ESI program ‘Rigidity and Flexibility’
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List of Abstracts

Last update: April 27

Victor ALEXANDROV: **Hidden symmetries of flexible polyhedra and the Strong Bellows Conjecture.**

According to the Strong Bellows Conjecture, if an imbedded polyhedron P_1 is obtained from an imbedded polyhedron P_0 by a continuous flex then P_1 and P_0 are scissor congruent, i. e., P_1 can be partitioned in a finite set of polyhedra $\{Q_j\}$, $j = 1, \ldots, n$, with the following property: for every j, there exists an isometry $F_j : \mathbb{R}^3 \to \mathbb{R}^3$ such that the set $\{F_j(Q_j)\}$, $j = 1, \ldots, n$, is a partition of P_0.
We propose a more general conjecture which implies the Strong Bellows Conjecture for imbedded polyhedra but makes sense for non-imbedded polyhedra either and refers to some symmetry-like properties of polyhedra. We confirm the truth of the new conjecture for the Bricard's flexible octahedra of types 1–3. In particular we prove the Strong Bellows Conjecture to be true for every imbedded flexible polyhedron obtained from Bricard's octahedra and non-flexible polyhedra by gluing faces, e. g., for the famous Steffen's flexible polyhedron. (30 minutes talk.)

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Robert CONNELLY: **The origins, background and conjectures related to bellows problems, rigidity and flexibility of polyhedral surfaces.**

In 1973 Herman Gluck highlighted the rigidity conjecture which says that any embedded polyhedral surface is rigid. There are several examples of singular polyhedral surfaces that are not rigid. For example, all flexible octahedra were classified by R. Bricard in 1896. They have some interesting properties. The volume they bound is zero, and the boundaries of some pair of the triangular faces either intersect or link. This and Cauchy's classical theorem about the rigidity of convex polyhedra lent credence to the rigidity conjecture. Nevertheless, in 1977 I proved this conjecture is false, and one of the interesting related questions was whether the volume bounded by any flexible surface is constant during the motion. In 1995 this was proved by I. Sabitov. But still many interesting questions remain. For example, it may be true that any flexible polyhedral surface is built from truly flexible surfaces and some rigid pieces, so that the truly flexible surfaces have zero volume and the only way to have an embedded flexible surface is to attach rigid pieces with non-zero volume. (60 minutes talk.)

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Nikolaj Dolbilin: **On the unfolding of a rectangle enlarging perimeter.**

In the talk it will be said on a solution of the following rather old problem. Given a rectangular sheet of paper, whether one can isometrically fold it into another flat polygon with a bigger perimeter. According to [1] and [2], where this problem stands under N 1, it was set by Arnold yet in 1956. More detailed discussion of this problem can be found in [3]. Recently ([4], see also my comment in [2]) my student A. Tarasov solved this problem. Indeed, there is a bunch of claims on solving this problem. Some have been checked and found wrong. It is quite possible that there are other correct constructions enlarging the perimeter. But what is most remarkable in a very nice Tarasov's construction that, in fact, *the explicitly presented corresponding unfolding enlarges the perimeter of a rectangular in arbitrarily large, a priori given number times.* In this presentation an improved (comparatively to [4]) construction will be given.

References: [1] *V.I.Arnold*, Arnold's problems (in Russian), Moscow: Phasis, 2000
[2] *V.I. Arnold*, Arnold's Problems (updated English translation), Springer–Phasis 2004
[3] *I.V. Yaschenko*, Make your dollar bigger now!!!, Math. Intell. (1998), **20**(2), 38–40
[4] *A.S.Tarasov*, Solving Arnold's problem on "rumpled rouble" [in Russian], Chebysheskii sbornik (2004), vol. 5, no. 1(9), 174–187. (45 minutes talk.)

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Nikolaj Dolbilin: **The Minkowski theorem on convex polyhedra and its role in the tiling theory.**

This talk would consist of, in part, commonly known material as well as of more special facts, including some new ones. (60 minutes talk.)

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François Fillastre: **Realization of polyhedral metric on compact surfaces.**

A famous statement of A.D. Alexandrov says that each Euclidean metric with conical singularities on the sphere is realised as an unique (convex) polyhedron in Euclidean space (up to isometries).

The aim of this talk is to present an analogous statement for a compact surface S of genus ≥ 2: a metric of constant curvature with conical singularities (with positive curvature) on S is realised as an unique convex so-called Fuchsian polyhedron in hyperbolic space (a Fuchsian polyhedron is a polyhedron equivariant under the action of isometries preserving a totally geodesic plane).

Furthermore, polyhedral metrics on S with others curvatures are realisable in Lorentzian space-forms. (45 minutes talk)

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Vasyl Gorkavyy: **Linear bendings of arbitrary star-like bipyramids** (joint work with Anatoliy Milka).

Linear bendings of particular right star-like bipyramids were applied by A.D. Milka to mathematically describe a physical instability of some star-like bipyramidal thin shells called model flexors [1], [2]. Actually we will go to discuss similar linear bendings for arbitrary star-like bipyramids without particular restrictions on their sizes. ¿From analytical viewpoint a linear bending of a given star-like bipyramid is represented by solutions of a system of 5 algebraic equations for 6 parameters. We analyze dynamical properties of linear bendings in the general case and discuss analytical reasons, which allow to mathematically distinguish the model flexors between star-like bypiramids. The linear bendings of model flexors describe a non-rigid loss of stability, whereas for all another star-like bipyramids the linear bendings in question represent an analogue of the rigid loss of stability. Particular examples of star-like bipyramids with rigid loss of stability were constructed by A.D. Alexandrov and S.M. Vladimirova in 1956.

References: [1] *A.D. Milka*, Nonrigid starlike bipiramids of A.D. Alexandrov and S.M. Vladimirova, Siberian Adv. Math. **12** (2) (2002) 56–72.

[2] *A.D. Milka*, Linear bendings of star-like pyramids, C. R. Mecanique, **331** (12) (2003) 805–810. (60 minutes talk)

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Tim Hoffmann: **On the integrability of infinitesimal and finite deformations of polyhedral surfaces** (joint work with W. Schief and A. Bobenko).

It is established that there exists an intimate connection between isometric deformations of polyhedral surfaces and discrete integrable systems. In particular, Sauer's kinematic approach is adopted to show that second-order infinitesimal isometric deformations of discrete surfaces composed of planar quadrilaterals (discrete conjugate nets) are determined by the solutions of an integrable discrete version of Bianchi's classical equation governing finite isometric deformations of conjugate nets. Moreover, it is demonstrated that finite isometric deformations of discrete conjugate nets are completely encapsulated in the standard integrable discretization of a particular nonlinear sigma-model subject to a constraint. The deformability of discrete Voss surfaces is thereby retrieved in a natural manner. (45 minutes talk)

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Ivan Izmestiev: **Alexandrov theorem, weighted Delaunay triangulations and mixed volumes** (joint work with Alexander Bobenko).

We present a new proof of a theorem by A.D. Alexandrov: *Any convex polyhedral metric on 2-sphere can be realized as the boundary of a convex polytope in* $\mathbb{R}^3$. The proof provides an algorithm for finding the polytope. This algorithm is implemented in a computer program. The idea is to start with a 'generalized polytope' with the given metric on the boundary and with 'curved' interior. Then this polytope is continuously deformed so that the curvature vanishes. (60 minutes talk)

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Bill JACKSON: **Rank and independence in the rigidity matroid of molecular graphs** (joint work with Tibor JORDÁN).

We consider the 3-dimensional rigidity matroid of squares of graphs. These graphs are also called *molecular graphs* due to their importance in the study of flexibility in molecules. The Molecular Conjecture, posed in 1984 by T-S. Tay and W. Whiteley, indicates that determining whether a molecular graph is rigid (or more generally, computing the rank of its rigidity matroid) may be tractable by combinatorial methods. A related conjecture of D. Jacobs would give a combinatorial characterization of when a molecular graph is *M-independent*, that is to say when its edge set is independent in its rigidity matroid. We introduce two partitions of the vertex set of a graph G, the *brick partition* and the *superbrick partition*, and use them to investigate properties of the rigidity matroid of G^2.

• We show that the conjectured characterization of rigidity of G^2 is equivalent to the apparently stronger conjectured characterization of the rank of G^2.

• We show that Jacobs conjectured characterization of M-independence of G^2 is equivalent to a statement about the superbricks of G, and prove that a strengthening of this statement is sufficient to imply M-independence of G^2.

• We show that the truth of the Molecular Conjecture would imply that the maximal rigid subgraphs of G^2 could easily be obtained from the bricks of G, and hence could be determined in polynomial time using standard algorithms in graph theory. (60 minutes talk.)

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Tibor JORDÁN: **Rank and independence in the 3-dimensional rigidity matroid** (joint work with Bill JACKSON).

It is a difficult open problem to characterize generic rigidity of graphs in 3 dimensions. We give a short survey of results concerning special families of graphs for which a good characterization of rigidity (or more generally, rank) is known, and for which the number of the degrees of freedom can be computed efficiently. We also discuss old and new conjectures for other special families of graphs.

In particular, we consider a bar-and-joint formulation of the Molecular Conjecture, which was originally posed in 1984 by T.-S. Tay and W. Whiteley in terms of body-and-hinge frameworks. This conjecture indicates that determining whether the square of a graph is rigid (or computing its rank) may be tractable by combinatorial methods. (Two vertices u, v are adjacent in the *square* G^2 of graph G if and only if they are adjacent or they have a common neighbour in G.) We prove that the conjectured formula for the rank of the square of a graph is indeed an upper bound on its rank. (45 minutes talk)

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Natalia Kopteva: **Blaschke addition and convex polyhedra** (joint work with V. Alexandrov and S. Kutateladze).

The aim of the talk is to give an overview of known results about the Blaschke addition of convex bodies and to describe a software for visualisation of the Blaschke sum of polyhedra with given outward normals and face areas. The paper related to the talk is available at http://www.arxiv.org/math.MG/0502345. (45 minutes talk)

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Hiroshi Maehara: **Reversing a polyhedral surface.**

We introduce a new variety of flexatube, a rhomboflexatube. It is obtained from a cardboard rhombohedron by removing a pair of opposite faces (rhombi), and then subdividing the remaining four faces by pairs of diagonals. It is reversible, that is, it can be turned inside out by a series of folds, using edges and diagonals of the rhombi. To turn a rhomboflexatube inside out is quite a challenging puzzle. We also consider the reversibility of general polyhedral surfaces. We show that if an orientable polyhedral surface with boundary is reversible, then its genus is 0 and, for every interior vertex, the sum of face angles at the vertex is at least 2π. After defining tube-attachment operation, we show that every polyhedral surface obtained from a rectangular tube by applying tube-attachment operations one after another can be subdivided so that it becomes reversible. (30 minutes talk)

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Yves Martinez-Maure:
Principles, problems and new tools for hedgehog theory.

First, I will recall the genesis of my main results on herissons and which principles and tools I used to obtain them. Next, I will present the main open problems I am interested in. Finally, I will propose new tools for studying these open problems. (60 minutes talk)

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Sergej Mikhalev:

A metric description of Bricard's flexible octahedra of the 3rd type.

Bricard's flexible octahedra (see [1, 2]) play an important role in the metric theory of polyhedra. For example, at the moment the most of known constructions of flexible polyhedra in Euclidian space are based on one of Bricard's octahedra. R.Bricard proves that there exist only three types of flexible octahedra. For the first two types their metric description is very simple (lengths of some edges must be equal to each other). But it turns out that the description of type 3 (3-octahedra) is rather complicated: it involves 12 different conditions which are not independent. It is not obvious how to operate with this "non-constructive" description. If we want to calculate some metric characteristics of 3-octahedra (say, the Sabitov's volume polynomial), it would not be an easy task.

In this work we obtain a system of 7 necessary independent polynomial equations for 12 lengths of edges of 3-octahedra. First six of these equations have very simple form. The last one consists of 24 monomials (the total degree is equal to 5, and the degree with respect to any variable is 3). In the 12-space of octahedron's edges the 6-manifold formed by first six equations admits a simple parametrization (six edges are parameters and other edges are some rational functions of the parameters). The last equation is the only relation between these parameters. This description allows to solve various metric problems for 3-octahedra. In particular we hope to find the Sabitov's volume polynomial and to solve the problem of isometric realization for an abstract 3-octahedron.

References: [1] *Bricard R.* Mémoire sur la théorie de l'octaèdre articulé. J. Math. Pures et Appl. 1897. T. 5, No. 3., 113–148

[2] *Lebesgue H.* Octaèdres articulés de Bricard. Enseign. Math. Ser. 2. 1967. T. 13, No. 3, 175–185. (30 minutes talk)

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Anatoliy Milka: **Linear bendings of star-like pyramids.**

We present a recently discovered surprising property of right star-like bipyramid of A.D. Alexandrov and S.M. Vladimirova, which contradicts to actual principles of geometry and mechanics. On one hand, models of these polyhedra admit free continuous invertible large bendings without visible distortions of materials. On the other hand, the polyhedra in question are mathematically rigid and do not admit continuous mathematical bendings as polyhedra in the sense of A. Cauchy. Polyhedra which possess similar properties are called model flexors, whereas flexible polyhedra invented by R. Connelly and others are called theoretical flexors. ¿From A.V. Pogorelov's geometric theory of shells viewpoint bendings of models are asymptotically exactly approximated by linear bendings of polyhedra which are determined uniquely by some mechanical principles. ¿From viewpoint of V.I. Arnold's analytical theory of dynamical systems bendigs in question represent non-rigid, soft or retarded, loss of stability. It corresponds to the loss of stability "in small” in the sense of static criterion by L. Euler. This new phenomenon in mechanics of deformable solid bodies may be considered as an original geometric machine of catastrophe which supplements known physical models by E.C. Zeeman and T. Poston. (60 minutes talk)

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Anatoliy MILKA: **Unidentified Egyptian geometry.**

Theorems we a going to discuss are well-known in mathematics. They are related to the foundations of geometry, to the geometry “in the large” and to the history of geometry. Namely, we are dealing with three beautiful ancient theorems whose authors are Archimedes (theorem on the drop of a stone), Euclid and an Egyptian writer Akhmes (problems from Egyptian papyruses). It seems to be a paradox that the mentioned theorem by Euclid went unnoticed as a generalization of the fundamental uniqueness theorems by A. Cauchy and H. Minkowski about convex closed polyhedra. Three discussed theorems are absolutely flawless, but their theoretical and historical interpretations are still rather inadequate. In our opinion, these theorems belong to the ancient civilizations of Babylon, Egypt and Sumer, which were superior to our modern civilization by numerous aspects. This opinion will be confirmed by generalizations, proofs and a precise reconstruction of ancient theorems. (45 minutes lecture dedicated to the memory of Veniamin Fedorovich KAGAN)

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Gajane PANINA: **Hyperbolic virtual polytopes and related topics.**

Hyperbolic virtual polytopes arose originally as a discrete tool for advanced study of a long-standing uniqueness conjecture for smooth convex surfaces (A.D. Alexandrov’s problem). They proved to unite different topics in metric geometry, combinatorics, topology and PDE. Hyperbolic virtual polytopes are very peculiar phenomena: in some way, they play the role and have the properties of a non-existent object — of a closed saddle polytopal surface. Program of the mini-course:

LECTURE 1. Virtual polytopes. Hyperbolic virtual polytopes. Examples (a surprise was that hyperbolic polytopes exist. Another surprise is their diversity). The contraposition ”convex polytopes hyperbolic polytopes”. Colored graphs on the sphere, generated by hyperbolic polytopes.

LECTURE 2. Related topics: The uniqueness conjecture, extrinsic geometry of saddle surfaces. First counterexample to the conjecture (by Yves Martinez-Maure). Advanced counterexamples. Non-isotopic hyperbolic polytopes with 4 horns.

LECTURE 3. Topological elimination: spanning of saddle surfaces by special linkages (in the sense of O. and Yu. Viro). A.D. Alexandrov’s uniqueness theorem for convex polytopes and its refinements.

REFERENCES: [1] *A. Khovanskii* and *A. Pukhlikov*, Finitely additive measures of virtual

polytopes, St. Petersburg Math. J. **4**, No. 2 (1993), 337–356.
[2] *Y. Martinez-Maure*, Contre-exemple à une caractérisation conjecturée de la sphère, C.R. Acad. Sci. Paris **332**, No. 1 (2001), 41–44.
[3] *G. Panina*, New counterexamples to A.D. Alexandrov's uniqueness hypothesis, Advances in Geometry, no. 5 (2005), 301–317.
[4] *G. Panina*, On hyperbolic virtual polytopes and hyperbolic fans, CESJM (2006), to appear.
[5] *G. Panina*, Isotopy problems for saddle surfaces, ESI preprint, to appear.
(Three 60 minutes lectures)

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Paul PENNING: **Some observations on Bricard Octahedrons.**

An abstract of this 30 minutes talk is in process.

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András RECSKI: **One-dimensional synthesis of graphs as tensegrity frameworks** (joint work with Offer SHAI)**.**

The edge set of a graph G is partitioned into two subsets C and S. A tensegrity framework with underlying graph G and with cables for C and struts for S is proved to be rigidly embedable into a 1-dimensional line if and only if G is 2-edge-connected and every 2-vertex-connected component of G intersects both C and S. Polynomial algorithms are given to find an embedding of such graphs and to check the rigidity of a given 1-dimensional embedding. (30 minutes talk)

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Idzhad SABITOV: **On rigid and non-rigid surfaces with flat metrics in S^3.**

It is known that the most famous unsolved problem in the metric theory of surfaces goes back to Euler: to prove the unbendability of any compact surface in R^3 (after the results by Nash-Kuiper we have to precise the class of smoothness of surfaces and deformations in question). It turns out that in the spherical space S^3 this conjecture is not true: we prove that almost all flat tori in S^3 are bendable. The presentation of the talk will be open by some comments concerning the bellow conjecture for surfaces. (45 minutes talk)

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Idzhad SABITOV:
On some approach to the problem of Bonnet's pairs of surfaces.

We want to propose a very short way to find the explicit formulae for coefficients of second fundamental form of a surface in function of its metric and mean curvature. Here there are some exclusions which are ones also for the problem of Bonnet's pairs of surfaces. This problem concerns the unicity of the compact surface with a given mean curvature among all surfaces with the same metric. It is solved positivily for sphere-type surfaces but is still open for surfaces of higher topological genus. (30 minutes talk)

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Jean-Marc SCHLENKER: **Rigidity results for Euclidean polyhedra through hyperbolic geometry.**

Let P be a polyhedron in Euclidean space, with all vertices lying outside an ellipsoid which intersects all edges. Suppose moreover that the interior of P can be cut into simplices without adding any vertex. Then P is infinitesimally rigid. The proof is based on hyperbolic geometry, and uses some convexity properties of the volume of hyperideal simplices. This statement also leads to some open questions. If possible I would like to explain also why this statement is related to other rigidity statements for different geometric objects (circle patterns). (45 minutes talk)

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Brigitte SERVATIUS: **Assembling a graph from globally rigid blocks.**

We describe a canonical decomposition of a graph into globally rigid blocks. The structure of the graph can be encoded in a tree labelled with assembly instructions. Given a generic realization of a rigid graph in the plane we can compute the number of different realizations on the same edge lengths. (30 minutes talk)

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Offer SHAI: **The equivalence between static (rigid) and kinematic (mobile) systems through the graph theoretic duality.**

The duality of static systems is widely reported in the literature, for example: reciprocity [1]; the duality between weavings and tensegrity [2]; between grillages and trusses through projective geometry [3] and through graph theory duality [2], and many more.

In the talk it will be shown that constructing the graphs in a different manner yields that kinematic (mobile) systems correspond to the dual graphs, and these systems reflect the behavior, theorems and methodology of the original static systems. Obviously, the equivalence is valid for both directions.

Applying this duality relation shows that Maxwell's force diagram of an isostatic framework (statically rigid) is fully identical to the image velocity of the dual mobile system (linkage). It was proved by Maxwell that reciprocal diagram transforms a static system into another static system, while in this talk it will be shown how to follow the same idea but to get, this time, a kinematic system. In this duality relation, each force in the rod is proportional to the relative linear velocity in the corresponding dual linkage, and vice versa.

In order to reflect all the physical entities governing both the kinematic and static behaviors, the graphs were constructed through different processes which yielded new correlations such as the duality between linkages and pillar systems in 2D, and the relation between serial robots and Stewart-Platforms in 3D. This result correlates with the results achieved using projective geometry duality [4].

This equivalence between static and kinematic systems enables to reveal special properties that exist in the dual domain while not explicitly present in the original domain. For example, it enables to observe the existence of new entities in statics, such as the so-called face forces and equimometal lines, where the former is dual to the absolute linear velocity and the latter to instant centers in kinematics. These two entities were found to be useful in defining the geometry in which frameworks have self-stresses indicating on the relation existing between equimomental lines and the compatible cross-section for testing whether a line drawing is a correct projection of a spherical polyhedron [5].

In the talk, among other ideas that were borrowed from kinematics into rigidity, a subset of Laman graphs will be introduced, termed — 'Assur Graphs'. These graphs are minimal rigid graphs with respect to **vertices**. One of the unique properties of these graphs is a conjecture that for each one of the Assur graphs, although being a rigid graph satisfying $e(G) = 2 * v(G) - 3$ there is a special geometry in which the corresponding framework may possess a self stress in **all** the rods, while at the same time **all** the rods have infinitesimal motions.

References: [1] *Maxwell J.C.,* On reciprocal figures and diagrams of forces, Phil. Mag. Ser. (4) **27** (1864) 250–261.

[2] *Whiteley W.,* Ridigidity and polarity II: Weaving lines and tensegrity frameworks, Geom. Dedicata **30** (1989) 255–279.

[3] *Tarnai T.,* Duality between plane trusses and grillages, Int. J. of Solids and Structures **25**(12) (1989) 1395–1409.

[4] *Waldron K.J.* and *Hunt K.H.,* Series-parallel dualities in actively coordinated mechanisms, Int. J. of Robotics Research **10**(5) (1991) 473–480.

[5] *Whiteley W.,* Weaving, sections and projections of spherical polyhedra, Discrete Appl. Math. **32**(3) (1991) 275–294. (40 minutes)

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Rabah Souam: **TBA.**

An abstract of this talk is in process

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Hellmuth Stachel:
A proposal for a proper definition of higher-order rigidity.

A framework F is called *infinitesimally flexible of order* n, if it admits a nontrivial n-th order flex $X_0 + tX_1 + \ldots + t^n X_n$. F is called *infinitesimally rigid of order* n, if any n-th order flex of F is trivial. According to R. Connelly and H. Servatius [1] these definitions cause problems when a flex with trivial velocity distribution X_1 is already called trivial. But otherwise, any nontrivial first order flex $X_0 + tX_1$ of F gives rise to the nontrivial second order flex $X_0 + t^2 X_1$. How to escape this dilemma? The lecture focusses on two items: (1) It demonstrates how nontrivial flexes can be identified. (2) The representation of a nontrivial flex of F is not unique. As the definition of a flex of F is invariant under regular polynomial parameter transformations, the problem of reducible representations of an n-th order flex is adressed where the exponents of t have a common factor greater than 1. As a conclusion, the definitions of n-th order infinitesimal flexibility and rigidity must be based on irreducible representations of flexes.

References: [1] *Connelly R.* and *Servatius H.,* Higher-order rigidity — What is the proper definition? Discrete Comput. Geom. **11**, no. 2, (1994) 193–200.

[2] *Tarnai T.,* Higher order infinitesimal mechanisms, Acta Technica Acad. Sci. Hung. **102** (3–4) (1989) 363–378. (30 minutes talk)

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Dragutin Svrtan: **On circumradius equations for cyclic polygons.**

An abstract of this 60 minutes talk is in process.

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Zoltan Szabadka: **Globally linked pairs of vertices in equivalent realizations of graphs** (joint work with Bill Jackson and Tibor Jordan)**.**

A 2-dimensional *framework* (G, p) is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^2$. We view (G, p) as a straight line realization of G in $\mathbb{R}^2$. Two realizations of G are *equivalent* if the corresponding edges in the two frameworks have the same length. A pair of vertices $\{u, v\}$ is *globally linked* in G if the distance between the points corresponding to u and v is the same in all pairs of equivalent generic realizations of G. The graph G is *globally rigid* if all of its pairs of vertices are globally linked. We extend the characterization of globally rigid graphs given by the first two authors by characterizing globally linked pairs in M-connected graphs, an important family of rigid graphs. As a byproduct we simplify the proof of a result of Connelly which is a key step in the characterization of globally rigid graphs. (30 minutes talk)

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Tibor Tarnai:

Constrained circle packings on a sphere and their (Danzerian) rigidity.

Many situations in the natural sciences and technological applications can be modelled as variants of the packing problem — the determination of efficient packings of objects in an appropriate space. In one version of the packing problem, the task is to arrange n equal circles (spherical caps) without overlap on a sphere, so that their angular radius r is a maximum. In applications it can be more useful to deal with a constrained version of this spherical circle packing problem. For example, the arrangement of packed circles may be required to exhibit a particular overall point-group symmetry; or a given number of circles form a morphological unit, and identical copies of these units should be packed on a sphere. A survey of families of multisymmetric packings, in tetrahedral, octahedral and icosahedral groups will be presented, and the polymorphism of these packings will be analysed.

Additionally, three problems will be studied here: How must kN non-overlapping equal circles forming N units be packed on a sphere so that the angular diameter of the circles will be as large as possible under the constraint that, within each unit, the k circle centres lie

(1) for $k = 2$, at the end points of a diameter;

(2) for $k = 3$, at the vertices of an equilateral triangle inscribed in a great circle;

(3) for $k = 4$, at the vertices of a regular tetrahedron. Computer-generated putative solutions have been calculated and will be presented for different values of N.

Finally, for $k = 2$, the problem of packing of twin circles is investigated, where a twin is defined as two circles that are constrained to touch. Here solutions to the constrained problem are mainly expected as perfect matchings in the graphs of the solutions to the respective unconstrained problem.

For constructing locally optimal circle packings, the rigidity theory can be useful. In the cases (2) and (3), the Danzerian degree of freedom of packing is defined as $f = 3N - e - 2$ where e denotes the number of contacts. Danzer's 'almost' conjecture is applied: If $f > 0$ then the packing can be improved, or the opposite: If $f \leq 0$ and all contacts are in compression, then the packing is rigid and locally optimal. (Few exceptions, however, can occur.) If the circles are represented by their centres as vertices, and the contacts are represented by edges connecting the centres of two circles that are in contact (so forming the packing graph), then the Danzerian degree of freedom can be interpreted as the normal degree of freedom of the assembly (union of the graph and the underlying triangles or tetrahedra) that is increased by one. This additional degree of freedom is coming from a uniform concerted expansion of the edges of the packing graph. In the case of antipodal and twin packings, the investigation can be reduced to the analysis of the Danzerian rigidity of the packing graph. (45 minutes talk)

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Tibor TARNAI: **Detecting first-order infinitesimal mechanisms in bar-and-joint assemblies.**

This lecture is concerned with both statically and kinematically indeterminate bar-and-joint assemblies. It will investigate whether the kinematic indeterminacy appears in the form of a first-order or higher-order infinitesimal mechanism. It will be shown that the method by Calladine and Pellegrino based on positive stiffness and the method by Kuznetsov based on stability of self-stress can be synthesized in a unified approach based on the Hellinger-Reissner principle. In the case of a single state of self-stress, examples will be presented. (30 minutes talk)

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Ljubica VELIMIROVIĆ: **Infinitesimal bending of curves and surfaces.**

We are giving a brief historical overview of some results at the bending theory. Especialy, infinitesimal bending of the first, the second and the higher order for the toroidal rotational surfaces with polygonal meridian is given. Variation of the volume at the level of infinitesimal bending for rotational surfaces is examined. Infinitesimal bending of a closed plane curve and variation of some geometric magnitudes (curvature, torsion, area and the volume) in relation with infinitesimal bending of a curve are considered. We also consider graphical presentation of surface bending. (30 minutes talk)

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Ljubica VELIMIROVIĆ: **Rigidity and flexibility at non-symmetric affine connection space** (joint work with Svetislav MINČIĆ)**.**

We consider infinitesimal deformations $f : x^i \to x^i + \varepsilon z^i(x^j)$ of a space L_N with non-symmetric affine connection L^i_{jk}. Deformations of geometric objects (basic tensor, curvature tensor, connection coefficients) are given by virtue of Lie derivative. Using the fact of non-symmetry of the connection, we consider four kinds of covariant derivative to express the Lie derivative and the deformations of geometric objects at L_N. (30 minutes talk)

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Walter Whiteley: **Some observations from the projective theory of rigidity and from parallel drawing.**

Drawing on discussions at the Oberwolfach workshop on Discrete Differential Geometry pointed to several themes which are important for than analysis of flexibility and rigidity of surfaces. One is the underlying projective geometry of first-order rigidity. The second is the connections with a related geometry topic: parallel drawings. We propose two talks which review (i) the current projective theory of rigidity; and (ii) the theory of parallel drawing in 3-spaced.

For first-order rigidity; we describe how first-order motions can be represented in projective coordinates [1]. From this projective basis, we present simple transformations between first-order rigidity in Euclidean Space, Spherical Space, and Hyperbolic Space (metrics derived from the shared underlying projective geometry) [2]. We combine these transformations with a general averaging principle which applies in all of these metrics, to give a simple decomposition of Pogorelov's formulas for taking pairs of non-congruent but equivalent frameworks in one metric to another pair in any of the alternate metrics.

For parallel drawings, frameworks are redrawn with the new edges parallel to the original edges. The trivial redrawings are translations and dilations (a space of dimension $d+1$ in dimension d). We begin with the definition and rigidity style matrix. For the plane there is a surprising connection to first-order rigidity [3]. This is an old engineer trick arising in folklore techniques developed at the drafting tables in the 19th century. In 3-space, the connection to first-order motions is more tenous, but non-trivial parallel redrawings still lead to non-trivial first-order motions. In general, parallel drawing in any dimension is the projective dual of the concepts of lifting and projection for polyhedral scences into the next lower dimension. These objects have been studied in scene analysis and there are known fast combinatorial algorithms for determining the space of parallel drawings at generic configurations come from studies in that field [4]. Again, the parallel drawing properties of a configuration are projectively invariant (in Euclidean spaces) and this underlying projective geometry offers additional insight.

References: [1] *H. Crapo* and *W. Whiteley,* Statics of frameworks and motions of panel structures: a projective geometric introduction, Structural Topology **6** (1982) 43–82. Available from `http://www-iri.upc.es/people/ros/StructuralTopology/`

[2] *F. Saliola* and *W. Whiteley*, Some notes on the equivalence of first-order rigidity in various geometries, Preprint, York University (2003).

[3] *B. Servatius* and *W. Whiteley*, Constraining plane configurations in CAD: combinatorics of lengths and directions, SIAM J. Discrete Math. **12** (1999) 136–153.

[4] *W. Whiteley*, A matroid on hypergraphs, with applications in scene analysis and geometry, Disc. and Comp. Geometry **4** (1988) 75–95.

[5] *W. Whiteley*, Rigidity and scene analysis (revised chapter), in: Handbook of Discrete and Computational Geometry, J. Goodman and J. O'Rourke (eds.), second edition, 2004, 1327–1354. (Two 45 minutes lectures)

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