 2(e_2e_3-e_0e_1) & e_0^2-e_1^2-e_2^2+e_3^2 \end{bmatrix}, \tag{1}$$

the translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and $K := e_0^2 + e_1^2 + e_2^2 + e_3^2$. Moreover it should be noted that K is used as homogenizing factor whenever it is suitable.

It is well known (see e.g. [4]) that the set of singular configurations is given by $Q := det(\mathbf{Q}) = 0$, where the i^{th} row of the 6×6 matrix $\mathbf{Q}$ equals the Plücker coordinates $(\mathbf{l}_i, \widehat{\mathbf{l}}_i) := (\mathbf{R}\cdot\mathbf{m}_i + \mathbf{t} - K\mathbf{M}_i, \mathbf{M}_i \times \mathbf{l}_i)$ of the carrier line of the i^{th} leg.

As we consider only manipulators with planar platform we may suppose $C_i = c_i = 0$ for $i = 1, \ldots, 6$. Moreover it was proven by Karger in [2] that for planar parallel manipulators with no four points on a line we can assume $A_1 = B_1 = B_2 = a_1 = b_1 = b_2 = 0$ and $A_2B_3B_4B_5a_2(a_4 - a_3)coll(3, 4, 5) \neq 0$ with

$$coll(i, j, k) := a_i(b_j - b_k) + a_j(b_k - b_i) + a_k(b_i - b_j). \tag{2}$$

$coll(i, j, k) = 0$ characterizes collinear platform anchor points $\mathbf{m}_i, \mathbf{m}_j$ and $\mathbf{m}_k$.

2 Preliminary Considerations

The set of Stewart Gough Platforms whose singularity set for any orientation is a cylindrical surface with rulings parallel to a given direction p also contains the set of architecturally singular manipulators. This is due to the fact that the singularity surface of these manipulators equals the whole space of translations for any orientation.

It can easily be seen from the following example that the above two sets are distinct:

The non-planar manipulator determined by $\mathbf{m}_1 = \mathbf{m}_2$, $\mathbf{m}_3 = \mathbf{m}_4$, $\mathbf{m}_5 = \mathbf{m}_6$ and $\overline{\mathbf{M}_1\mathbf{M}_2} \parallel \overline{\mathbf{M}_3\mathbf{M}_4} \parallel \overline{\mathbf{M}_5\mathbf{M}_6} \parallel p$ has for any orientation of the platform a cylindrical surface with rulings parallel to the direction p without being

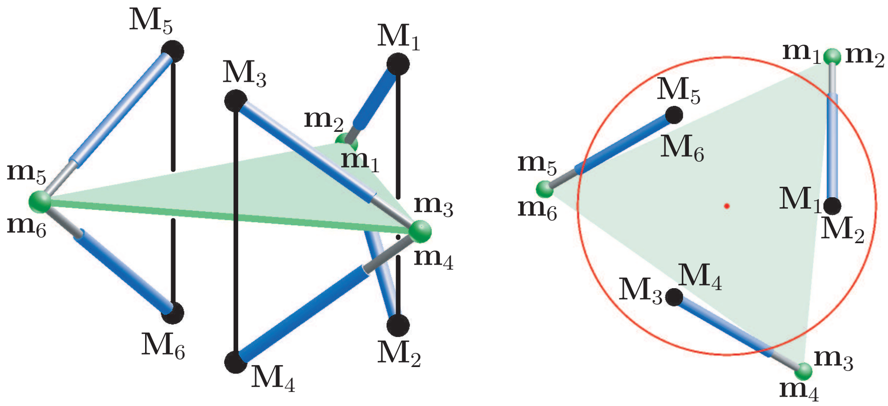

Figure 1: Non-planar manipulator with cylindrical singularity surface: (a) Axonometric view. (b) Projection in direction p: The singularity surface (with respect to the barycenter of the platform) is displayed as conic.

architecturally singular (see Fig. 1). This manipulator is in a singular configuration if and only if the three planes $[\mathbf{M}_1, \mathbf{M}_2, \mathbf{m}_1]$, $[\mathbf{M}_3, \mathbf{M}_4, \mathbf{m}_3]$ and $[\mathbf{M}_5, \mathbf{M}_6, \mathbf{m}_5]$ have a common intersection line.

As the direct kinematics of this manipulator can be put down to that of a 3-dof RPR parallel manipulator, a rational parametrization of its singularity surface according to [1] can be given. The singularity surface is a quadratic cylinder due to the singular affine correspondence between the base and the platform (cf. [3]).

Moreover, if $\mathbf{M}_1, \ldots, \mathbf{M}_6$ are coplanar we get an example for a planar parallel manipulator with this property. Now the question arises, whether there exist non-architecturally singular planar manipulators with no four anchor points on a line possessing such a singularity surface. In the following section we prove that such manipulators do not exist.

3 The Main Theorem and its Proof

Theorem *The set of planar parallel manipulators with no four anchor points on a line which posses a cylindrical singularity surface with rulings parallel to a given fixed direction p for any orientation of the platform equals the set of planar architecture singular manipulators (with no four anchor points on* *a line).*

The analytical proof of this theorem is based on the following idea: We choose a Cartesian frame in the base such that one axis t_i is parallel to the given direction p. Then $Q = 0$ must be independent of t_i for all $e_0, \ldots, e_3, t_j, t_k$ with $j \neq k \neq i \neq j$. Our proof is based on the resulting equations and Theorem 1 of [2].

We have to distinguish between two cases, depending on whether the base of the manipulator is parallel to p or not.

Base is not parallel to p:

The proof of the case where the base is orthogonal to p is hidden in the proof of Theorem 1 of [2]. For all other cases the proof was given by the author in [5].

Base is parallel to p:

In this case we take as translation vector $\mathbf{t} := (\cos\phi t_1 - \sin\phi t_2, \sin\phi t_1 + \cos\phi t_2, t_3)^T$. After performing the same elementary operations with the matrix $\mathbf{Q}$ as described on page 1154 of [2], we can replace the sixth row of $\mathbf{Q}$ by

$$\begin{aligned}(r_{11}K_1 + r_{12}A_2K_2, r_{21}K_1 + r_{22}A_2K_2, r_{31}K_1 + r_{32}A_2K_2,\\ 0, r_{31}A_2K_3 + r_{32}A_2K_4, -r_{21}A_2K_3 - r_{22}A_2K_4)D^{-1}\end{aligned} \tag{3}$$

with $D := A_2B_3B_4B_5coll(3,4,5)$. $K_1 = K_2 = K_3 = K_4 = 0$ are the four conditions given in [2] which are satisfied iff a planar manipulator (with no four points on a line) is architecturally singular. We distinguish between the following two cases:

- $\mathbf{M}_1\mathbf{M}_2$ **is parallel to p.** The proof of this part can be done by considering only the four equations (12-15) given in [5].
- $\mathbf{M}_1\mathbf{M}_2$ **is not parallel to p.** This part of the proof is the primary concern of this paper, because it was to long to be given in [5]. In the cited paper only the two solutions were given which fulfill all equations resulting from the coefficients of t_1 of Q without contradicting

$$A_2B_3B_4B_5a_2(a_4 - a_3)coll(3,4,5)K_2\sin\phi \neq 0. \tag{4}$$

These two solutions S_1 and S_2 are

$$S_1: \quad A_i = B_i \cot\phi,\ A_j = B_j \cot\phi,\ A_k = A_2 + B_k \cot\phi, \tag{5}$$
$$b_k = 0,\ a_2 = a_k,\ a_i = K_1 b_i/(K_2 A_2),\ a_j = K_1 b_j/(K_2 A_2), \tag{6}$$
$$K_3 = 0 \quad \text{and} \quad K_4 = 0 \tag{7}$$

$$S_2: \quad A_i = A_2 + B_i \cot\phi,\ A_j = A_2 + B_j \cot\phi,\ A_k = B_k \cot\phi, \tag{8}$$
$$a_i = a_2 + b_i K_3/K_4,\ a_j = a_2 + b_j K_3/K_4,\ a_k = b_k = 0, \tag{9}$$
$$A_2 K_2 + K_4 = 0 \quad \text{and} \quad K_1 + K_3 = 0 \tag{10}$$

for $i, j, k \in \{3, 4, 5\}$ and $i \neq j \neq k \neq i$. In the following we proof that for $K_2 \neq 0$ all coefficients of t_1 can only vanish for the above two given solutions. Moreover we prove that for $K_2 = 0$ all coefficients of t_1 only vanish for architecturally singular manipulators, i.e., $K_1 = K_3 = K_4 = 0$. The proof is split into the following two cases given in subsection 3.1 and 3.2, respectively.

1 $\mathbf{M}_1\mathbf{M}_2$ is orthogonal to p

We denote the coefficients of $t_1^i t_2^j t_3^k$ from Q by $Q^{i,j,k}$. From all $Q^{i,j,k}$ with $i > 0$ we can factor out K. From $Q^{1,0,0}$ we can even factor out K^2.

For this case we set $\phi = \pi/2$ and eliminate t_1 from Q. Now we can additionally factor out $(e_0 e_1 - e_2 e_3)$ of $Q^{2,j,k}$. We denote the coefficient of $e_0^a e_1^b e_2^c e_3^d$ of $Q^{i,j,k}$ by $P^{i,j,k}_{a,b,c,d}$ and compute the following 15 polynomials:

$$P_1[18] := P^{1,1,1}_{2,2,0,0} \quad P_2[42] := P^{2,0,1}_{1,0,1,0} \quad P_3[12] := P^{1,1,0}_{3,3,0,0} \tag{11}$$
$$P_4[42] := P^{2,0,0}_{2,0,2,0} \qquad P_5[72] := P^{2,0,0}_{2,1,1,0} \tag{12}$$

$$P_6[36] := P^{1,0,0}_{3,2,1,0} - P^{1,0,0}_{2,3,0,1} - P^{1,0,0}_{1,0,3,2} + P^{1,0,0}_{0,1,2,3} \tag{13}$$
$$P_7[42] := P^{1,0,1}_{4,1,1,0} - P^{1,0,1}_{1,4,0,1} - P^{1,0,1}_{1,0,4,1} + P^{1,0,1}_{0,1,1,4} \tag{14}$$
$$P_8[30] := P^{1,0,1}_{4,2,0,0} + P^{1,0,1}_{2,4,0,0} + P^{1,0,1}_{0,0,4,2} + P^{1,0,1}_{0,0,2,4} \tag{15}$$
$$P_9[30] := P^{1,0,1}_{4,2,0,0} + P^{1,0,1}_{2,4,0,0} - P^{1,0,1}_{0,0,4,2} - P^{1,0,1}_{0,0,2,4} \tag{16}$$

$$P_{10}[18] := P^{1,1,1}_{2,1,1,0} - P^{1,1,1}_{1,2,0,1} \qquad P_{11}[42] := P^{1,0,1}_{3,1,1,1} + P^{1,0,1}_{1,3,1,1} \tag{17}$$
$$P_{12}[36] := P^{1,1,0}_{3,2,1,0} - P^{1,1,0}_{2,3,0,1} \qquad P_{13}[24] := P^{1,1,0}_{3,1,2,0} - P^{1,1,0}_{2,0,3,1} \tag{18}$$
$$P_{14}[12] := P^{1,0,0}_{3,3,0,0} + P^{1,0,0}_{0,0,3,3} \qquad P_{15}[24] := P^{1,0,0}_{2,1,2,1} - P^{1,0,0}_{1,2,1,2} \tag{19}$$

It should be noted that the number in the square brackets denotes the number of terms in the expression. In the first step we compute the resultant of P_3 and P_{14} with respect to A_3 which yields

$$a_2^2 b_3 b_4 b_5 B_3 B_4 B_5 K_2 coll(3,4,5)[K_2(A_4B_5 - A_5B_4) + K_4(B_5 - B_4)]. \tag{20}$$

Therefore we have to distinguish between the following three cases:

Case I) $K_2 = 0$

We set K_2 equal to zero and compute P_1 and P_{10} which factor into $K_1F_1[6]$ and $K_1F_{10}[6]$, respectively.

Part [A] $K_1 \neq 0$: The resultant of F_1 and F_{10} with respect to B_3 yields

$$b_3 B_4 B_5 coll(3,4,5)(b_4B_5 - b_5B_4). \tag{21}$$

(i) $b_3 = 0$ implies $b_4b_5 \neq 0$. From $P_1 = 0$ we get $B_4 = B_5$. Substituting this into P_{10} yields $K_1B_3B_5coll(3,4,5)$ and therefore a contradiction.

(ii) So we set $b_5 = b_4B_5/B_4$ and plug this into P_1 and P_{10} which yields:

$$K_1B_5b_4\,(B_4 - B_5)\,(B_4b_3 - B_3b_4) \quad \text{and} \quad K_1B_5\,(a_4 - a_5)\,(B_4b_3 - B_3b_4)\,. \tag{22}$$

If we set $a_4 = a_5$ and $B_4 = B_5$ we get $b_4 = b_5$ and thus $coll(3,4,5) = 0$, a contradiction. For $b_4 = B_4b_3/B_3$ the polynomial P_{14} factors into $a_2b_3B_4B_5K_4coll(3,4,5)$ which implies $K_4 = 0$. Now P_8, which splits into $b_3B_4B_5K_1coll(3,4,5)$, yields a contradiction.

Part [B] $K_1 = 0$: We compute

$$P_3 = a_2K_4F_1, \quad P_8 = a_2K_4F_{10}, \quad P_{11} = a_2K_3F_1, \quad P_{13} = a_2K_3F_{10}, \tag{23}$$

which implies that $K_3 = K_4 = 0$ or $F_1 = F_{10} = 0$ must hold. We assume $K_3 \neq 0$ and $K_4 \neq 0$ and consider again the resultant of F_1 and F_{10} with respect to B_3 given in Eq. (21).

(i) For $b_3 = 0$ we get $B_4 = B_5$ from $P_3 = 0$. Substituting this into P_8 and P_{13} yields $a_2B_3B_5K_4coll(3,4,5)$ and $a_2B_3B_5K_3coll(3,4,5)$.

(ii) If we plug $b_5 = b_4B_5/B_4$ into $P_3 = 0$, $P_8 = 0$, $P_{11} = 0$ and $P_{13} = 0$ we see that these equations can only vanish for $b_4 = B_4b_3/B_3$ or $a_4 = a_5$ and $B_4 = B_5$. The later contradicts again $coll(3,4,5) \neq 0$. Therefore we set $b_4 = B_4b_3/B_3$ and substitute this into P_{14} and P_{15} which yields $a_2b_3B_3B_5K_4coll(3,4,5)$ and $a_2b_3B_3B_5K_3coll(3,4,5)$, respectively.

Case II) $b_i = 0$**,** $K_2 \neq 0$

Without loss of generality we can say $b_3 = 0$, which implies $b_4b_5 \neq 0$. Now P_3 factors into $a_2B_3b_4b_5[K_2(A_4B_5 - A_5B_4) + K_4(B_5 - B_4)]$. From the last factor we compute A_5. Now the resultant of P_8 and P_1 with respect to A_3 yields $K_2a_2B_3B_4B_5coll(3,4,5)R_1$ with

$$R_1 := K_2A_2(a_4b_5B_5 - a_5b_4B_4) + K_1b_4b_5(B_4 - B_5) + K_4a_2(b_5B_5 - b_4B_4). \quad (24)$$

From $R_1 = 0$ we compute a_5. Then P_1 simplifies to

$$B_5\,(b_4 - b_5)\,[K_2B_4(a_2A_3 - a_3A_2) - a_2B_3(K_4 + K_2A_4)]\,. \quad (25)$$

If $b_4 = b_5$ the equation $P_9 = 0$ can only vanish (w.c.) for $A_4 = -K_4/K_2$. Now $P_8 = 0$ implies $a_3 = a_2A_3/A_2$ and $P_{10} = 0$ yields a contradiction. Therefore we set $a_3 = a_2[K_2(A_3B_4 - A_4B_3) - K_4B_3]/(K_2A_2B_4)$. Now $P_{10} = 0$ can only vanish (w.c.) for $K_2A_2a_4 - K_1b_4 + K_4a_2 = 0$. From this equation we compute a_4. $P_9 = 0$ implies $A_4 = -K_4/K_2$. Then P_7 factors into $a_2B_3(K_2A_3 + K_4)F_7[8]/(K_2^2A_2)$. As $K_2A_3 + K_4 = 0$ yield $coll(3,4,5) = 0$ we set F_7 equal to zero:

Part [A] $K_1K_4 - K_2K_3A_2 \neq 0$: Under this assumption we can compute b_4 from $F_7 = 0$. $P_{12} = 0$ splits into several factor, where only one does not lead to a direct contradiction. From this factor we compute

$$b_5 = a_2[K_2^2A_3B_5(A_2 - A_3) + K_4B_3(K_2A_2 + K_4)]/[B_3(K_1K_4 - K_2K_3A_2)]. \quad (26)$$

Finally $P_6 = 0$ yields a contradiction.

Part [B] $K_1K_4 - K_2K_3A_2 = 0$:

(i) Assuming $K_3 \neq 0$ we can compute A_2. Now F_7 factors into $a_2K_4^2(B_5 - B_4)(K_1 + K_3)/K_3$.

- Firstly we consider the case $K_1 = -K_3$. Now $P_{12} = 0$ implies $A_3 = 0$ and we get solution S_2 for $\cos\phi = 0$ and $k = 3$.

- $B_4 = B_5, K_1 + K_3 \neq 0$: From $P_{12} = 0$ we compute B_3 as

$$B_3 = K_2 A_3 B_4 (K_2 K_3 A_3 - K_1 K_4)/(K_4^2 (K_1 + K_3)). \tag{27}$$

  Plugging this into $P_6 = 0$ yields the contradiction.

(ii) Assuming $K_3 = 0$ yields $K_1 K_4 = 0$.

- We start with $K_4 = 0$. $P_{12} = 0$ can only vanish (w.c.) for $A_2 = A_3$ which yields solution S_1 for $\cos\phi = 0$ and $k = 3$.

- $K_1 = 0, K_4 \neq 0$: We compute P_{13} which factors into

$$a_2^3 K_4 B_3 (K_2 A_2 + K_4)(K_2 A_3 + K_4)(B_4 - B_5)/(K_2^2 A_2^2). \tag{28}$$

  $K_2 A_3 + K_4 = 0$ contradicts $coll(3,4,5) \neq 0$.

  (α) For $A_2 = -K_4/K_2$ we get from $P_{12} = 0$ the condition $A_3 = 0$. We get solution S_2 for $\cos\phi = 0$ and $k = 3$ with the additional condition $K_1 = 0$.

  (β) $B_4 = B_5, K_4 + K_2 A_2 \neq 0$: Now $P_6 = 0$ can only vanish (w.c.) for $A_3 = (K_2 A_2 (B_5 - B_3) - K_4 B_3)/(B_5 K_2)$ or $A_3 = 0$. For both cases we get a contradiction from $P_{12} = 0$.

Case III) $K_2(A_4 B_5 - A_5 B_4) + K_4(B_5 - B_4) = 0$**,** $b_3 b_4 b_5 K_2 \neq 0$

From the above condition and $P_3 = 0$ we compute A_3 and A_4 as

$$A_i = [K_2 B_i A_5 + K_4(B_i - B_5)]/(K_2 B_5) \quad \text{for} \quad i = 3, 4. \tag{29}$$

In the next step we calculate the resultant of P_1 and P_{10} with respect to a_2 which yields $B_3 B_4 B_5 coll(3,4,5) K_4 R_2$[12].

Part [A] $K_4 = 0$: Now P_9 equals $K_2 A_2 B_3 B_4 A_5 coll(3,4,5)$ which implies $A_5 = 0$. Then P_2 simplifies to $A_2 K_3 F_2$ with

$$F_2 := B_3(a_5 b_4 - a_4 b_5) + B_4(a_3 b_5 - a_5 b_3) + B_5(a_4 b_3 - a_3 b_4). \tag{30}$$

(i) $F_2 = 0$:

- If we assume $b_4B_5 - b_5B_4 \neq 0$ we can compute a_3 from F_2. Now the polynomials P_4 and P_5 factors into

$$K_3A_2(a_4b_5 - a_5b_4)F_4[8] \quad \text{and} \quad K_3A_2(a_4b_5 - a_5b_4)F_5[6], \tag{31}$$

respectively. The factor $a_4b_5 - a_5b_4 = 0$ implies $coll(3,4,5) = 0$.

(α) Therefore we assume $K_3 \neq 0$ and compute the resultant of F_4 and F_5 with respect to B_3, which yields

$$b_3B_4B_5(B_4 - B_5)(b_3 - b_4)(b_4B_5 - b_5B_4)(b_5 - b_3)(a_4b_5 - a_5b_4). \tag{32}$$

For the cases $B_4 = B_5$ or $b_3 = b_i$ for $i = 4, 5$ equation $F_5 = 0$ yields a contradiction. The last factor of Eq. (32) implies $coll(3,4,5) = 0$.

(β) $K_3 = 0$: Now the P_1 and P_{10} factors into F_5C and F_4C with

$$C := K_2A_2(a_4B_5 - a_5B_4) - K_1(b_4B_5 - b_5B_4). \tag{33}$$

For $C = 0$ we compute a_4 from this equation and plug the obtained expression into $P_8 = 0$, which already yields a contradiction. Therefore we consider again the resultant of F_4 and F_5 with respect to B_3 given in Eq. (32). For all possible cases ($B_4 = B_5$ or $b_3 = b_i$ for $i = 4, 5$) the equation $P_1 = 0$ can only vanish (w.c.) for $C = 0$.

- We proceed with $b_4 = B_4b_5/B_5$. Now the polynomial F_2 equals

$$(a_4B_5 - a_5B_4)(b_3B_5 - b_5B_3)/B_5. \tag{34}$$

(α) $a_4 = B_4a_5/B_5$: $P_5 = 0$ can only vanish (w.c.) for $K_3 = 0$. Then $P_8 = 0$ implies $a_5 = K_1b_5/(K_2A_2)$. $P_1 = 0$ yields a contradiction.

(β) $b_3 = b_5B_3/B_5$: Now we consider $P_1 = A_2K_2b_5F_1[6]/B_5$ and $P_{10} = K_2A_2F_{10}[6]$. The resultant of F_1 and F_{10} with respect to a_3 yields

$$B_3(B_3 - B_4)(B_3 - B_5)(a_4B_5 - a_5B_4). \tag{35}$$

For $B_3 = B_i$ for $i = 4, 5$ the equation $P_1 = 0$ yields the contradiction. If we set $a_4 = B_4a_5/B_5$ the equation $P_5 = 0$ implies $K_3 = 0$. Now P_8 equals $B_3B_4coll(3,4,5)(K_1b_5 - a_5K_2A_2)$. From the last factor we compute a_5 and plug this into $P_1 = 0$ which yields the contradiction.

(ii) $K_3 = 0, F_2 \neq 0$: Computing the resultant of P_1 and P_{10} with respect to A_2 yields $K_1K_2B_3B_4B_5coll(3,4,5)F_2$. This implies $K_1 = 0$.

- Assuming $b_4 \neq b_5$ we can compute a_3 from $P_1 = 0$. Now P_{10} splits into $K_2A_2B_3(a_4B_5 - a_5B_4)coll(3,4,5)/(b_4 - b_5)$. Plugging $a_4 = B_4a_5/B_5$ into $P_8 = 0$ yields a contradiction.

- $b_4 = b_5$: Now P_1 factors into $K_2A_2B_3(b_3 - b_5)(a_4B_5 - a_5B_4)$. As $b_3 = b_5$ contradicts $coll(3,4,5) \neq 0$ we set $a_4 = B_4a_5/B_5$. $P_{10} = 0$ can only vanish (w.c.) for $a_3 = B_3a_5/B_5$. Again $P_8 = 0$ yields a contradiction.

Part [B] $R_2 = 0, K_4 \neq 0$:

(i) If we assume $B_4 \neq B_5$ we can compute a_3 from $R_2 = 0$. Now P_1 splits up into the two factors C and F_1 with

$$C := K_4a_2(B_4 - B_5) + K_1(b_4B_5 - b_5B_4) + K_2A_2(a_5B_4 - a_4B_5) \tag{36}$$
$$F_1 := b_3B_3(B_4 - B_5) + b_4B_4(B_5 - B_3) + b_5B_5(B_3 - B_4). \tag{37}$$

If we compute a_2 from $C = 0$, the resulting equation $P_8 = 0$ cannot vanish without contradiction. If we compute b_3 from $F_1 = 0$ the polynomial P_{10} splits into 4 factors. Three of them yield $coll(3,4,5) = 0$. The fourth factor equals C.

(ii) We get $R_2 = (B_3 - B_5)[K_2A_2(a_5 - a_4) + K_1(b_4 - b_5)]$ for the remaining case $B_4 = B_5$. If $B_3 = B_5$ the equation $P_8 = 0$ yields a contradiction. Therefore we compute a_4 from the second factor. Now P_{15} factors into $a_2B_5coll(3,4,5)(B_3 - B_5)(K_1b_5 - K_2A_2a_5)$. This implies $a_5 = K_1b_5/(K_2A_2)$ and finally $P_8 = 0$ yields the contradiction. $\square$

2 $\mathbf{M_1M_2}$ is not orthogonal to p

Due to the above studied cases we can assume $\cos\phi \neq 0$ and $\sin\phi \neq 0$ when eliminating t_1 from Q. For the proof of this part we need the following 20 polynomials:

$$P_1[12] := (P^{1,0,0}_{3,3,0,0} + P^{1,0,0}_{0,0,3,3})/(a_2 \sin\phi) \qquad P_3[78] := P^{1,0,1}_{0,4,0,2} - P^{1,0,1}_{2,0,4,0} \quad (38)$$
$$P_2[36] := (P^{1,0,0}_{3,0,3,0} - P^{1,0,0}_{0,3,0,3})/(a_2 \cos\phi) \qquad P_4[66] := P^{1,0,1}_{4,0,2,0} - P^{1,0,1}_{2,0,4,0} \quad (39)$$
$$P_5[30] := (P^{1,0,1}_{4,2,0,0} + P^{1,0,1}_{0,0,4,2})/\sin\phi \qquad P_7[36] := P^{1,0,1}_{4,2,0,0} - P^{1,0,1}_{0,0,2,4} \quad (40)$$
$$P_6[66] := (P^{1,0,1}_{4,0,2,0} + P^{1,0,1}_{0,4,0,2})/\cos\phi \qquad P_8[42] := P^{1,0,1}_{0,0,4,2} - P^{1,0,1}_{0,0,2,4} \quad (41)$$
$$P_9[18] := (P^{1,0,2}_{3,1,0,0} + P^{1,0,2}_{1,3,0,0})/\sin\phi \qquad P_{11}[108] := P^{1,0,1}_{3,1,1,1} - P^{1,0,1}_{1,3,1,1} \quad (42)$$
$$P_{10}[18] := (P^{1,0,2}_{2,0,1,1} - P^{1,0,2}_{1,1,2,0})/\cos\phi \qquad P_{12}[102] := P^{1,0,1}_{3,1,1,0} - P^{1,0,1}_{1,4,0,1} \quad (43)$$

$$P_{13}[24] := (P^{1,0,0}_{3,2,1,0} - P^{1,0,0}_{0,1,2,3} - P^{1,0,0}_{2,3,0,1} + P^{1,0,0}_{1,0,3,2})/(a_2 \sin\phi) \quad (44)$$
$$P_{14}[42] := (P^{1,0,0}_{3,2,1,0} + P^{1,0,0}_{0,1,2,3} + P^{1,0,0}_{2,3,0,1} + P^{1,0,0}_{1,0,3,2})/A_2 \quad (45)$$
$$P_{15}[48] := P^{1,0,0}_{3,2,1,0} + P^{1,0,0}_{0,1,2,3} - P^{1,0,0}_{2,3,0,1} - P^{1,0,0}_{1,0,3,2} \quad (46)$$
$$P_{16}[36] := P^{1,0,0}_{3,1,2,0} - P^{1,0,0}_{0,2,1,3} - P^{1,0,0}_{2,0,3,1} + P^{1,0,0}_{1,3,0,2} \quad (47)$$
$$P_{17}[66] := P^{1,0,1}_{4,1,1,0} + P^{1,0,1}_{1,4,0,1} + P^{1,0,1}_{1,0,4,1} + P^{1,0,1}_{0,1,1,4} \quad (48)$$
$$P_{18}[54] := P^{1,0,1}_{4,1,1,0} - P^{1,0,1}_{1,4,0,1} + P^{1,0,1}_{1,0,4,1} - P^{1,0,1}_{0,1,1,4} \quad (49)$$
$$P_{19}[48] := P^{1,0,1}_{3,2,0,1} - P^{1,0,1}_{2,3,1,0} - P^{1,0,1}_{0,1,3,2} + P^{1,0,1}_{1,0,2,3} \quad (50)$$
$$P_{20}[150] := P^{1,0,1}_{3,2,0,1} + P^{1,0,1}_{2,3,1,0} - P^{1,0,1}_{0,1,3,2} - P^{1,0,1}_{1,0,2,3} \quad (51)$$

Firstly we compute the resultant of P_9 and P_{10} with respect to A_5, which yields the expression $a_2B_3B_4B_5K_2coll(3,4,5)R_1$ with

$$R_1 := K_2B_3(A_2a_4 - A_4a_2) + K_2B_4(A_3a_2 - A_2a_3) + K_1(b_3B_4 - b_4B_3). \quad (52)$$

In the following section we show that for $K_2 = 0$ the equations $P_i = 0$ for $i = 1, \ldots, 20$ can only be fulfilled for $K_1 = K_3 = K_4 = 0$.

$\boldsymbol{K_2 = 0}$

Now the polynomials P_9 and P_{10} factors into K_1F_9 and K_1F_{10} with

$$F_9 := B_3B_4b_5(a_4 - a_3) + B_4B_5b_3(a_5 - a_4) + B_3B_5b_4(a_3 - a_5), \quad (53)$$
$$F_{10} := B_3B_4b_5(b_4 - b_3) + B_4B_5b_3(b_5 - b_4) + B_3B_5b_4(b_3 - b_5). \quad (54)$$

(i) $K_1 \neq 0$: We compute the resultant of F_9 and F_{10} with respect to B_3 which yields $b_3B_4B_5(b_4B_5 - b_5B_4)coll(3,4,5)$. We start with $b_3 = 0$. Now

$F_{10} = 0$ implies $B_4 = B_5$ and F_9 equals $K_1 B_3 B_5 coll(3,4,5)$. Therefore we set $b_4 = B_4 b_5/B_5$. Now F_9 splits up into $B_4(a_4 - a_5)(b_3 B_5 - b_5 B_3)$. If we set $b_3 = B_3 b_5/B_5$ the equation $P_5 = 0$ yields the contradiction. For $a_4 = a_5$ the equation $F_{10} = 0$ implies $b_3 = B_3 b_5/B_5$, which yields via $P_5 = 0$ the contradiction.

(ii) $K_1 = 0, K_4 \neq 0$: Now P_5 equals $K_4 a_2 F_9$. Moreover from P_1 also K_4 factors out. We compute the resultant of F_9 and $F_1 := P_1/K_4$ with respect to B_3 which yields

$$b_3 B_4 B_5 coll(3,4,5)(a_4 b_5 B_4 - a_5 b_4 B_5). \tag{55}$$

- For $b_3 = 0$ we get $P_1 = K_4 a_3 b_4 b_5 (B_4 - B_5)$.

  (α) For $a_3 = 0$ the equation $P_5 = 0$ implies $b_5 = b_4 a_5 B_5/(a_4 B_4)$. Now $P_{14} = 0$ and $P_8 = 0$ yield $A_3 = B_3 = 0$, a contradiction.

  (β) For $B_4 = B_5$ we get the contradiction from $P_5 = 0$.

- Now we set $a_4 b_5 B_4 - a_5 b_4 B_5 = 0$.

  (α) We assume $b_i = 0$ which yields $a_i = 0$ for $i, j \in \{4,5\}$ and $i \neq j$. Then equation $P_5 = 0$ can only vanish (w.c.) for $a_3 = a_j b_3 B_4/(b_j B_3)$. The equations $P_{14} = 0$ and $P_8 = 0$ yield $A_i = B_i = 0$, a contradiction.

  (β) For $b_4 b_5 \neq 0$ we can compute a_4. Now the polynomial P_1 factorize into $K_4 b_4 (B_4 - B_5)(a_3 b_5 B_3 - a_5 b_3 B_5)$. For $a_3 = b_3 a_5 B_5/(B_3 b_5)$ we get $P_{15} = a_5 B_5 K_4 F_{15}/(b_5 B_3 B_4)$ ($a_5 = 0$ implies $a_3 = a_4 = 0$ a contradiction) and $P_8 = K_4 F_8/(B_3 B_4)$. Computing $F_8 - 2F_{15} = 0$ yields the contradiction. For $B_4 = B_5$ the condition $P_5 = 0$ also implies $a_3 = b_3 a_5 B_5/(B_3 b_5)$ and we can construct the same contradiction as before.

(iii) $K_1 = 0, K_4 = 0, K_3 \neq 0$: Now the polynomials P_{18} and P_{15} split into $K_3 a_2 \sin\phi F_9$ and $K_3 A_2 \sin\phi F_1$. We consider again the resultant which is given in Eq. (55).

- For $b_3 = 0$ the polynomial F_1 splits into $a_3 b_4 b_5 (B_4 - B_5)$. As for $B_4 = B_5$ the equation $F_9 = 0$ yields a contradiction, we set $a_3 = 0$. Now $F_9 = 0$ implies $b_5 = b_4 a_5 B_5/(a_4 B_4)$. From $P_6 = 0$ we get $A_4 = A_5$. $P_{17} = 0$ yields the contradiction.

- Now we set $a_4b_5B_4 - a_5b_4B_5 = 0$:

  (α) We assume $b_i = 0$ which yields $a_i = 0$ for $i, j \in \{4, 5\}$ and $i \neq j$. Now $P_{18} = 0$ implies $a_3 = a_jb_3B_4/(b_jB_3)$. Then $P_6 = 0$ can only vanish (w.c.) for $A_3 = A_j$. Finally $P_{17} = 0$ yields the contradiction.

  (β) $b_4b_5 \neq 0$: We compute a_4 and factorize P_2 and F_1 which yield

  $$K_3b_4(A_4-A_5)(a_3b_5B_3-a_5b_3B_5) \quad \text{and} \quad b_4(B_4-B_5)(a_3b_5B_3-a_5b_3B_5). \quad (56)$$

  For $A_4 = A_5$ and $B_4 = B_5$ the equation $P_6 = 0$ can only vanish (w.c.) for $A_3 = A_5$. Now $P_{17} = 0$ implies $B_3 = B_5$, a contradiction. Therefore we set $a_3 = b_3a_5B_5/(B_3b_5)$. P_6 factors into $K_3a_2F_6$ with

  $$F_6 := b_3b_4B_5(A_3 - A_4) + b_3b_5B_4(A_5 - A_3) + b_4b_5B_3(A_4 - A_5). \quad (57)$$

  Assuming $B_4b_3 - b_4B_3 \neq 0$ we can compute A_5 from $F_6 = 0$. Inserting this into $P_{17} = 0$ yields the contradiction. For $b_3 = b_4B_3/B_4$ the equation $P_6 = 0$ implies $A_3 = A_4$. Again $P_{17} = 0$ yields the contradiction. Hence, we can assume $K_2 \neq 0$ for the rest of the proof.

$R_1 = 0, K_2 \neq 0$

We proceed by setting R_1 of Eq. (52) equal to zero. We compute A_3 from $R_1 = 0$ and plug this into P_9 which splits into $B_3(a_3 - a_4)F_9$ with

$$F_9 := K_2B_4(A_5a_2 - A_2a_5) + K_2B_5(A_2a_4 - A_4a_2) + K_1(b_5B_4 - b_4B_5). \quad (58)$$

From $F_9 = 0$ we can compute A_5. Now the resultant of P_1 and P_5 with respect to A_2 simplifies to $K_2B_3B_4B_5coll(3, 4, 5)R_2[12]/a_2$.

Case I) $a_3b_4b_5[K_4a_2(a_4 - a_5) + K_1(b_4a_5 - b_5a_4)] \neq 0$

Under this assumption we can compute B_3 from $R_2 = 0$.

Part [A] Assuming $a_4a_5(b_4B_5 - b_5B_4) \neq 0$ we can compute A_2 from the common factor of P_1 and P_5. Now

$$P_8 = B_3B_5coll(3, 4, 5)F_8[8] \quad \text{and} \quad P_7 = B_3B_5coll(3, 4, 5)F_7[27].$$

$F_7 = 0$ and $F_8 = 0$ are homogeneous linear equations in the unknowns $\sin\phi$ and $\cos\phi$. So we compute the determinant of then coefficient-matrix which yields $K_2(b_4B_5 - b_5B_4)B_4a_2a_5D[21]$. From $D[21] = 0$ we can compute A_4. Now we get $F_7 = F_8 = a_4a_5(b_4B_5 - b_5B_4)C[8]$. From $C = 0$ we compute B_4 and plug the expression into P_{11}, which splits into $coll(3,4,5)F_{11}$. The factor F_{11} is quadratic in the unknown B_5. Therefore we obtain two solutions for

$$B_5 = \frac{2[K_1(a_ib_5 - b_ia_5) + K_4a_2(a_5 - a_i)]K_1b_5\sin\phi}{(K_4a_2 - K_1b_i)K_2a_2a_5\cos\phi} \tag{59}$$

with $i = 3, 4$. If we plug B_5 into B_i we get $B_i = 0$, a contradiction.

Part [B] $a_i = 0$ for $i, j \in \{4, 5\}$ and $i \neq j$. We compute P_5 which splits into $B_jb_3F_5[8]$. As $b_3 = 0$ yields $B_3 = 0$ we can assume $b_3 \neq 0$.

(i) Assuming $d := K_4(a_3 - a_j)(b_iB_j - b_jB_i) \neq 0$ we can compute

$$a_2 = K_1[B_ib_j(a_jb_3 - a_3b_j) + b_i^2B_j(a_3 - a_j)]/d \tag{60}$$

from $F_5 = 0$. Inserting this into $P_1 = 0$ yields the contradiction.

(ii) $K_4 = 0$: Now P_1 equals $b_3b_iB_jK_1(a_3b_j - a_jb_3)(b_iB_j - b_jB_i)/(a_2b_2)$. For both possible cases (i.e. $a_3 = b_3a_j/b_j$ and $b_i = b_jB_i/B_j$) the equation $P_5 = 0$ yields the contradiction.

(iii) $B_j = b_jB_i/b_i, K_4 \neq 0$: Now $P_5 = 0$ implies $K_1 = 0$ and $P_1 = 0$ yields the contradiction.

(iv) $a_3 = a_j$: Again $P_5 = 0$ yields $K_1 = 0$ and $P_1 = 0$ the contradiction.

Part [C] $b_4 = b_5B_4/B_5, a_4a_5 \neq 0$: $P_5 = 0$ implies $K_1 = 0$ and $P_1 = 0$ yields the contradiction.

Case II) $K_4a_2(a_4 - a_5) + K_1(b_4a_5 - b_5a_4) = 0$

We do this case without the assumption $a_3 - a_4 \neq 0$, such that a later reindexing can be done without loss of generality.

Part [A] Assuming $K_4(a_4 - a_5) \neq 0$ we can compute a_2. Now the factor R_2 simplifies to $K_1a_4a_5b_3(b_4B_5 - b_5B_4)coll(3,4,5)$.

(i) For $b_3 = 0$ we compute A_2 from $P_5 = 0$ which yields $A_2 = K_1(b_4 - b_5)/[K_2(a_4 - a_5)]$. Now $P_7 = 0$ can only vanish (w.c.) for

$$B_4 = (K_4 + A_4K_2)\sin\phi/(K_2\cos\phi). \tag{61}$$

We compute P_3 which splits up into $coll(3,4,5)K_1B_3F_3[18]$.

- Assuming $K_1(b_4a_5 - b_5a_4) + K_3b_4(a_5 - a_4) + K_4a_4(a_4 - a_5) \neq 0$ we can compute B_5 from $F_3 = 0$. Now $P_2 = 0$ can only vanish (w.c.) for $a_3 = 0$ and a factor $F_2[14] = 0$. As for $a_3 = 0$ the equation $P_{20} = 0$ yields a contradiction, we compute B_3 from $F_2 = 0$. Again $P_{20} = 0$ yields a contradiction.

- $K_1(b_4a_5 - b_5a_4) + K_3b_4(a_5 - a_4) + K_4a_4(a_4 - a_5) = 0$:

  (α) We can compute b_5 from this equation for $a_4 \neq 0$. Now $F_3 = 0$ can only vanish (w.c.) for $K_1 = -K_3$. Then $P_2 = 0$ implies $a_3 = 0$. This yields solution S_2 for $k = 3$.

  (β) For $a_4 = 0$ we get $b_4a_5(K_1 + K_3) = 0$ which implies $K_1 = -K_3$. $P_3 = 0$ can only vanish (w.c.) for $b_4 = (K_1b_5 + K_4a_5)/K_1$. $P_{18} = 0$ implies $a_3 = 0$. We get solution S_2 for $k = 3$ with the additional condition $a_4 = 0$.

(ii) $a_i = 0, b_3 \neq 0$ for $i, j \in \{4, 5\}$ and $i \neq j$. As $P_5 = 0$ yields a contradiction if we set $a_3 = 0$ or $b_3 = b_jB_3/B_j$, we can assume $a_3(b_jB_3 - b_3B_j) \neq 0$. Now we can compute A_2 from $P_5 = 0$. $P_8 = 0$ can only vanish (w.c.) for $b_j = 0$ and a second factor $F_8 = 0$.

- For $b_j = 0$ we compute B_4 from the only factor of $P_7 = 0$ which does not yield a contradiction.

  (α) Assuming $K_1 + K_3 \neq 0$ we can compute B_3 from the only factor of $P_3 = 0$ which does not yield a contradiction. From the factor of $P_{13} = 0$ which does not yield a contradiction we compute B_5. Then $P_6 = 0$ yields the contradiction.

  (β) For $K_1 = -K_3$ we compute b_3 from $F_3 = 0$. Plugging this into $P_{13} = 0$ yields the contradiction.

- $F_8 = 0, b_j \neq 0$: We compute A_4 from $F_8 = 0$. Then $P_{11} = 0$ yields the contradiction.

(iii) $b_4 = b_5B_4/B_5, a_4a_5b_3 \neq 0$: As $P_5 = 0$ yields a contradiction if we set $a_3 = 0$ or $b_5 = b_3B_5/B_3$, we can assume $a_3(b_5B_3 - b_3B_5) \neq 0$. Now we can compute A_2 from $P_5 = 0$. We compute A_4 from the only non-contradicting factor of $P_8 = 0$. Finally $P_{11} = 0$ yields a contradiction.

Part [B1] $K_4 = 0$ and $K_1 = 0$:

(i) Assuming $B_4a_5b_5(a_4b_3 - a_3b_4) + B_5a_4b_4(a_3b_5 - a_5b_3) \neq 0$ we can compute B_3 from $P_1 = 0$. Now $P_5 = 0$ can only vanish (w.c.) for $(B_4b_5 - B_5b_4) = 0$ or $a_i = 0$ with $i = 4, 5$.

- For $a_i = 0$ $(i, j \in \{4, 5\},\ i \neq j)$ we compute $P_{16} = 0$, which can only vanish (w.c.) for $K_3 = 0$ or $F_{16} = 0$. As $P_8 = 0$ yields a contradiction if we compute B_4 from $F_{16} = 0$, we set $K_3 = 0$. Now $P_{19} = 0$ implies $a_3 = [a_2(b_j - b_3) + a_jb_3]/b_j$. $P_{14} = 0$ can only vanish (w.c.) for $F_{16} = 0$.

- $b_4 = B_4b_5/B_5$: $P_7 = 0$ can only vanish (w.c.) for
$$A_4 = (\sin\phi A_2a_4 + \cos\phi B_4a_2)/(a_2\sin\phi).$$
$P_8 = 0$ yields the contradiction.

(ii) $B_4a_5b_5(a_4b_3 - a_3b_4) + B_5a_4b_4(a_3b_5 - a_5b_3) = 0$:

- Assuming $b_4b_5(a_4B_5 - a_5B_4) \neq 0$ we can compute a_3. Now $P_1 = 0$ vanish (w.c.) for $b_3 = 0$, $b_5B_4 - B_5b_4 = 0$ or $a_i = 0$ with $i = 4, 5$.

  (α) $a_i = 0$: From $P_7 = 0$ we get B_4. $P_8 = 0$ yields a contradiction.

  (β) $b_3 = 0,\quad a_4a_5 \neq 0$: We get $a_4 = a_2(\sin\phi A_4 - \cos\phi B_4)/(A_2\sin\phi)$ from $P_{15} = 0$. Now $P_5 = 0$ can vanish (w.c.) for $B_4 = \sin\phi A_4/\cos\phi$ or $b_4 = b_5B_4/B_5$. For both cases $P_8 = 0$ yields the contradiction.

  (γ) $b_4 = b_5B_4/B_5,\quad b_3a_4a_5 \neq 0$: Now $P_7 = 0$ implies a_4 of (β). Plugging this into $P_8 = 0$ yields the contradiction.

- $b_i = 0$ $(i = 4, 5)$: Eq. **(ii)** can only vanish (w.c.) for $a_4 = 0$ or $a_5 = 0$.

  (α) For $a_i = 0$ we compute B_4 from $P_{15} = 0$. Now $P_5 = 0$ only vanish (w.c.) for $a_3 = 0$, $a_j = 0$ or $b_3 = b_jB_3/B_j$. For all three cases equation $P_8 = 0$ yields the contradiction.

(β) $a_j = 0, a_i \neq 0$: $P_5 = 0$ can only vanish (w.c.) for $a_3 = 0$. From the only non-contradicting factor of $P_7 = 0$ we compute B_4. Now $P_{15} = 0$ can only vanish (w.c.) for $K_3 = 0$ or $B_3 = B_j$. For $K_3 = 0$ the equation $P_6 = 0$ implies $a_2 = a_i$. This yields solution S_1 for $k = i$ with the additional condition $K_1 = 0$. For $B_3 = B_j$ and $K_3 \neq 0$ the equation $P_4 = 0$ yields the contradiction.

- $a_4 = a_5 B_4/B_5, b_4 b_5 \neq 0$:

  (α) Assuming $b_5 B_4 - b_4 B_5 \neq 0$ we can compute a_5 from $P_5 = 0$. Now $P_1 = 0$ implies $b_4 = [B_3 b_5 + b_3(B_4 - B_5)]/B_3$. $P_7 = 0$ implies $B_4 = \sin\phi A_4/\cos\phi$ and $P_8 = 0$ yields the contradiction.

  (β) $b_4 = b_5 B_4/B_5$: Now $P_5 = 0$ can only vanish for $a_3 = 0$ or $b_3 = B_3 b_5/B_5$. In both cases we compute A_2 from the only non-contradicting factor of $P_7 = 0$. $P_8 = 0$ yields the contradiction.

Part [B2] $K_4 = 0$ and $b_4 a_5 - b_5 a_4 = 0, K_1 \neq 0$:

(i) With $a_5 \neq 0$ we can set $b_4 = b_5 a_4/a_5$. Now $P_1 = 0$ vanishes without contradiction for $a_4 = 0$, $a_4 B_5 - B_4 a_5 = 0$ or $K_1 b_5 - K_2 A_2 a_5 = 0$.

- $a_4 = 0$: As $P_5 = 0$ yields a contradiction if we set $a_3 = 0$ or $b_3 = b_5 B_3/B_5$, we can assume $a_3(b_3 B_5 - b_5 B_3) \neq 0$. Now we can compute A_2 from $P_5 = 0$. $P_7 = 0$ implies A_4. $P_{14} = 0$ yields a contradiction.

- $B_4 = a_4 B_5/a_5, a_4 \neq 0$: For the same reason as above we can again assume $a_3(b_3 B_5 - b_5 B_3) \neq 0$ and compute A_2 from $P_5 = 0$. From the only non-contradicting factor of $P_{15} = 0$ we compute

  $$B_5 = (K_2 A_4 a_2 a_5 - K_3 a_4 b_5)\sin\phi/(K_2 a_2 a_4 \cos\phi).$$

  $P_8 = 0$ can only vanish without contradiction for $b_3 = 0$ or $F_8[8] = 0$.

  (α) $P_7 = 0$ implies $K_3 = 0$ for $b_3 = 0$. Then $P_6 = 0$ implies $a_2 = a_3$ which corresponds with solution S_1 for $k = 3$ with the additional condition $B_4 = a_4 B_5/a_5$.

  (β) Now we can assume $b_3 \neq 0$ for the case $F_8 = 0$. We can compute B_3 from $F_8 = 0$. Plugging this into $P_2 = 0$ yields the contradiction.

- $K_1b_5 - K_2A_2a_5 = 0$, $a_4(a_4B_5 - B_4a_5) \neq 0$: We set $A_2 = K_1b_5/(K_2a_5)$ and compute $P_5 = 0$ which yields $b_3 = 0$. From $P_7 = 0$ we get $B_4 = \sin\phi A_4/\cos\phi$ and plug this into $P_2 = 0$, which can only vanish (w.c.) for $K_3 = 0$ or $B_5 = A_4 \sin\phi/\cos\phi$, respectively.

  (α) For $K_3 = 0$ we obtain from $P_6 = 0$ the condition $a_2 = a_3$, which corresponds with solution S_1 for $k = 3$.

  (β) Now we can assume $K_3 \neq 0$ and set $B_5 = A_4 \sin\phi/\cos\phi$. Again $P_6 = 0$ implies $a_2 = a_3$, but now $P_4 = 0$ yields a contradiction.

(ii) Assuming $a_5 = 0$ yields $b_5 = 0$ or $a_4 = 0$.

- $b_5 = 0$: We obtain $B_4 = [K_2(a_2A_4 - a_4A_2) + K_1b_4]\sin\phi/(a_2K_2\cos\phi)$ from $P_{15} = 0$. Now $P_7 = 0$ can only vanish for $K_1b_i - K_2A_2a_i = 0$ with $i = 3, 4$ or $F_7[4] = 0$. If we set $a_i = K_1b_i/(K_2A_2)$ the equation $P_8 = 0$ yields the contradiction. Therefore we compute B_3 from $F_7 = 0$ which yields $B_3 = b_3[K_2(A_4a_2 - A_2a_4) + K_1b_4)]\sin\phi/(a_2b_4K_2\cos\phi)$. Plugging this into $P_5 = 0$ yields the contradiction.

- $a_4 = 0, b_5 \neq 0$: Now $P_1 = 0$ can only vanish (w.c.) for $b_4 = 0$ or $b_5 = b_4B_5/B_4$. For both cases $P_5 = 0$ yields the contradiction.

Part [C] $a_4 = a_5, K_4 \neq 0$

Now the condition of case II can only vanish (w.c.) for $K_1 = 0$ and $a_4 = 0$.

(i) $K_1 = 0$: $P_5 = 0$ can only vanish (w.c.) for $a_4 = -a_2K_4/(A_2K_2)$ or $b_4 = B_4b_5/B_5$.

- $a_4 = -a_2K_4/(A_2K_2)$: We get $B_4 = (K_2A_4 + K_4)\sin\phi/(K_2\cos\phi)$ from $P_7 = 0$. Now $P_8 = 0$ can only vanish (w.c.) for $b_3 = 0$.

  (α) $K_3 \neq 0$: We can compute b_4 from the only non-contradicting factor $F_4[9] = 0$ of $P_4 = 0$. Moreover if we assume $a_3(a_2 - a_3) \neq 0$ we can compute B_5 from $F_6[5] = 0$, which is the only factor of $P_6 = 0$, which does not yield a direct contradiction. Plugging this into $P_2 = 0$ yields the contradiction. For both remaining cases ($a_3 = 0$ and $a_2 = a_3$) the equation $F_6 = 0$ already yields the contradiction.

(β) $K_3 = 0$: Now $F_4 = 0$ can only vanish (w.c.) for $A_2 = -K_4/K_2$ or $B_5 = \sin\phi(K_2A_4 + K_4)/(K_2\cos\phi)$. For $A_2 = -K_4/K_2$ the equation $P_6 = 0$ implies $a_3 = 0$, which yields solution S_2 for $k = 3$ with the additional condition $K_1 = K_3 = 0$. Now we can assume $A_2 \neq -K_4/K_2$ and set $B_5 = \sin\phi(K_2A_4 + K_4)/(K_2\cos\phi)$. From $P_6 = 0$ we can compute B_3. Plugging this into $P_2 = 0$ yields the contradiction.

- $b_4 = B_4b_5/B_5, a_4 \neq -a_2K_4/(A_2K_2)$: We get $B_4 = (a_2A_4 - a_4A_2)\sin\phi/(a_2\cos\phi)$ from $P_7 = 0$. Now $P_4 = 0$ can only vanish (w.c.) for $a_4 = 0$ or $a_4 = a_2$. For both cases we compute B_3 from the non-contradicting factor of $P_{18} = 0$ and plug the obtained expression into $P_1 = 0$ which yields the contradiction.

(ii) $a_4 = 0, \quad K_1 \neq 0$: As $P_5 = 0$ yields the contradiction if we set $b_4 = b_5B_4/B_5$, we can assume $b_4B_5 - b_5B_4 \neq 0$. Now we can compute a_2 from $P_5 = 0$. $P_1 = 0$ can only vanish (w.c.) for $b_i = 0$ ($i, j \in \{4, 5\}$, $i \neq j$). Then $P_{14} = 0$ can only vanish (w.c.) for$K_1(b_j - b_3) + A_2a_3K_2 = 0$ and a second factor $F_{14} = 0$.

- If we solve $F_{14} = 0$ for B_4 we get from $P_4 = 0$ the condition $K_1 = -K_3$. From $P_3 = 0$ we compute b_3 and plug the obtained expression into $P_{17} = 0$, which can only vanish (w.c.) for $A_2 = -K_4/K_2$. This corresponds with solution S_2 for $k = i$ with the additional condition $a_j = 0$.
- $A_2 = K_1(b_3 - b_j)/(K_2a_3), \quad F_{14} \neq 0$: From $P_{18} = 0$ we get $K_1 = -K_3$. $P_7 = 0$ yields the contradiction.

Case III) $b_i = 0$ **for** $i = 4, 5$

For $b_i = 0$ the factor R_2 simplifies to

$$R_2 := b_3b_ja_iB_i[K_4a_2(a_3 - a_j) + K_1(b_3a_j - b_ja_3)] \tag{62}$$

with $i, j \in \{4, 5\}$ and $i \neq j$. Therefore the two possibilities are $a_i = 0$ or $K_4a_2(a_3 - a_j) + K_1(b_3a_j - b_ja_3) = 0$. The latter was already done in case II just for another indexing. Therefore we obtain the same solutions as in case II just for another index k.

The remaining discussion of $a_i = 0$ can be done under the assumption $K_4a_2(a_4-a_5)+K_1(b_4a_5-b_5a_4) \neq 0$ due to case II. If we consider $P_{15} = 0$ and $P_{16} = 0$ we see that these equations can only vanish (w.c.) for $K_1 = -K_3$ and $A_2 = -K_4/K_2$ or for the common factor $G = 0$.

Part [A] $G = 0$: From this equation we compute B_4.

(i) $K_1 \neq 0$: We can compute b_3 from $P_8 = 0$. From $P_5 = 0$ we get a_j.

- Assuming $K_2A_2(a_2K_4 + b_jK_3) - K_4(K_1b_j - a_2K_4) \neq 0$ we can compute B_3 from $P_6 = 0$. $P_4 = 0$ yields the contradiction.
- $K_2A_2(a_2K_4 + b_jK_3) - K_4(K_1b_j - a_2K_4) = 0$:

  (α) Assuming $A_2K_2 + K_4 \neq 0$ we can compute a_2. $P_3 = 0$ yields the contradiction.

  (β) $A_2 = -K_4/K_2$ implies $K_1 = -K_3$. We get solution S_2 for $k = i$.

(ii) $K_1 = 0$: Now we can compute a_3 from $P_8 = 0$. From $P_5 = 0$ we get $a_2 = -K_2A_2a_j/K_4$.

- $K_3 \neq 0$: We compute a_j from $P_4 = 0$. Then $P_6 = 0$ yields a contradiction.
- $K_3 = 0$: Now $P_4 = 0$ can only vanish for $A_2 = -K_4/K_2$, which yield solution S_2 for $k = i$ with the additional condition $K_1 = K_3 = 0$, or $B_3 = B_j$. For the later $P_{20} = 0$ yields the contradiction under the assumption $A_2 \neq -K_4/K_2$.

Part [B] $K_1 = -K_3, A_2 = -K_4/K_2, G \neq 0$:

(i) Assuming $b_3a_jB_j - b_jB_3a_3 \neq 0$ we can compute a_2 from $P_5 = 0$. Now $P_8 = 0$ can only vanish (w.c.) for $K_3(b_j - b_3) - K_4(a_j - a_3) = 0$ or a second factor $F_8 = 0$.

- $a_j = (K_3(b_j - b_3) + K_4a_3)/K_4$: Now $P_{12} = 0$ yields $K_3 = 0$. Then the equation $P_7 = 0$ yields the contradiction.
- $F_8 = 0, K_3(b_j - b_3) - K_4(a_j - a_3) \neq 0$: From $F_8 = 0$ we compute A_4. $P_4 = 0$ implies $a_3 = 0$ and $P_6 = 0$ yields the contradiction.

(ii) $a_3 = b_3 a_j B_j/(b_j B_3)$: From $P_5 = 0$ we compute $B_j = b_j B_3[K_3(b_3 - b_j) + K_4 a_j]/(K_4 a_j b_3)$. Now $P_{14} = 0$ implies $a_2 = (a_j K_4 - b_j K_3)/K_4$. $P_7 = 0$ yields the contradiction.

Case IV) $a_3 = 0, \quad N := b_4 b_5[K_4 a_2(a_4 - a_5) + K_1(b_4 a_5 - b_5 a_4)] \neq 0$

Now R_2 splits up into $a_4 a_5 b_3(K_1 b_3 - K_4 a_2)(b_4 B_5 - b_5 B_4)$.

Part [A] $b_3 = 0$: As $P_5 = 0$ yields a contradiction if we set $a_5 = 0$ or $b_4 = b_5 B_4/B_5$ we can assume $a_5(b_4 B_5 - b_5 B_4) \neq 0$. Now we can compute A_2 from $P_5 = 0$. P_{14} and P_8 factors into $Ncoll(3,4,5)B_3 B_5 F_{14}[8]$ and $Ncoll(3,4,5)B_3 B_5 F_8[8]$. Computing $F_8 - 2F_{14} = 0$ yields the equation $a_2 a_5 B_4 K_2(b_4 B_5 - b_5 B_4)\cos\phi = 0$ and therefore a contradiction.

Part [B] $a_5 = 0, b_3 \neq 0$: Assuming $K_1 b_3 - K_4 a_2 \neq 0$ we can compute B_5 from $P_5 = 0$. $P_1 = 0$ yields the contradiction. Therefore we assume $K_4 \neq 0$ and set $a_2 = K_1 b_3/K_4$. $P_5 = 0$ implies $b_3 = b_4$ and $P_1 = 0$ yields the contradiction. For $K_4 = 0$ we get $K_1 b_3$ which is a contradiction.

Part [C] $b_4 = b_3 B_4/B_5, a_5 b_3 \neq 0$: We can solve $P_5 = 0$ for B_3. $P_1 = 0$ yields the contradiction.

Part [D] $K_1 b_3 - K_4 a_2 = 0, b_3 a_5(b_4 B_5 - b_5 B_4) \neq 0$: We can assume $K_4 \neq 0$ otherwise we get a contradiction. So we can set $a_2 = K_1 b_3/K_4$. From $P_5 = 0$ we can compute A_2. Then we compute A_4 from the only factor of $P_8 = 0$ which does not yield a direct contradiction. Now we can compute B_4 from $P_7 = 0$. $P_{11} = 0$ yields the contradiction. End of all cases.

The close of the proof was already done by the author in [5], by showing that the solutions S_1 and S_2 imply contradictions for the choice of $\mathbf{M}_6$ and $\mathbf{m}_6$, respectively. This finishes the proof of the given Theorem. □

4 Conclusion

We proved that there do not exist non-architecturally singular Stewart Gough Platforms with planar base and platform and no four anchor points collinear which possess a cylindrical singularity surface with rulings parallel to a given fixed direction p in the space of translations.

A complete list of planar parallel manipulators with such a singularity surface is in preparation [6].

References

[1] Husty, M.L., Hayes, M.J.D., and Loibnegger, H.: The General Singularity Surface of Planar Three-Legged Platforms, Advances in Multibody Systems and Mechantronics (A. Kecskemethy, ed.), Duisburg, Germany, 203–214 (1999).

[2] Karger, A.: Architecture singular planar parallel manipulators, Mechanism and Machine Theory **38** (11) 1149–1164 (2003).

[3] Karger, A.: Stewart-Gough platforms with simple singularity surface, Advances in Robot Kinematics: Mechanisms and Motion (J. Lenarcic, B. Roth eds.), 247–254, Springer (2006).

[4] Merlet, J.-P.: Singular Configurations of Parallel Manipulators and Grassmann Geometry, International Journal of Robotics Research **8** (5) 45–56 (1992).

[5] Nawratil, G.: Results on Planar Parallel Manipulators with Cylindrical Singularity Surface, Advances in Robot Kinematics - Analysis and Design (J. Lenarcic, P. Wenger eds.), 321–330, Springer (2008).

[6] Nawratil, G.: All Planar Parallel Manipulators with Cylindrical Singularity Surface, Technical Report **191**, August 2008, Geometry Preprint Series, Vienna University of Technology.

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