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Characterization and a Construction of Equiframed Curves

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1 Introduction

In this contribution, which is based on the paper [4], we give a constructive description of centrally symmetric *equiframed curves*, i.e., centrally symmetric closed convex curves that are touched at each of their points by some circumscribed parallelogram of smallest area. Equiframed curves and their higher dimensional analogues were introduced by Pełczynski and Szarek [7]. This class of curves properly contains the Radon curves introduced by Radon [8]. For a survey on them see [5, 6].

Clearly, any regular polygon with $2n$ sides is equiframed, and if n is even, the boundary is not a Radon curve. Equiframed curves occur as the unit circles of two-dimensional norms for which equality holds in a certain inequality in Minkowski Geometry, recently found in [6]. This inequality bounds the ratio between the area of the unit circle and a circumscribed parallelogram of smallest area ((2) below) in terms of the circumference of the unit circle. It can be seen as a dual to an old inequality of Lenz [3], bounding the ratio between the area of the unit circle and an inscribed parallelogram of largest area in terms of the circumference (see (1)). In Section 2 we also give characterizations of equiframed curves (Proposition 2.2) similar to known ones of Radon curves (Proposition 2.3).

In Section 3 we describe the construction of a general equiframed curve based on the known construction of Radon curves. It turns out that equiframed curves are Radon curves with certain triangles added to the boundary.

2 Inequalities for Radon and equiframed curves

We let C_0 denote a centrally symmetric convex body in the plane and denote its *boundary curve* by ∂C_0. The curve ∂C_0 is a *Radon curve* if each point of ∂C_0 is a vertex of some inscribed parallelogram of maximum area. Dually, ∂C_0 is an *equiframed curve* if each point of ∂C_0 is touched by some circumscribed parallelogram of minimum area. Note that our definition of a Radon curve is not the standard one, but chosen so as to be dual to the definition of an equiframed curve.

For any centrally symmetric C_0 we have

$$\tfrac{4}{U} \leq p^-(C_0) = q^-(C_0) \text{ with equality iff } \partial C_0 \text{ is a Radon curve.} \tag{1}$$

This was first proved by Lenz [3] and subsequently rediscovered by Yaglom [9]. We now consider the dual of (1); see [6].

Theorem 2.1. *For any centrally symmetric convex body C_0 in the plane,*

$$q^+(C_0) = p^+(C_0) \leq \tfrac{8}{U} \textit{ with equality iff } \partial C_0 \textit{ is an equiframed curve.} \tag{2}$$

The next two propositions give various characterizations of the equality case in (1) and (2).

Proposition 2.2. *The following are equivalent for a centrally symmetric curve ∂C_0:*

1. *∂C_0 is equiframed,*

2. *each right semi-tangent of C_0 is a side of a circumscribed parallelogram of least area,*

3. *$\alpha(t)$ is constant,*

4. *$\alpha(t) = \frac{1}{4}p^+(C_0)|C_0|$ for all $t \in [0, U)$,*

5. *$p^+(C_0) = \frac{8}{U}$, i.e., in (2) equality holds.*

6. *each boundary point of C_0 lies on some circumscribed quadrilateral of smallest area.*

The following proposition shows the subtle difference between equiframed curves and Radon curves. All of these characterizations are well-known [5, 6]. Note that 1–5 correspond to 1–5 of Proposition 2.2.

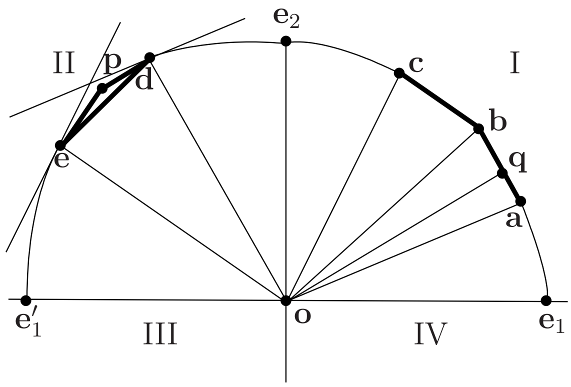

Figure 1: Gluing a triangle to a Radon curve with wedges.

Proposition 2.3. *The following are equivalent for a centrally symmetric curve ∂C_0:*

1. *∂C_0 is a Radon curve,*

2. *each supporting line of C_0 is a side of some circumscribed parallelogram of smallest area,*

3. *$|\det[\mathbf{u}(t), \mathbf{v}]|$ is constant for all $t \in [0, U)$ and all unit vectors $\mathbf{v}$ such that the line through $\mathbf{u}(t)$ parallel to $\mathbf{v}$ supports C_0,*

4. $|\det[\mathbf{u}(t), \mathbf{v}]| = \frac{1}{4}p^+(C_0)|C_0|$ *for all* $t \in [0, U)$ *and all unit vectors* $\mathbf{v}$ *as in 3,*

5. $p^-(C_0) = \frac{4}{U}$, *i.e., in* (1) *equality holds.*

6. $\alpha(t) = \frac{1}{2}p^-(C_0)|C_0|$ *for all* $t \in [0, U)$,

7. *each boundary point of* C_0 *is the midpoint of some circumscribed parallelogram of smallest area,*

3 Construction of equiframed curves

Since equiframed curves turn out to be Radon curves, we first describe a construction of the latter. The standard construction [8, 1] is based on the fact that the "second quadrant" of a Radon curve is in some sense the rotated polar of the "first quadrant". Since the polar of a convex body lies strictly speaking in the dual of the underlying space, one first has to fix a polarity, or in the language of linear algebra, identify the space and its dual via a nondegenerate bilinear form. In the usual construction of a Radon curve, the bilinear form is chosen to be symmetric (with the corresponding geometry being Euclidean), and then, as a second step in the construction, one has to rotate by a right angle in the Euclidean structure determined by the bilinear form. We simplify this construction by using a skew-symmetric bilinear form instead of a symmetric one (i.e., corresponding to symplectic geometry instead of Euclidean geometry). The details are as follows.

We fix the bilinear form $[\mathbf{x}, \mathbf{y}]$ on $\mathbb{R}^2$ to be the determinant $\det[\mathbf{x}, \mathbf{y}]$, i.e., after choosing a unit of area we define $[\mathbf{x}, \mathbf{y}]$ to be the signed area of the parallelogram with vertices $\mathbf{o}, \mathbf{x}, \mathbf{x}+\mathbf{y}, \mathbf{y}$. If the line ℓ is polar to the point $\mathbf{x}$ in the polarity defined by this form, then for any line m parallel to ℓ we write $[\mathbf{x}, m] := [\mathbf{x}, \mathbf{y}]$ for any $\mathbf{y} \in m$, with a similar definition for $[m, \mathbf{x}]$.

1 Construction of an arbitrary Radon curve

Let $\alpha > 0$ and fix a parallelogram P of area 4α centred at $\mathbf{o}$. The two lines through $\mathbf{o}$ parallel to the sides of P divide the plane into four *quadrants*, which we label $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ in any way such that $\mathrm{III} = -\mathrm{I}$, $\mathrm{IV} = -\mathrm{II}$, and

$[\mathbf{x}, \mathbf{y}] \geq 0$ for any $\mathbf{x} \in \mathrm{I}$ and $\mathbf{y} \in \mathrm{II}$. We consider the quadrants to be closed sets. Let $\mathbf{e}_1$ be the vector on a side of P on the boundary of I and IV, and $\mathbf{e}_2$ be the vector on a side of P on the boundary of I and II. Let $\mathbf{e}_1' = -\mathbf{e}_1$.

Now choose any convex curve Γ_{I} joining $\mathbf{e}_1$ to $\mathbf{e}_2$ such that $\Gamma_{\mathrm{I}} \cup \mathbf{oe}_1 \cup \mathbf{oe}_2$ is a closed convex curve. We now define a curve Γ_{II} in II. For each direction $\mathbf{v} \in \mathbf{e}_2\mathbf{e}_1'$ we let $\lambda\mathbf{v} \in \Gamma_{\mathrm{II}}$ be such that $[\ell, \lambda\mathbf{v}] = \alpha$, where ℓ is the line parallel to $\mathbf{v}$ supporting Γ_{I}. To finish the construction we let $\Gamma_{\mathrm{III}} = -\Gamma_{\mathrm{I}}$ and $\Gamma_{\mathrm{IV}} = -\Gamma_{\mathrm{II}}$. Then the Radon curve is $\Gamma_{\mathrm{I}} \cup \Gamma_{\mathrm{II}} \cup \Gamma_{\mathrm{III}} \cup \Gamma_{\mathrm{IV}}$.

Proposition 3.1. *The above construction gives a Radon curve C_0 with $p^-(C_0) = \alpha$. Furthermore, any Radon curve can be obtained in this way.*

This is due to Radon [8]. A proof may be found in [2].

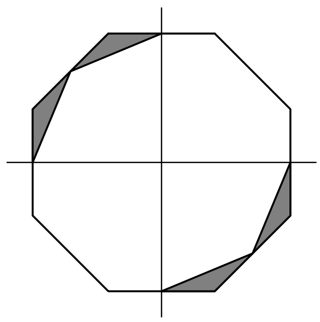

Figure 2: Constructing the regular octagon by gluing triangles to a Radon curve.

2 Construction of an arbitrary equiframed curve

Choose $\alpha > 0$ and Γ_{I} and proceed as in the construction of a Radon curve. Then glue triangles to Γ_{II} as follows. We define a *wedge* on a convex curve to be the union of two nonparallel segments on the curve joined at a common endpoint, called the *corner* of the wedge. Each wedge $\mathbf{ab} \cup \mathbf{bc} \subseteq \Gamma_{\mathrm{I}}$ with corner $\mathbf{b}$ on Γ_{I} corresponds to a segment $\mathbf{de}$ on Γ_{II} with nonregular endpoints (Figure 1).

Here $\mathbf{d}$ is parallel to $\mathbf{ab}$, $\mathbf{e}$ is parallel to $\mathbf{bc}$, $\mathbf{de}$ is parallel to $\mathbf{b}$, and there are supporting lines to Γ_{II} at $\mathbf{d}$ and $\mathbf{e}$ parallel to $\mathbf{a}$ and $\mathbf{c}$, respectively. For each such segment **de**, choose any point **p** in the closed triangle bounded by **de**, the supporting line at **d** parallel to **a**, and the supporting line at **e** parallel to **c**. Replace the segment **de** on Γ_{II} by $\mathbf{dp} \cup \mathbf{pe}$. This may be done arbitrarily for each wedge on Γ_{I}, provided that we keep the modified Γ_{II} convex. Note that it is possible for a countable infinity of triangles to be added, since there may be an infinite number of wedges on Γ_{I}.

We denote the modified Γ_{II} by Δ_{II}, and let $\Delta_{\mathrm{IV}} = -\Delta_{\mathrm{II}}$ (and $\Gamma_{\mathrm{III}} = -\Gamma_{\mathrm{I}}$ as before). Then the equiframed curve is $\Gamma_{\mathrm{I}} \cup \Delta_{\mathrm{II}} \cup \Gamma_{\mathrm{III}} \cup \Delta_{\mathrm{IV}}$.

Theorem 3.2. *The above contruction gives an equiframed curve. Furthermore, any equiframed curve can be obtained in this way.*

Corollary 3.3. *An equiframed curve that does not contain any wedge is a Radon curve. In particular, an equiframed curve that is smooth or strictly convex is a Radon curve.* □

Corollary 3.4. *An equiframed hexagon must be an affine regular hexagon or a parallelogram* (*if the hexagon is degenerate*).

In Figure 2 we show how a regular octagon arises as an equiframed curve in our construction.

Two shaded triangles are added to the Radon curve in the second quadrant.

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