Scientific Report on the ESI-Program Rigidity and Flexibility

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Preprints contributed: [1791], [1796], [1869], [1868]

Report on the program

This ESI Program brought together highly qualified researchers working on different aspects of rigidity and flexibility. Such a convene is of high importance as the solution of problems in this field requires different techniques ranging from classical differential geometry to geometry 'in the large', convex geometry, algebraic geometry, geometric analysis, kinematics, combinatorics, scientific computations and algorithmics. This was the first international meeting dedicated to rigidity and flexibility.

The program was aimed at the study of various flexible and rigid structures such as flexible polyhedra and frameworks, polyhedral herissons and virtual polytopes, smooth herissons and smooth surfaces. To recall, a polyhedron is said to be flexible if its spatial form can be changed analytically with respect to a parameter while its intrinsic metric remains unaltered. A few outstanding rather recent results were: R. Connelly constructed a piecewise linear flexible embedding of the 2-sphere into the Euclidean 3-space E^3, a flexing sphere. R. Alexander proved that every flexible polyhedron in E^3 preserves its total mean curvature during the flex. And I. Sabitov proved the famous Bellows Conjecture stating that for every flexible polyhedron in E^3 the volume keeps constant during the flex. This surprising result is based on a generalized Heron formula: The squared volume of any triangulated sphere-like polyhedron is root of a polnomial with coefficients depending on the edge lengths only.

The main topics of the program were:

- Finite deformation of polyhedra
- Infinite bendings of polyhedra and surfaces
- Bricard's flexible octahedra
- Combinatorial methods for characterizing rigidity
- Hedgehog theory and virtual polyhedra
- On A.D. Alexandrov's theorem on the realization of Euclidean metrics
- Applications of rigidity

In the following we describe briefly some of the main subjects of the program.

1. **Finite deformation of polyhedra:**
   The meeting started with an overview on flexible polyhedra and an outlook to open problems (R. Connelly). J. Schlenker reported about how to use hyperbolic geometry for obtaining rigidity theorems on polyhedra in the Euclidean space. Until recent only a small number of flexible polyhedra is known. And each of these examples can be realized by a real-world model. A. Milka presented a series of almost flexible examples thus initiating a discussion on what flexibility means in our physical world. Another form of weakening flexibility is that of regarding polyhedra with boundary. H. Maehara gave an example of such a polyhedron which can be reversed. Bendings of polyhedra with boundary play an important role in discrete differential geometry for discrete integrable systems (T. Hoffmann). As a limiting case of a polyhedron with boundary, A. Dolbilin presented A. Tarasov's sophisticated solution of V.I. Arnold's problem: How to fold a rectangular sheet of paper into a flat polygon with larger perimeter.

2. **Infinite bendings of polyhedra and surfaces:**
   L. Velimirovic gave a historical overview of the theory of infinitesimal bendings of surfaces and discussed toroidal surfaces with polygonal meridian. I. Sabitov showed that all flat tori are bendable in the spherical space S^3 while these surfaces are unbendable in the Euclidean space. T. Hoffmann demonstrated the application of infinitesimal flexibility of polyhedra in discrete differential geometry. The fundamental question of how to define higher order rigidity or flexibility was adressed by H. Stachel.

3. **Combinatorial methods for characterizing rigidity:**
   The rigidity of generic 3D graphs is of high importance for studying the flexibility of molecules. The Molecular Conjecture, posed in 1984 by T-S. Tay and W. Whiteley, indicates that determining whether a molecular graph is rigid (or more generally, computing the rank of its rigidity matroid) may be tractable by combinatorial methods. Interesting results in this direction were presented by B. Jackson, T. Jordan, B. Servatius and Z. Szabadka. These methods consist mainly in decompositions of the graph into rigidity blocks. O. Shai addressed the equivalences between rigidity or flexibility of structures and particular properties of graphs. A. Recski presented polynomial algorithms for checking the rigidity of tensegrity frameworks.

4. **Bricard's flexible octahedra:**
   These most famous flexible polyhedra date back to Roul Bricard (1896). They are the basis for most of the known flexible polyhedra, in particular for Connelly's flexing sphere and for Steffen's flexible polyhedron. There are

three types to distinguish, one with a planar symmetry, one with line symmetry and one totally unsymmetric. Nevertheless, V. Alexandrov proved that also this third type has local symmetries at each vertex, and that all types fulfill the strong Bellows Conjecture stating the scissor congruence of any two flexes. P. Penning gave a summary on different kinematic properties of the relative motions showing up at these octahedra. And S. Mikhalev presented 7 independent conditions on the edge lengths of any octahedron for obtaining one of Bricard type 3.

5. **Hedgehog theory and virtual polyhedra:**
   There are attempts to weaken the demand of convexity in the classical Cauchy rigidity theorem. Recent examples are herissons (= hedgehogs) which can be seen as Minkowki differences of convex bodies. In an overview Y. Martinez-Maure explained the genesis of main results on herissons and listed a series of open problems in this fields. At the same time he proposed new tools for studying these open problems. G. Panina presented a mini-course on hyperbolic virtual polytopes with emphasis on A.D. Alexandrov's problem and on closed saddle polytopal surfaces.

6. **On A.D. Alexandrov's theorem:**
   The theorem under consideration states that any convex polyhedral metric on the 2-sphere can be realized as the boundary of a convex polytope in E^3. I. Izmestiev provided an algorithmic solution for this polyhedron. This theorem was also addressed in G. Panina's lecture series on hyperbolic virtual polytopes. F. Fillastre presented an analogous statement for a compact surface S of genus ≥ 2.

7. **Applications of rigidity:**
   The applications of rigidity theory in chemistry and in discrete differential geometry have already been mentioned above. Beyond these topics, there are applications, e.g., at circle packings on the sphere (T. Tarnai). Here rigidity characterizes optimal packings, and the Danzerian degree corresponds to the degree of freedom of the assembly. When addressing application in mechanics, T. Tarnai showed that different principles concerning stiffness or stability of self-stress at bar-and-joint assemblies can be unified using the Hellinger-Reissner principle. W. Whiteley focussed on relations between rigidity theory and projective geometry.

Invited scientists

Victor Alexandrov, Robert Connelly, Nikolaj Dolbilin, François Fillastre, Tim Hoffmann, Manfred Husty, Ivan Izmestiev, Bill Jackson, Tibor Jordan, Natalia Kopteva, Hiroshi Maehara, Yves Martinez-Maure, Sergej Mikhalev, Anatoliy Milka, Georg Nawratil, Gajane Panina, Paul Penning, András Recski, Otto

Röschel, Idzhad Sabitov, Jean-Marc Schlenker, Bernd Schultze, Brigitte Servatius, Offer Shai, Hellmuth Stachel, Dragutin Svrtan, Zoltan Szabadka, Tibor Tarnai, Johannes Wallner, Dominik Walter, Gunter Weiss, Walter Whiteley, Ljubica Velimirović.