Continuous deformations of polyhedra that do not alter the dihedral angles

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Abstract

We prove that, both in the Lobachevskij and spherical 3-spaces, there exist nonconvex compact boundary-free polyhedral surfaces without selfintersections which admit nontrivial continuous deformations preserving all dihedral angles and study properties of such polyhedral surfaces. In particular, we prove that the volume of the domain, bounded by such a polyhedral surface, is necessarily constant during such a deformation while, for some families of polyhedral surfaces, the surface area, the total mean curvature, and the Gauss curvature of some vertices are nonconstant during deformations that preserve the dihedral angles. Moreover, we prove that, in the both spaces, there exist tilings that possess nontrivial deformations preserving the dihedral angles of every tile in the course of deformation.
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1 Introduction

We study polyhedra (more precisely, boundary-free compact polyhedral surfaces) the spatial shape of which can be changed continuously in such a way that all dihedral angles remain constant.

These polyhedra may be considered as a natural 'dual object' for the flexible polyhedra. The latter are defined as polyhedra whose spatial shape can be changed continuously due to changes of their dihedral angles only, i. e., in such a way that every face remains congruent to itself during the flex. Since 1897, it was shown that flexible polyhedra do exist and have numerous nontrivial properties. Many authors contributed to the theory of flexible polyhedra, first of all we should mention R. Bricard, R. Connelly, I.Kh. Sabitov, H. Stachel, and A.A. Gaifullin. For more details, the reader is referred to the survey article [9] and references given there.

In 1996, M.Eh. Kapovich [6] brought our attention to the fact that polyhedra, admitting nontrivial deformations that keep all dihedral angles fixed, may be of some interest in the theory of hyperbolic manifolds, where Andreev's theorem [1] plays an important role. The latter reads that, under the restriction that the dihedral angles must be nonobtuse, a compact convex hyperbolic polyhedron is uniquely determined by its dihedral angles.

The case of the Euclidean 3-space is somewhat special and we do not study it here. The reader may consult [7] and references given there to be acquainted with the progress in solving old conjectures about unique determination of Euclidean polytopes by their dihedral angles that may be dated back to J.J. Stoker's paper [11].

Coming back to continuous deformations of hyperbolic or spherical polyhedra which leave the dihedral angles fixed, we can immediately propose the following example. Consider the boundary P of the union of a convex polytope Q and a small tetrahedron T (the both are treated as solid

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bodies for a moment) located so that (i) a face τ of T lies inside a face of Q and (ii) T and Q lie on the different sides of the plane containing τ.

Obviously, the nonconvex compact polyhedron P has no selfintersections and admits nontrivial (i.e., not generated by a rigid motion of the whole space) continuous deformations preserving all dihedral angles. In order to construct such a deformation we can keep Q fixed and continuously move (e.g., rotate) T in such a way that the conditions (i) and (ii) are satisfied. In this example, many quantities associated with P remain constant. To name a few, we can mention

• the volume;

• the surface area;

• the Gauss curvature of every vertex (i.e., the difference between 2π and the sum of all plane angles of P incident to this vertex);

• the total mean curvature of P (i.e., the sum

$$\frac{1}{2}\sum_{\ell}\big(\pi-\alpha(\ell)\big)|\ell| \tag{1}$$

calculated over all edges ℓ of P, where $\alpha(\ell)$ stands for the dihedral angle of P attached to ℓ and $|\ell|$ for the length of ℓ);

• every separate summand $\big(\pi-\alpha(\ell)\big)|\ell|$ in (1).

In the Lobachevskij and spherical 3-spaces, we study whether the above example provides us with the only possibility to construct nonconvex compact polyhedra that admit nontrivial continuous deformations preserving all dihedral angles and prove that the answer is negative.

We study also what quantities associated with nonconvex compact polyhedra necessarily remain constant during such deformations and show that the volume of the domain bounded by the polyhedral surface is necessarily constant, while the surface area, the total mean curvature, and the Gauss curvature of a vertex may be nonconstant.

At last, we prove that there exist tilings that possess nontrivial deformations leaving the dihedral angles of every tile fixed in the course of deformation. Here we use a construction originally proposed (but never published) in the 1980th by A.V. Kuz'minykh at the geometry seminar of A.D. Alexandrov in Novosibirsk, Russia, which was proposed for the study of a similar problem for flexible polyhedra.

2 Polyhedra in the Lobachevskij 3-space

THEOREM 1. *In the Lobachevskij 3-space, there exists a nonconvex sphere-homeomorphic polyhedron P with the following properties*:

1. *P has no selfintersections*;

2. *P admits a continuous family of nontrivial deformations which leave the dihedral angles fixed*;

3. *the surface area of P is nonconstant during the deformation*;

4. *the total mean curvature of P is nonconstant during the deformation*;

5. *the Gauss curvature of some vertex of P is nonconstant during the deformation.*

Proof is divided in a few steps.

Step I: Let S be a unit 2-sphere in the Lobachevskij 3-space, $C\subset S$ be a circle on the 2-sphere, and $a\in S$ be a point lying outside the convex disk on S bounded by C. Draw two geodesic lines L^1 and L^2 on S that pass through a and are tangent to C (see Figure 1). Now rotate the figure $L^1\cup L^2$ to an arbitrary angle around the center of C and denote the image of L^j by $\overline{L^j}$, $j=1,2$. Let c be the image of the point a under the above rotation, $b=L^1\cap\overline{L^1}$, and $d=L^2\cap\overline{L^2}$. The resulting configuration is schematically shown on Figure 1.

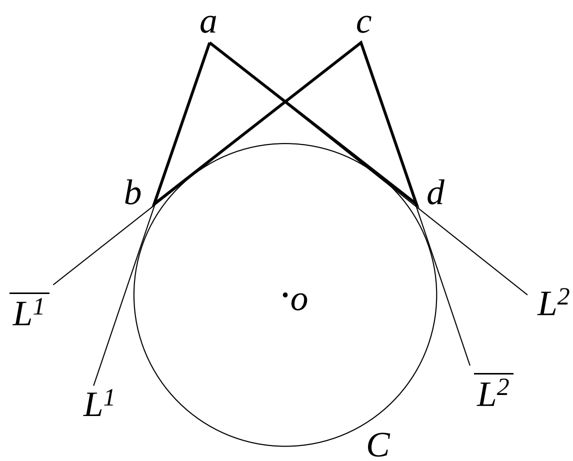

Figure 1: Constructing a spherical antiparallelogram *abcd*

As a result, we get an antiparallelogram *abcd* on S that is circumscribed around the circle C. We call the selfintersecting quadrilateral *abcd* the antiparallelogram because its opposite sides have equal lengths and we say that it is circumscribed around the circle C because its sides lie on the lines that are tangent to C.

Step II: Let C be the circle from Step I, $o \in S$ be the center of C, and r_C be the radius of C. Consider a continuous family of circles $C_r \subset S$ such that

(a) o is the center of C_r for every r;

(b) C_r has radius r for every r;

(c) the circle C belongs to this family.

For every r, we repeat Step I. More precisely, we first select a continuous family of points a_r such that the angle between the two (geodesic) lines L_r^1 and L_r^2 on S, that pass through a_r and are tangent to C_r, is equal to the angle between the two (geodesic) lines on S that pass through a_{r_C} and are tangent to the circle $C = C_{r_C}$. Then we select such a rotation of the figure $L_r^1 \cup L_r^2$ around the center of C_r that the angle between the lines L_r^1 and $\overline{L_r^1}$ is equal to the angle between the lines $L^1 = L_{r_C}^1$ and $\overline{L^1} = \overline{L_{r_C}^1}$. Here $\overline{L_r^j}$, $j = 1, 2$, stands for the image of the line L_r^j under the rotation. At last, let c_r be the image of the point a_r under the above rotation, $b_r = L_r^1 \cap \overline{L_r^1}$, and $d_r = L_r^2 \cap \overline{L_r^2}$.

As a result, we obtain a continuous family of antiparallelograms $a_r b_r c_r d_r$ on S such that, for every r, $a_r b_r c_r d_r$ is circumscribed around the circle C_r and has the same angles as *abcd*. The fact that the angles at the vertices a_r, b_r, and c_r are equal to the corresponding angles of the antiparallelogram *abcd* at the vertexes a, b, and c, obviously, holds true by construction. Due to the symmetry of $a_r b_r c_r d_r$, the angle at the vertex d_r is equal to the angle at the vertex b_r and, thus, is equal to the angle of the antiparallelogram *abcd* at the vertices b and d.

Step III: Let $a_r b_r c_r d_r$ be an antiparallelogram constructed in Step II that lies on the sphere S, is circumscribed around the circle C_r of radius r, depends continuously on r, and whose angles at the vertices a_r, b_r, c_r, and d_r are independent of r.

Consider an infinite cone K_r over the antiparallelogram $a_r b_r c_r d_r$ with apex at the center O of the sphere S. Draw a plane $\pi(s)$ in the 3-space that is perpendicular to the line joining O with the center o of the circle C_r and such that $\pi(s)$ is at distance s from O. We claim that there is a continuous function $s(r)$ such that the value β of the dihedral angle between the plane Π_r of the triangle $a_r b_r O$ and the plane $\pi(s(r))$ is independent of r.

In fact, this statement immediately follows from the trigonometric relations for hyperbolic right triangles. Consider the hyperbolic right triangle Opq, where p stands for the nearest point of the plane $\pi(s(r))$ to the point O and q stands for the nearest point of the line $\pi(s(r))\cap\Pi_r$ to the point O. Then

$$\cos\beta = \cosh s(r)\sin r, \tag{2}$$

see, e.g., [8, formula (3.5.16)]. Using this equation, we find $s(r)$.

Since the antiparallelogram $a_r b_r c_r d_r$ is circumscribed around the circle C_r, it follows that the dihedral angle between the plane $\pi(s(r))$ and each of the planes containing one of the triangles $b_r c_r O$, $c_r d_r O$, and $a_r d_r O$, is equal to β and, in particular, is independent of r.

Step IV: Consider a continuous family of cones K_r constructed in Step III. Let σ_r be the half-space determined by the plane $\pi(s(r))$ such that $O\in\sigma_r$. By definition, put $K_r^+ = K_r\cap\sigma_r$. Let K_r^- be obtained from K_r^+ by reflecting it in the plane σ_r and let $M_r = K_r^+\cup K_r^-$.

For every r, M_r is a boundary-free polyhedral surface with selfintersections that is combinatorially equivalent to the surface of the regular octahedron.

The reader familiar with the theory of flexible polyhedra [9] can observe that the construction of M_r has very much in common with the construction of the Bricard octahedra of type II.

In the next Step we will finalize the construction of a selfintersection-free polyhedron, whose existence is proclaimed in Theorem 1, using the trick that was originally proposed by R. Connelly in the construction of his famous selfintersection-free flexible polyhedron [3].

Step V: Let $\widetilde{a}_r$ be the point of the intersection of the line $a_r O$ and the plane $\pi(s(r))$. Similarly we define the points $\widetilde{b}_r$, $\widetilde{c}_r$, and $\widetilde{d}_r$. Let $\widetilde{O}_r$ be symmetric to the point O with respect to the plane $\pi(s(r))$. Note that the points O, $\widetilde{a}_r$, $\widetilde{b}_r$, $\widetilde{c}_r$, $\widetilde{d}_r$, and $\widetilde{O}_r$ are the vertices of the polyhedron M_r.

Observe that, if we remove the triangles $\widetilde{a}_r\widetilde{d}_r O$ and $\widetilde{a}_r\widetilde{d}_r\widetilde{O}_r$ from the polyhedron M_r, we get a disk-homeomorphic selfintersection-free polyhedral surface. Denote it by N_r.

Recall that the dihedral angle between the triangles $\widetilde{a}_r\widetilde{d}_r O$ and $\widetilde{a}_r\widetilde{d}_r\widetilde{O}_r$ is equal to 2β, where β is the dihedral angle between the plane Π_r of the triangle $a_r b_r O$ and the plane $\pi(s(r))$ constructed in Step III. In particular, this dihedral angle is independent of r.

Now, consider a tetrahedron $T = WXYZ$ such that T is sufficiently large in comparison with the dimensions of the polyhedron N_r and the dihedral angle of T, attached to the edge XY, is equal to 2β. Let us select the points x_r, y_r, z_r and w_r such that

(1) x_r lies on the geodesic segment XY sufficiently far from its end-points and depends continuously on r;

(2) y_r lies on the geodesic segment XY, depends continuously on r, and the distance between x_r and y_r is equal to the distance between O and $\widetilde{O}$;

(3) z_r belongs to the triangle XYZ and its distances from the points x_r and y_r are equal to the distance between the points $\widetilde{a}_r$ and O (and, thus between the points $\widetilde{a}_r$ and $\widetilde{O}_r$);

(4) w_r belongs to the triangle XYW and its distances from the points x_r and y_r are equal to the distance between the points $\widetilde{d}_r$ and O.

From the fact that the dihedral angle of T, attached to the edge XY, is equal to 2β and the conditions (2)–(4), it obviously follows that there is an isometry φ of the Lobachevskij 3-space such that $\varphi(\widetilde{O}_r) = x_r$, $\varphi(O) = y_r$, $\varphi(\widetilde{d}_r) = z_r$, and $\varphi(\widetilde{a}_r) = w_r$.

For every r, let us remove the triangles $x_r y_r z_r$ and $x_r y_r w_r$ from the polyhedral surface T and replace the union of these triangles by the polyhedral surface $\varphi(N_r)$, see Figure 2. Denote the resulting polyhedral surface by P_r. We may also describe this transformation of T into P_r as follows: first, we produce a quadrilateral hole on the polyhedral surface T and, second, we glue this hole with an isometric copy of the polyhedral surface N_r.

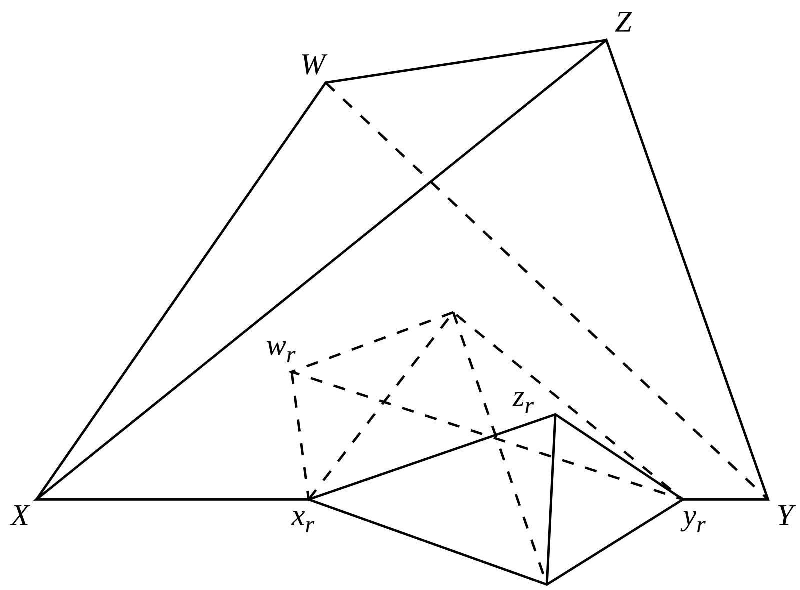

Figure 2: Polyhedral surface P_r

Obviously, for some open interval $I \subset \mathbb{R}$, the family $\{P_r\}_{r\in I}$ is a continuous family of nonconvex sphere-homeomorphic selfintersection-free polyhedral surfaces such that every dihedral angle of P_r is independent of r.

In the rest part of the proof we show that all the statements of Theorem 1 are fulfilled for any polyhedron $P = P_r$, $r \in I$, i.e., that, as we vary r, the deformation of P_r is nontrivial, the surface area and total mean curvature of P_r as well as the Gauss curvature of some vertex of P_r are nonconstant in r.

Step VI: In order to prove that the above constructed deformation of the polyhedral surface P_r is nontrivial, it is sufficient to prove that the (spatial) distance between the points x_r and y_r is not constant in r.

Observe that this distance is equal to $2s(r)$, where the function $s(r)$ is defined in Step III as the distance from the point O to a plane. In particular, the function $s(r)$ satisfies the equation (2) and, obviously, is nonconstant on every interval of the reals.

Thus, the deformation of the polyhedral surface P_r is nontrivial.

Step VII: Let's prove that the surface area of P_r is not constant for $r \in I$.

According to Step V, for $r \in I$, P_r is obtained from T by replacing the union of triangles $x_ry_rz_r$ and $x_ry_rw_r$ with an isometric copy of the polyhedral surface N_r. The latter consists of the triangle $\widetilde{O}_rO\widetilde{d}_r$ (which is isometric to the triangle $x_ry_rz_r$), the triangle $\widetilde{O}_rO\widetilde{a}_r$ (which is isometric to the triangle $x_ry_rw_r$), and four mutually isometric triangles (each of which is isometric to the triangle $O\widetilde{a}_r\widetilde{b}_r$) whose surface area S_r is positive. Hence, the surface area of P_r exceeds the surface area of T by the strictly positive number $4S_r$.

Using the notation of Step III, we can say that the distance between the points x_r and y_r is equal to $2s(r)$ and satisfies the equation (2), namely, $\cos\beta = \cosh s(r)\sin r$, where 2β stands for the dihedral angle of P_r at the edge $\widetilde{a}_r\widetilde{b}_r$. Taking into account that β is independent of r, pass to the limit in the equation (2) as $r \to (\pi/2 - \beta) - 0$. As a result we get $s(r) \to 0$. Moreover, if we consider the hyperbolic right triangle Opq from Step III, we conclude that the distance between every pair of the vertices of the polyhedral surface N_r tends to zero as $r \to (\pi/2 - \beta) - 0$. Hence, for all values of the parameter r that are less than $\pi/2 - \beta$ but sufficiently close to $\pi/2 - \beta$, the surface area of P_r is arbitrarily close to the surface area of T.

So, we see that the difference of the surface area of P_r and the surface area of T is strictly positive for every $r \in I$ and tends to zero as $r \to (\pi/2 - \beta) - 0$. Hence, the surface area of P_r is nonconstant. As far as this function is analytic in r, it is nonconstant on every interval, in particular, on I. Thus the surface area of P_r is nonconstant for $r \in I$.

Step VIII: The proof of the fact that the total mean curvature of the polyhedral surface P_r is nonconstant for $r \in I$ is similar to Step VII. More precisely, we observe that, for a given $r \in I$, the total mean curvature of P_r is strictly greater than the total mean curvature of T but their difference tends to zero as $r \to (\pi/2 - \beta) - 0$ and, thus, is nonconstant.

Step IX: Here we prove that the Gauss curvature of the vertex $y_r \in P_r$ is nonconstant in r.

Recall that the Gauss curvature of the vertex y_r is equal to the difference between 2π and the sum of all plane angles of P incident to y_r.

Using the notation introduced in Step V and the fact that P_r is obtained from T by replacing the union of the triangles $x_r y_r z_r$ and $x_r y_r w_r$ with an isometric copy of the polyhedral surface N_r, we observe that the Gauss curvature of P_r at the vertex y_r is equal to $-2\angle \widetilde{a}_r O \widetilde{b}_r = -2 a_r b_r$, where $\angle \widetilde{a}_r O \widetilde{b}_r$ stands for the angle of the triangle $\widetilde{a}_r O \widetilde{b}_r$ attached to the vertex O and $a_r b_r$ stands for the spherical distance between the points a_r, b_r on the unit sphere. This means that all we need is to prove that $a_r b_r$ is nonconstant in r.

Arguing by contradiction, suppose we have two quadrilaterals $a_r b_r c_r d_r$ and $a_t b_t c_t d_t$, constructed according to Step I, such that $a_r b_r = a_t b_t$. Since the corresponding angles of $a_r b_r c_r d_r$ and $a_t b_t c_t d_t$ are equal to each other, we conclude that these quadrilaterals are mutually congruent. Hence, their circumradii are equal to each other, i.e., $r = t$. Thus, $a_r b_r \neq a_t b_t$ for $r \neq t$ and the Gauss curvature of the vertex $y_r \in P_r$ is nonconstant in r. This completes the proof of Theorem 1. $\square$

THEOREM 2. *For every compact bounary-free oriented polyhedral surface P in the Lobachevskij 3-space and every smooth deformation that leaves the dihedral angles of P fixed, the volume bounded by P remains constant in the course of deformation.*

Proof. Let $\{P_r\}_{r \in I}$ be a smooth deformation of P leaving the dihedral angles fixed and let ℓ_r^j, $j = 1, \dots, J$, stand for the edges of the polyhedron P_r. If we denote the length of the edge ℓ_r^j by $|\ell_r^j|$ and the dihedral angle of P_r attached to ℓ_r^j by α_r^j then the classical Schläfli differential formula [10] reads as follows

$$\frac{d}{dr} \operatorname{vol} P_r = -\frac{1}{2} \sum_j |\ell_r^j| \frac{d}{dr} \alpha_r^j. \tag{3}$$

This completes the proof of Theorem 2 since, by assumption, $\frac{d}{dr}\alpha_r^j = 0$ for all $j = 1, \dots, J$. $\square$

REMARK. The statement and the proof of Theorem 2 hold true both for polyhedra with or without selfintersections. For more details, including a formal definition of a polyhedron with selfintersections, the reader is referred to the theory of flexible polyhedra [9].

THEOREM 3. *In the Lobachevskij 3-space, there exists a tiling composed of congruent polyhedral tiles that possesses a nontrivial continuous deformation such that, in the course of deformation, the dihedral angles of every tile are left fixed, and the union of the deformed tiles produces a tiling composed of congruent polyhedral tiles again.*

Proof makes use of the so-called Böröczky tiling of the Lobachevskij 3-space by congruent polyhedra and, for the reader's convenience, we start with a short description of this tiling. Figure 3 illustrates the construction of the Böröczky tiling in the upper half-space model.

Suppose the Lobachevskij 3-space has curvature -1 and let Σ_0 be an horosphere. It is well-known that Σ_0 is isometric to the Euclidean plane. Let π_0 be an edge-to-edge tiling of Σ_0 with pairwise equal geodesic squares with edge length 1. Fix one of the squares of π_0 and denote its vertices by A_0, B_0, C_0, and D_0 as shown in Figure 3. Let γ be an oriented line through A_0 that is

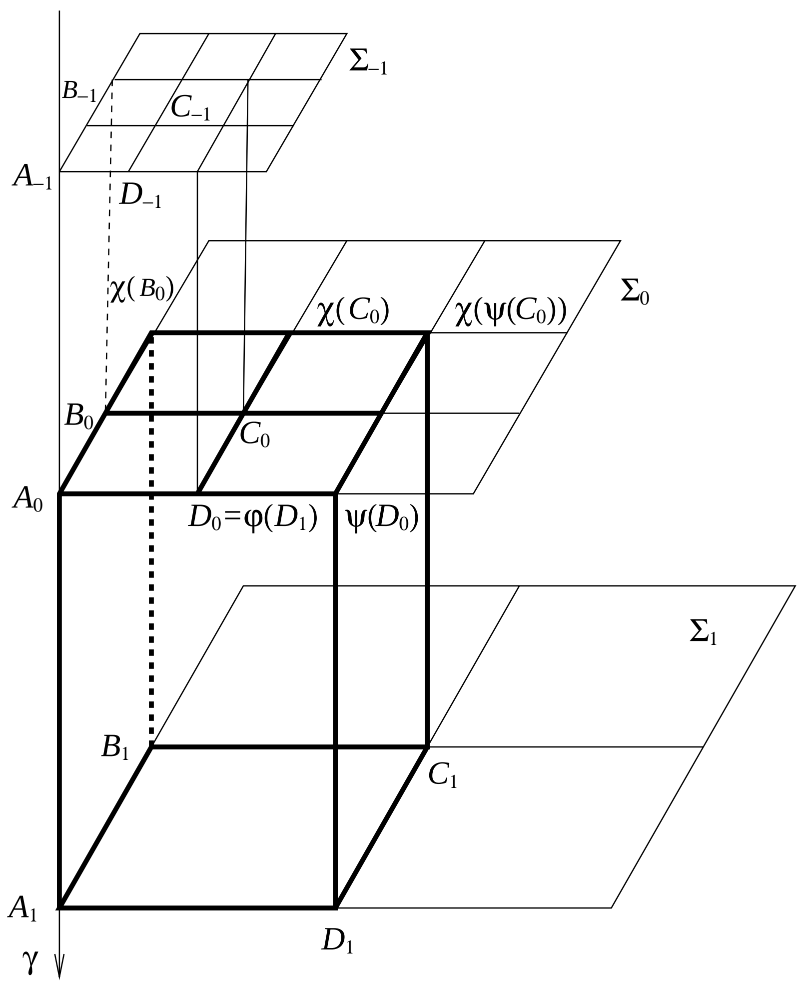

Figure 3: Constructing the Böröczky tiling in the upper half-space model of the Lobachevskij 3-space. The tile $\varkappa$ is shown by bold lines

orthogonal to Σ_0. Starting from A_0, place points A_k, $k \in \mathbb{Z}$, on γ such that A_k precedes A_{k+1} on the oriented line γ and the hyperbolic distance between A_k and A_{k+1} is equal to $\ln 2$ for all $k \in \mathbb{Z}$. Through every point A_k draw an horocycle Σ_k orthogonal to γ.

Let φ be an orientation preserving isometric mapping of the Lobachevskij 3-space onto itself that maps the point A_0 into the point A_{-1}, maps the line γ onto itself and maps the plane through γ and B_0 onto itself. Starting from the tiling π_0 of the horosphere Σ_0, define the tiling π_k of the horosphere Σ_k by putting $\pi_k = \varphi(\pi_{k+1})$ for all $k \in \mathbb{Z}$. Note that $A_k = \varphi(A_{k+1})$ for all $k \in \mathbb{Z}$. By definition, put $B_k = \varphi(B_{k+1})$, $C_k = \varphi(C_{k+1})$, and $D_k = \varphi(D_{k+1})$.

Let ψ (respectively, χ) be an orientation preserving isometry of the Lobachevskij 3-space onto itself that maps the square $A_0B_0C_0D_0$ onto its neighbour in the tiling π_0 in such a way that $\psi(A_0) = D_0$ and $\psi(B_0) = C_0$ (respectively, $\chi(A_0) = B_0$ and $\chi(D_0) = C_0$).

A cell of the Böröczky tiling is a (nonconvex solid) polyhedron with the following 13 vertices: A_1, B_1, C_1, D_1, A_0, B_0, C_0, D_0, $\psi(C_0)$, $\psi(D_0)$, $\chi(B_0)$, $\chi(C_0)$, and $\chi(\psi(C_0)) = \psi(\chi(C_0))$. Its combinatorial structure is shown in Figure 3 with bold lines. Denote this polyhedron by $\varkappa$.

The main observation, allowing to build the Böröczky tiling is that, since the hyperbolic distance between the points A_0 and A_1 is equal to $\ln 2$ and the curvature of the space is equal to -1, the Euclidean distance between the points A_0 and $\psi(D_0)$ (measured in the Euclidean plane Σ_0) is twice the Euclidean distance between the points A_0 and D_0 and, thus, twice the distance between the points A_1 and D_1. Similarly, the Euclidean distance between the points A_0 and $\chi(B_0)$ is twice the Euclidean distance between the points A_0 and B_0. If we observe now that, for every $k \in \mathbb{Z}$, φ maps the tiling π_k of the horosphere Σ_k onto the tiling π_{k-1} of the horosphere Σ_{k-1} and both ψ and χ map the tiling π_k of the horosphere Σ_k onto itself, we can complete the description of the Böröczky tiling as follows.

We apply iterations φ^k, $k \in \mathbb{Z}$, of the hyperbolic isometry φ to the polyhedron $\varkappa$ and get a sequence of polyhedra $\varphi^k(\varkappa)$ whose 9 'upper' vertices lie on the horosphere Σ_k (and belong to the set of the vertices of the tiling π_k) and 4 'bottom' vertices lie on the horosphere Σ_{k+1} (and belong to the set of the vertices of the tiling π_{k+1}). Then we fix $k \in \mathbb{Z}$ and apply iterations ψ^p, $p \in \mathbb{Z}$, and χ^q, $q \in \mathbb{Z}$, of the isometries ψ and χ to the polyhedron $\varphi^k(\varkappa)$. As a result we get a tiling of a 'polyhedral layer' with vertices on Σ_k and Σ_{k+1}. These 'polyhedral layers' fit together and produce the Böröczky tiling of the whole 3-space.

In short, we can say that the Böröczky tiling is produced from the cell $\varkappa$ by isometries φ, ψ, and χ.

For more details about the Böröczky tiling the reader may consult [4] and references given there.

We now turn to the proof of Theorem 3 directly. Let's construct a polyhedral surface $P \subset \mathbb{S}^3$ that is described in the statement of Theorem 1 such that all dimensions of P are sufficiently small in comparison with the size of the polyhedron $\varkappa$. For the vertices of P, we use the notations from Step V of the proof of Theorem 1 (or, identically, from Figure 2).

On the face $A_1A_0B_0\chi(B_0)B_1$ of the boundary $\partial\varkappa$ of the solid polyhedron $\varkappa$, find a (hyperbolic) triangle Δ that is congruent to the triangle WYZ of the polyhedral surface P (see Figure 2) and lies sufficiently far from all the vertices of $\varkappa$. Remove Δ from the polyhedral surface $\partial\varkappa$ and glue the hole obtained by an isometric copy Σ of the disk-homeomorphic polyhedral surface remaining after the removal of WYZ from the polyhedral surface P. From the resulting sphere-homeomorphic surface remove a triangle $\psi(\Delta)$ and glue the hole obtained by the disk-homeomorphic surface $\psi(\Sigma)$. Denote the resulting sphere-homeomorphic polyhedral surface by $\partial\overline{\varkappa}$.

The compact part $\overline{\varkappa}$ of the Lobachevskij 3-space bounded by the sphere-homeomorphic polyhedral surface $\partial\overline{\varkappa}$ is the cell of the tiling whose existence is proclaimed in Theorem 3. The tiling itself is produced from the cell $\overline{\varkappa}$ by isometries φ, ψ, and χ precisely in the same way as these isometries produce the Böröczky tiling from its cell $\varkappa$. The nontrivial continuous deformation that preserves the dihedral angles of the cell is produced by the corresponding deformation of P. $\square$

3 Polyhedra in the spherical 3-space

In the spherical 3-space $\mathbb{S}^3$ we may prove the same statements as in the Lobachevskij 3-space. For the reader's convenience, below we formulate Theorems 4–6 that hold true in the spherical 3-space and are similar to Theorems 1–3 and give brief comments on their proofs.

THEOREM 4. *In the spherical 3-space, there exists a nonconvex sphere-homeomorphic polyhedron P with the following properties*:

1. *P has no selfintersections*;
2. *P admits a continuous family of nontrivial deformations which leave the dihedral angles fixed*;
3. *the surface area of P is nonconstant during the deformation*;
4. *the total mean curvature of P is nonconstant during the deformation*;
5. *the Gauss curvature of some vertex of P is nonconstant during the deformation.*

Proof is similar to the proof of Theorem 1 with an obvious replacement of theorems of hyperbolic trigonometry by the corresponding theorems of spherical trigonometry. $\square$

THEOREM 5. *For every compact bounary-free oriented polyhedral surface P in the spherical 3-space and every smooth deformation that leaves the dihedral angles of P fixed, the volume bounded by P remains constant in the course of deformation.*

Proof follows directly from the classical Schläfli differential formula for the spherical 3-space [10], which may be obtained from the formula (3) if we multiplay the right-hand side by -1. $\square$

REMARK. Note that polyhedra admitting continuous deformations do not changing dihedral angles are not applicable for the study of the following open problem [2, 5]: *prove that if all dihedral angles of a polyhedron in the spherical 3-space $\mathbb{S}^3$ are rational multiples of π then the volume of this polyhedron is a rational multiple of π^2*. One may be tempted to construct a polyhedron in $\mathbb{S}^3$, with all dihedral angles being rational multiples of π, admitting continuous deformations that preserve its dihedral angles but change its volume. In this case, the volume, being a continuous function, takes every value in some interval of real numbers and, among others, values that are not rational multiples of π^2. Hence, this argument, if it is correct, will easily result in a negative solution to the above problem. Nevertheless, Theorem 5 shows that this argument is not applicable because the volume is necessarily constant.

THEOREM 6. *In the spherical 3-space $\mathbb{S}^3$, there exists a tiling composed of congruent polyhedral tiles that possesses a nontrivial continuous deformation such that, in the course of deformation, the dihedral angles of every tile are left fixed, and the union of the deformed tiles produces a tiling composed of congruent polyhedral tiles again.*

Proof is even simpler then the proof of Theorem 3 because now we can use a finite partition of $\mathbb{S}^3$ instead of the Böröczky tiling of the Lobachevskij 3-space.

For example, let's use the tiling of $\mathbb{S}^3$ with 12 mutually congruent polyhedra obtained in the following way. Let's treat $\mathbb{S}^3$ as the standard unit sphere in $\mathbb{R}^4$ centered at the origin. Let $N = (0,0,0,1)$, $S = (0,0,0,-1)$ and L be the subspace of $\mathbb{R}^4$ orthogonal to the vector $(0,0,0,1)$. By definition, put $\mathcal{S} = \mathbb{S}^3 \cap L$. Then $\mathcal{S}$ is a unit 2-sphere located in the 3-space L. Let $C \subset L$ be a 3-dimensional cube inscribed in $\mathcal{S}$. Then the images of the 2-faces of C under the central projection from the origin $(0,0,0,0)$ into the 2-sphere $\mathcal{S}$ form a tiling of $\mathcal{S}$ with 6 mutually congruent spherical convex polygons that we denote by T_j, $j = 1,\dots,6$. Joining each vertex of T_j with N we get six mutually congruent spherical convex polyhedra T_j^N, $j = 1,\dots,6$. Similarly, joining each vertex of T_j with S we get six mutually congruent spherical convex polyhedra T_j^S, $j = 1,\dots,6$. Obviously, 12 mutually congruent polyhedra T_j^N, T_j^S, $j = 1,\dots,6$, tile the spherical 3-space $\mathbb{S}^3$.

Observe that, for every $j = 1, \dots, 6$, the two polyhedra T_j^N and T_j^S are centrally symmetric to each other with respect to the center of symmetry O_j of the polygon T_j.

Let's construct a polyhedron $P \subset \mathbb{S}^3$ that is described in Theorem 4 such that all dimensions of P are sufficiently small in comparison with the size of T_1^N. For the vertices of P, we use the same notations as at Step V of the proof of Theorem 1 (or, equivalently, as on Figure 2).

On the polygon T_1, find a (spherical) triangle Δ that is congruent to the triangle WYZ of the polyhedral surface P (see Figure 2) and lies sufficiently far from all the vertices of T_1^N and from the point O_1. Remove the triangle Δ from T_1^N and glue the hole obtained by an isometric copy Σ of the disk-homeomorphic polyhedral surface remaining after the removal of WYZ from P. From the resulting sphere-homeomorphic surface remove a triangle Δ' that is symmetric to Δ with respect to the point O_1 and glue the hole obtained by a disk-homeomorphic surface that is symmetric to Σ with respect to O_1. Denote the resulting sphere-homeomorphic surface by $\overline{T_1^N}$.

Denote by $\overline{T_1^S}$ the sphere-homeomorphic surface that is symmetric to $\overline{T_1^N}$ with respect to O_1.

Let $\varphi_1 : \mathbb{S}^3 \to \mathbb{S}^3$ be the identity mapping and, for every $j = 2, \dots, 6$, $\varphi_j : \mathbb{S}^3 \to \mathbb{S}^3$ be an isometry such that $\varphi_j(T_1) = T_j$, $\varphi_j(N) = N$ and $\varphi_j(S) = S$.

Consider the union of 12 sphere-homeomorphic surfaces $\varphi_j(\overline{T_1^N})$ and $\varphi_j(\overline{T_1^S})$, $j = 1, \dots, 6$. Obviously, these surfaces define a tiling of $\mathbb{S}^3$ by pairwise congruent polyhedral tiles that possesses a nontrivial continuous deformation (which is produced by the corresponding deformation of P) such that, in the course of deformation, the dihedral angles of every tile are left fixed, and the union of the deformed tiles produces a tiling composed of congruent tiles again. This concludes the proof of Theorem 6. □

4 Final remark

Polyhedra that admit continuous deformations leaving all the dihedral angles unaltered are worth studying as much as the flexible polyhedra.

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