On Orthogonality-preserving Plücker transformations of Hyperbolic Spaces

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Abstract

A complete overview of all orthogonality-preserving Plücker transformations in finite dimensional hyperbolic spaces with dimension other than three is given. In the Cayley-Klein model of such a hyperbolic space all Plücker transformations are induced by collineations of the ambient projective space.

1 Introduction

Let G be an arbitrary non-empty set and $\sim$ a binary relation on G which is symmetric and reflexive. Following [3] we call the structure $(G, \sim)$ a *Plücker space*, if for each pair $a, b \in G$ there exists a finite number of elements $c_1, c_2, \ldots, c_n \in G$ with

$$a \sim c_1 \sim c_2 \sim \ldots \sim c_n \sim b.$$

A *Plücker transformation* of $(G, \sim)$ is a bijection $\varphi : G \to G$ with

$$a \sim b \Longleftrightarrow a^\varphi \sim b^\varphi \text{ for all } a, b \in G. \tag{1}$$

All such Plücker transformations form the *Plücker group* of $(G, \sim)$.

Orthogonality-preserving Plücker transformations of Euclidean spaces have been discussed by W. Benz and E.M. Schröder in [3, 4]. Here G is the set of lines and $a \sim b$ means that either $a = b$ or that a and b meet orthogonally. Similar results for elliptic and symplectic spaces are due to H. Havlicek [10, 11]. One result of all these papers is that dimension three is, in some sense, exceptional. For example the 3-dimensional elliptic spaces are the only ones with Plücker transformations not induced by collineations and dualities [10]. More examples of Plücker spaces can be found in [3, 5, 13, 15].

In this paper we discuss orthogonality-preserving Plücker transformations of finite dimensional hyperbolic spaces with a Euclidean ground field. For dimension other than three all Plücker transformations are determined. In the Cayley-Klein model they are induced by collineations of the ambient projective space. Moreover, condition (1) can be reduced to

$$a \sim b \Longrightarrow a^\varphi \sim b^\varphi \qquad \forall a, b \in G.$$

2 Plücker spaces on hyperbolic spaces

Let $\Pi := \Pi(\mathcal{P}, \mathcal{G})$ be a Pappian projective space $(2 \leq \dim \Pi := n < \infty)$ with point set $\mathcal{P}$, line set $\mathcal{G}$ and Euclidean ground field $\mathbb{K}$. The characteristic of a Euclidean field is always 0. Moreover, the set $\mathcal{H}$ of internal points of an oval quadric $\mathcal{Q}$ in Π never is empty [2, p.54]. Now the linear space $\Pi_h(\mathcal{H}, \overline{\mathcal{G}})$ with

$$\overline{\mathcal{G}} := \{\overline{g} = g \cap \mathcal{H} \mid \overline{g} \neq \varnothing, g \in \mathcal{G}\}$$

is the *Cayley-Klein model* of the n-dimensional hyperbolic space over $\mathbb{K}$; cf. [8] or [14] for an axiomatic approach. We call $\mathcal{Q}$ the *absolute quadric* and denote its polarity by π. Since $\mathbb{K}$ is Euclidean, each hyperbolic line $\overline{g}$ has two ideal points $\{A, B\} := g \cap \mathcal{Q}$ [2, p.54]. We define a mapping of the lattice of subspaces onto itself by setting

$$\mathcal{U} \mapsto \bigcap \{P^\pi \mid P \in \mathcal{U}\} \text{ for all subspaces } \mathcal{U} \neq \varnothing \text{ and } \varnothing \mapsto \Pi.$$

This mapping is again denoted by π.

Since the ground field $\mathbb{K}$ of Π is Euclidean, it can be ordered uniquely. Therefore only one separation function can be defined on $\mathcal{P}$ [2, p.60]. Pairs A, B and C, D with $A, B \neq C, D$ on a line or a conic are called *separated*, denoted by $(A, B \mid C, D) = -1$, if and only if the cross-ratio $CR(A, B, C, D) < 0$. Otherwise we write $(A, B \mid C, D) = 1$. Two coplanar hyperbolic lines with ideal points A, B and C, D intersect in $\mathcal{H}$ if and only if $(A, B \mid C, D) = -1$ [2, p.62ff].

In the following we will distinguish between a secant $g \in \mathcal{G}$ of $\mathcal{Q}$ and the hyperbolic line $\overline{g} := g \cap \mathcal{H}$.

The polarity π gives rise to the following binary relations $\sim$ and $\approx$ on $\overline{\mathcal{G}}$: For $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ put

$$\begin{array}{ll} \overline{a} \approx \overline{b} :\Longleftrightarrow a \cap b^\pi \neq \varnothing \text{ and } \overline{a} \cap \overline{b} \neq \varnothing & \text{(orthogonally intersecting lines)} \\ \overline{a} \sim \overline{b} :\Longleftrightarrow \overline{a} \approx \overline{b} \text{ or } a = b & \text{(related lines).} \end{array}$$

Both relations are symmetric and, by definition, $\sim$ is reflexive. Now we can show:

Proposition 1. *The structure $(\overline{\mathcal{G}}, \sim)$ is a Plücker space.*

Proof. Let $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ be distinct. First we assume that a, b are in a plane ε with $\mathcal{Q}_\varepsilon := \mathcal{Q} \cap \varepsilon$. We are looking for a finite sequence of lines $\overline{n}_1, \overline{n}_2, \ldots, \overline{n}_k \in \overline{\mathcal{G}}$ with

$$\overline{a} \sim \overline{n}_1 \sim \ldots \sim \overline{n}_k \sim \overline{b}. \tag{2}$$

1. For hyperparallel lines $\overline{a}, \overline{b}$, the intersection point $a \cap b$ is an external point. Hence the line $n := (a \cap b)^\pi \cap \varepsilon$ fulfills $\overline{a} \sim \overline{n} \sim \overline{b}$.

2. Now let $\overline{a}, \overline{b}$ be parallel and $A \in a \cap \mathcal{Q}_\varepsilon$, $B \in b \cap \mathcal{Q}_\varepsilon$ and $C := a \cap b \in \mathcal{Q}_\varepsilon$ with $A \neq C \neq B$. Furthermore we choose $D, E \in \mathcal{Q}_\varepsilon$ with $D \neq E$ such that the pairs (A, D), (C, B) and (A, E), (C, B) are separated. From $(C, B \mid D, A) = -1$, $(C, B \mid A, E) = -1$, $(A, C \mid D, B) = 1$, and

$(A, C \mid B, E) = 1$ the multiplication theorem for separation functions gives $(C, B \mid D, E) = 1$ and $(A, C \mid D, E) = 1$. So the line $\overline{n}_2 := DE \cap \mathcal{H}$ is hyperparallel to $\overline{a}$ and $\overline{b}$. Now we have reduced the problem to case 1.

3. $\overline{a}, \overline{b}$ intersect: We choose B in $b \cap \mathcal{Q}_\varepsilon$. Then $\overline{n}_1 := ((a^\pi \cap \varepsilon)B) \cap \mathcal{H}$ is parallel to $\overline{b}$ and $\overline{a} \sim \overline{n}_1$. Again, we have reduced the problem to case 2.

If a, b are skew lines we choose a line $\overline{c}$ intersecting $\overline{a}$ and $\overline{b}$. So the assertion follows from case 3. Any two lines in $\overline{\mathcal{G}}$ are either hyperparallel, or parallel, or intersecting, or skew. □

In the proof of Proposition 1 we saw that for any two skew lines $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ there exists a sequence of orthogonally intersecting lines that connect $\overline{a}$ and $\overline{b}$. By [7, p.64, (1)] it is always possible to reduce this sequence to one line:

Lemma 1. *Let $\overline{a}$, $\overline{b}$ be two given skew lines of a hyperbolic space Π_h. Then there exists a line $\overline{n}$ intersecting $\overline{a}$ and $\overline{b}$ orthogonally.*

Now we want to discuss Plücker transformations of $(\overline{\mathcal{G}}, \sim)$. We use the term *$\mathcal{Q}$-collineation* for any collineation of Π leaving the quadric $\mathcal{Q}$ invariant. It is obvious that $\mathcal{Q}$-collineations induce Plücker transformations:

Proposition 2. *Let $\psi : \Pi \to \Pi$ be a $\mathcal{Q}$-collineation. Then the line mapping $\varphi : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$, $\overline{g} \mapsto g^\psi \cap \mathcal{H}$ which is induced by ψ is a Plücker transformation of $(\overline{\mathcal{G}}, \sim)$.*

Now the question is the following: Are all Plücker transformations induced by $\mathcal{Q}$-collineations.

3 Plücker transformations in hyperbolic spaces with $\dim \Pi_h \geq 4$

Let $\Pi_h(\mathcal{H}, \overline{\mathcal{G}})$ be a hyperbolic space ($\dim \Pi_h \geq 2$) with the relations $\sim$ and $\approx$.

Lemma 2. *Given mutually distinct $\overline{a}, \overline{b}, \overline{c} \in \overline{\mathcal{G}}$ with $\overline{a}, \overline{b}$ concurrent and $\overline{c}$ intersecting $\overline{a}$ and $\overline{b}$ orthogonally, then $\overline{a} \cap \overline{b} \subset \overline{c}$.*

Proof. Since $\overline{a}, \overline{b}, \overline{c}$ are mutually distinct, $|\overline{a} \cap \overline{b}| = |\overline{a} \cap \overline{c}| = |\overline{b} \cap \overline{c}| = 1$. In Π_h there exists no triangle with two right angles. Therefore the point $\overline{a} \cap \overline{b}$ is on $\overline{c}$. □

Proposition 3. *Let $\dim \Pi_h \geq 4$ and $\varphi : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ be a bijection*[1] *with*

$$\overline{a} \sim \overline{b} \Longrightarrow \overline{a}^\varphi \sim \overline{b}^\varphi \qquad \forall \overline{a}, \overline{b} \in \overline{\mathcal{G}}. \tag{3}$$

Then for every point $A \in \mathcal{H}$ there exists an A' such that $\overline{g} \in \mathcal{G}$ and $A \in \overline{g}$ implies $A' \in \overline{g}^\varphi$.

[1] The mapping φ may also be seen as a bijection on secants $s \in \mathcal{G}$. By abuse of notation, we define the line s^φ as the unique projective line such that $\overline{s}^\varphi = s^\varphi \cap \mathcal{H}$.

Proof. We choose $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ with $\overline{a} \cap \overline{b} = A$ and $\overline{a} \approx \overline{b}$. Hence $\overline{a}^\varphi \approx \overline{b}^\varphi$ and we are able to define $A' := \overline{a}^\varphi \cap \overline{b}^\varphi$. Now it remains to be shown that[2] $\overline{g} \in \overline{\mathcal{G}}_A$ implies $\overline{g}^\varphi \in \overline{\mathcal{G}}_{A'}$.

1. If $\overline{g}$ is related to $\overline{a}$ and $\overline{b}$, then $\overline{a}^\varphi \sim \overline{g}^\varphi \sim \overline{b}^\varphi$. By Lemma 2, $A' \in \overline{g}^\varphi$.

2. Let $\overline{g}$ be not related to $\overline{a}$ and $\overline{b}$. Since $\dim \Pi_h = n \geq 4$, all lines passing A, and orthogonal to $\overline{a}$ and $\overline{b}$, span a subspace β of Π of dimension $n-2 \geq 2$. We put $a \vee b =: \alpha$ and choose $c, d \in \beta$ such that $\overline{c} \approx \overline{d}$ and $A \in c, d$ (see Figure 1). Additionally, there exist lines $e, f \ni A$ with $e \in \alpha, f \in \beta$ and $\overline{e} \approx \overline{g}$, $\overline{f} \approx \overline{g}$. Now the lines $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ and $\overline{e}, \overline{f}, \overline{g}$ as well as $\overline{a}, \overline{b}, \overline{f}$ and $\overline{c}, \overline{d}, \overline{e}$ are mutually orthogonal. This is also true for their φ-images. Together with Lemma 2 we get step by step: $A' \in \overline{c}^\varphi$, $A' \in \overline{d}^\varphi$, whence $A' \in \overline{e}^\varphi$, $A' \in \overline{f}^\varphi$, and finally $A' \in \overline{g}^\varphi$.

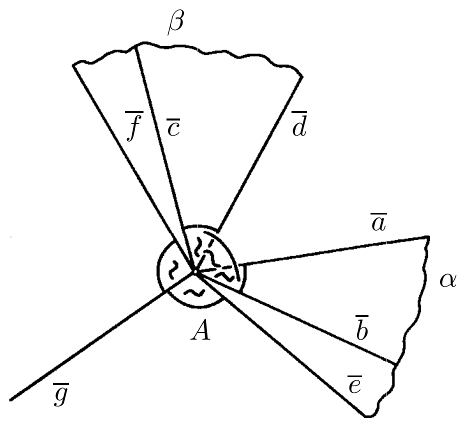

Figure 1: Step 2 of the proof of Proposition 3

□

In step 2 we used $\dim \Pi_h \geq 4$. Therefore, we cannot use the same methods for solving the 2- and 3-dimensional case.

With the help of φ we are able to define a mapping $\overline{\psi}$ on the point set of Π_h:

Proposition 4. *Let* Π_h *be a hyperbolic space with* $\dim \Pi_h \geq 4$. *If* $\varphi : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ *is a bijection satisfying property (3), then* φ *is induced by a collineation* $\overline{\psi}$ *of* Π_h.

Proof. We define $\overline{\psi} : \mathcal{H} \to \mathcal{H}$, $A \mapsto A^{\overline{\psi}} := A'$. This $\overline{\psi}$ is well defined by Proposition 3 and collinearity of points is invariant under φ.

1. Assume to the contrary that there exist two different points $A, B \in \mathcal{H}$ with $A^{\overline{\psi}} = B^{\overline{\psi}}$. For all $X \in \mathcal{H} \backslash AB$ we obtain $\overline{XB}^\varphi \neq \overline{XA}^\varphi$ by the injectivity

[2] By $\overline{\mathcal{G}}_A$ we denote the star of hyperbolic lines centered in A.

of φ. Further $X^{\overline{\psi}} = A^{\overline{\psi}} = B^{\overline{\psi}}$. So, for every $\overline{g} \in \overline{\mathcal{G}}$ we get $A^{\overline{\psi}} \in \overline{g}^{\varphi}$, which is a contradiction to the surjectivity of φ. Thus the mapping $\overline{\psi}$ is injective.

2. Let the points $A, B, C \in \mathcal{H}$ be non-collinear and let their images $A^{\overline{\psi}}, B^{\overline{\psi}}$, $C^{\overline{\psi}}$ be on a line $\overline{g}$. Since $\overline{\psi}$ is injective, these points are mutually distinct. Now $\overline{AB} \neq \overline{AC}$ and
$$\overline{AB}^{\varphi} = \overline{A^{\overline{\psi}}B^{\overline{\psi}}} = \overline{g} = \overline{A^{\overline{\psi}}C^{\overline{\psi}}} = \overline{AC}^{\varphi}$$
is a contradiction to the injectivity of φ.

3. The surjectivity of $\overline{\psi}$ remains to be shown:
   (a) First we prove that the restriction of φ to $\overline{\mathcal{G}}_A$ $(A \in \mathcal{H})$ is a bijection onto $\overline{\mathcal{G}}_{A^{\overline{\psi}}}$. For any two lines $\overline{a} \neq \overline{b}$ through A it follows that $\overline{a}^{\varphi} \neq \overline{b}^{\varphi}$ and so $A^{\overline{\psi}} = \overline{a}^{\varphi} \cap \overline{b}^{\varphi}$. Supposing $A^{\overline{\psi}} \in \overline{c}^{\varphi}$ but $A \notin \overline{c}$, we can also assume without loss of generality that $\overline{c}$ intersects $\overline{a}$ and $\overline{b}$. Hence $\overline{a} \cap \overline{c} = B \neq A$ and $A^{\overline{\psi}} = B^{\overline{\psi}}$. This contradicts the injectivity of $\overline{\psi}$.

   (b) Now we will show that the φ-preimages of parallel lines are again parallel: If two lines are hyperparallel or skew, then they have a common orthogonal line (Proposition 1 and Lemma 1), intersecting the lines in two different points. This is also true for their images. Concurrent and parallel lines do not have such a common orthogonal line (Lemma 2). Therefore, their φ-preimages are again concurrent or parallel. In Proposition 3 we proved that φ maps intersecting lines to intersecting lines. Hence the assertion follows.

   (c) In the next step we prove that
$$\overline{a}^{\overline{\psi}} = \{X^{\overline{\psi}} \mid X \in \overline{a}\} = \overline{a}^{\varphi} \quad \text{for all} \quad \overline{a} \in \overline{\mathcal{G}}.$$
   Let us take a look at a star with center $A \notin \overline{a}$. In (b) we saw that a line $\overline{b}$, with $\overline{b}^{\varphi} \cap \overline{a}^{\varphi} \neq \varnothing$, is necessarily parallel or concurrent to $\overline{a}$. But the only two lines in $\overline{\mathcal{G}}_A$ being parallel to $\overline{a}$ are the φ-preimages of the parallel lines to $\overline{a}^{\varphi}$. Therefore $\overline{a}$ and $\overline{b}$ intersect and $\overline{\psi}|\overline{a} : \overline{a} \to \overline{a}^{\varphi}$ is surjective.

   (d) If B' is an arbitrary point in $\mathcal{H}$, then we are able to choose a line $\overline{a}^{\varphi} \ni B'$. In (c) we proved the existence of a point $B \in \overline{a}$ with $B^{\overline{\psi}} = B'$.

□

Finally, we extend the collineation $\overline{\psi} : \mathcal{H} \to \mathcal{H}$ into the ambient projective space Π. The main tool will be a Theorem due to R. Frank [6].

Proposition 5. *Let φ be a bijection satisfying (3) in a hyperbolic space Π_h with $\dim \Pi_h \geq 4$. Then φ is induced by a $\mathcal{Q}$-collineation ψ of Π. Moreover, φ is a Plücker transformation.*

Proof. We already know that φ is induced by a collineation $\overline{\psi} : \mathcal{H} \to \mathcal{H}$. Using the terminology of [6], such a collineation can be extended to a *projection* $\overline{\psi} : \mathcal{H} \to \Pi$.

The Euclidean ground field $\mathbb{K}$, together with the order topology, is a topological field [17]. So Π becomes a topological projective space with the coordinate topology τ [16]. The set of internal points of any oval quadric, for example $\mathcal{H}$, is an open set of τ. Since $\mathcal{H}$ is not contained in a hyperplane, $\operatorname{span} \mathcal{H}^{\overline{\psi}} = \operatorname{span} \mathcal{H} = \Pi$. The induced topologies on the lines of Π form a *linear topology* in the sense of [6]. Because $\mathcal{H}$ is an open set, the intersection set of every line g with $\mathcal{H}$ is an open set with respect to the induced topology on g. So $\mathcal{H}$ is linearly open. If $\overline{g}$ is a hyperbolic line, then $\overline{g}^{\overline{\psi}} = (\mathcal{H} \cap g)^{\overline{\psi}} = \mathcal{H} \cap g^{\varphi} \neq \varnothing$ is again open with respect to the linear topology of Π. Therefore we can use Theorem 2 of [6]:

There exist subspaces $Z \subset \Pi \backslash \mathcal{H}$ and $D \subset \Pi \backslash Z$ with $\operatorname{span} D \vee Z = \Pi$. Moreover there exists a collineation $\psi : D \to D'$ with $\overline{\psi} = p\psi\iota$ where $p : \mathcal{H} \to D$ is a central projection with center Z, D' is a subspace of Π and $\iota : D' \to \Pi$ is the *canonical injection.* In our case Z is empty, otherwise each hyperbolic line $\overline{g}$ with $g \cap Z \neq \varnothing$ would be mapped onto a point. Furthermore, $\operatorname{span} D \vee Z = \Pi$ implies $D = \Pi = D'$ and ι is the identity. Hence the central projection $p : \mathcal{H} \to \Pi$ is the canonical injection and $\psi | \mathcal{H} = \overline{\psi}$.

Under the collineation ψ hyperplanes are mapped onto hyperplanes. There is a one-to-one correspondence between external points A of $\mathcal{Q}$ and hyperplanes $\varepsilon = A^{\pi}$ which contain internal points. If $I_1, I_2 \in \varepsilon$ are two different internal points, ε is spanned by all lines $a \in \mathcal{G} \cap \varepsilon$ with $\overline{a} \approx \overline{AI_1}$, $\overline{a} \approx \overline{AI_2}$ respectively. Orthogonality is invariant under ψ. That means $\overline{A^{\psi} I_1^{\psi}}$ ($\overline{A^{\psi} I_2^{\psi}}$) is the only line through I_1^{ψ} (I_2^{ψ}), that is orthogonal to all lines of the star with center I_1^{ψ} (I_2^{ψ}) in ε^{ψ}. Therefore $A^{\psi} = \varepsilon^{\psi\pi}$ is an external point. Since ψ and $\overline{\psi}$ are collineations, ψ yields also a bijection on the set of external points of $\mathcal{Q}$. So, ψ is a $\mathcal{Q}$-collineation. Proposition 2 shows that $\varphi : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ is a Plücker transformation. $\square$

Remark. For real hyperbolic spaces we could show Proposition 5 also with Theorem 2 in [12] by R. Höfer.

4 The 2-dimensional case

4.1 A characterization of Plücker transformations

If $\dim \Pi_h = 2$, then the absolute quadric $\mathcal{Q}$ is a conic with polarity π. For an arbitrary line $\overline{g} \in \overline{\mathcal{G}}$ all orthogonal lines are running through the point g^{π}. By $\dim \Pi_h = 2$, orthogonal hyperbolic lines are intersecting. Therefore there exists no common orthogonal transversal for $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ being parallel or concurrent. But the sequence (2) of related lines connecting $\overline{a}, \overline{b}$ can be reduced to two lines $\overline{n_1}, \overline{n_2}$:

Lemma 3. *In a hyperbolic plane let* $\overline{a}, \overline{b} \in \overline{\mathcal{G}}$ *be two different lines, that are parallel or intersecting, but not orthogonal. Then there exist* $\overline{n_1}, \overline{n_2} \in \overline{\mathcal{G}}$ *with*

$$\overline{a} \approx \overline{n_1} \approx \overline{n_2} \approx \overline{b}.$$

Proof. In both cases we will show the existence of a line $\overline{n_1}$ with $\overline{a} \approx \overline{n_1}$ that is hyperparallel to $\overline{b}$.

1. $\overline{a}, \overline{b}$ are parallel: Let $A \in a \cap \mathcal{Q}$, $B \in b \cap \mathcal{Q}$ and $C := a \cap b \in \mathcal{Q}$ with $A \neq C \neq B$. There exists a $D \in \mathcal{Q}$ such that the pairs (A, B) and (C, D) are seperated. The line $a^\pi D$ meets $\mathcal{Q}$ residually at a point E, say. Then $(A, C \mid D, E) = -1$, because $\overline{AC}$ and $\overline{DE}$ intersect orthogonally. Thus

   $$(C, D \mid B, E) = (C, D \mid B, A) \cdot (C, D \mid A, E) = (-1) \cdot 1 = -1$$

   and $(B, C \mid D, E) = 1$, which means, $\overline{b}$ and $\overline{n_1} := \overline{DE}$ are hyperparallel.

2. $\overline{a}, \overline{b}$ are intersecting (Figure 2): Let $A \neq B$ and $C \neq D$ be the intersection points of a and b with $\mathcal{Q}$. Because $\overline{a}, \overline{b}$ intersect, $(A, B \mid C, D) = -1$. Choose $E \in a^\pi D \cap \mathcal{Q}$ with $D \neq E$. So the lines $\overline{AB}$ and $\overline{DE}$ intersect, i.e. $(A, B \mid D, E) = -1$. From $a \not\sim b$ follows $C \neq E$. Without loss of generality we can assume that $(A, C \mid D, E) = -1$: If $(A, C \mid D, E) = 1$ then the multiplication theorem for seperation functions gives:

   $$(B, C \mid D, E) = (B, A \mid D, E) \cdot (A, C \mid D, E) = (-1) \cdot 1 = -1.$$

   Moreover, we choose F such that $(A, D \mid C, F) = -1$. For the second intersection point $G \in a^\pi F \cap \mathcal{Q}$ we get $(E, D \mid F, G) = 1$, since ED, FG intersect in the external point a^π of $\mathcal{Q}$. Now we get step by step:

   $$\begin{aligned} (A, D \mid E, F) &= (A, D \mid C, F) \cdot (A, D \mid E, C) = (-1) \cdot 1 = -1, \\ (D, E \mid A, G) &= (D, E \mid F, G) \cdot (D, E \mid A, F) = 1 \cdot 1 = 1, \\ (D, E \mid C, G) &= (D, E \mid A, G) \cdot (D, E \mid C, A) = 1 \cdot (-1) = -1, \\ (D, E \mid C, F) &= (D, E \mid C, G) \cdot (D, E \mid G, F) = (-1) \cdot 1 = -1. \end{aligned}$$

   Thus $(C, D \mid F, G) = (C, D \mid E, G) \cdot (C, D \mid F, E) = 1 \cdot 1 = 1$ and therefore $\overline{b}$ and $\overline{n_1} := \overline{GF}$ are hyperparallel.

$\square$

Proposition 6. *In every hyperbolic plane* Π_h *a bijection*

$$\varphi : \overline{\mathcal{G}} \to \overline{\mathcal{G}} \text{ with } \overline{g} \sim \overline{h} \Longrightarrow \overline{g}^\varphi \sim \overline{h}^\varphi$$

is a Plücker transformation of $(\overline{\mathcal{G}}, \sim)$.

Proof. For two arbitrary lines $\overline{g}, \overline{h} \in \overline{\mathcal{G}}$ we show

$$\overline{g} \not\sim \overline{h} \Longrightarrow \overline{g}^\varphi \not\sim \overline{h}^\varphi.$$

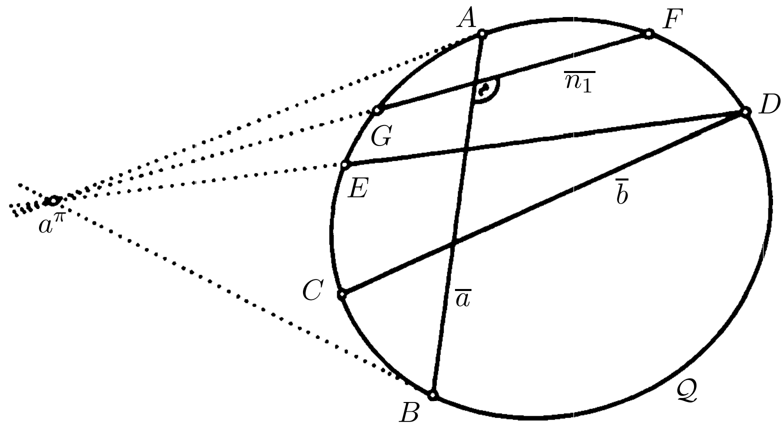

Figure 2: $\overline{a}, \overline{b}$ are intersecting

1. For every two hyperparallel lines $\overline{g}, \overline{h}$ there exists a line $\overline{n_1} \in \overline{\mathcal{G}}$ with $\overline{g} \approx \overline{n_1} \approx \overline{h}$ and furthermore $\overline{g}^\varphi \approx \overline{n_1}^\varphi \approx \overline{h}^\varphi$. Therefore $n_1^{\varphi\pi} = g^\varphi \cap h^\varphi$ is an external point of $\mathcal{Q}$ and $\overline{g}^\varphi, \overline{h}^\varphi$ are hyperparallel as well.

2. If $\overline{g}, \overline{h}$ are parallel or intersecting then, by Lemma 3, there exist lines $\overline{n_1}, \overline{n_2} \in \overline{\mathcal{G}}$ with $\overline{g} \approx \overline{n_1} \approx \overline{n_2} \approx \overline{h}$. Our assumptions lead to $\overline{g}^\varphi \approx \overline{n_1}^\varphi \approx \overline{n_2}^\varphi \approx \overline{h}^\varphi$. But in a hyperbolic plane there exists no rectangle.

So in both cases $\overline{g}^\varphi \not\sim \overline{h}^\varphi$ is true. □

In this proof the crucial point is that $\dim \Pi_h = 2$. Otherwise two intersecting lines have a common orthogonal line and there is even the possibility of $\overline{g}, \overline{h}$ being skew. Therefore we cannot use the same methods for $\dim \Pi_h \geq 3$.

4.2 Induced collineations on Π_h and Π

Together with every Plücker transformation φ of $(\overline{\mathcal{G}}, \sim)$ we have the bijection

$$(\pi|A_\mathcal{Q})\varphi\pi : A_\mathcal{Q} \to A_\mathcal{Q}$$

on the set of external points $A_\mathcal{Q}$. We denote this bijection again by φ. From now on let φ be the mapping

$$\varphi : \overline{\mathcal{G}} \cup A_\mathcal{Q} \to \overline{\mathcal{G}} \cup A_\mathcal{Q}$$

with $\overline{\mathcal{G}}^\varphi = \overline{\mathcal{G}}$ and $A_\mathcal{Q}^\varphi = A_\mathcal{Q}$. For every $A \in A_\mathcal{Q}$ and every secant g of $\mathcal{Q}$ there is

$$A \in g \iff \overline{g} \approx \overline{A^\pi} \iff \overline{g}^\varphi \approx \overline{A^\pi}^\varphi \iff A^{\pi\varphi\pi} \in g^\varphi :\iff A^\varphi \in g^\varphi. \tag{4}$$

Proposition 7. *Assume that φ satisfies the conditions of Proposition 6. Then for each point $A \in \mathcal{H}$ there exists an $A' \in \mathcal{H}$ with $\overline{\mathcal{G}}_A^\varphi \subset \overline{\mathcal{G}}_{A'}$.*

Using the polarity π we are able to translate Proposition 7 into an equivalent proposition concerning external points of $\mathcal{Q}$:

Proposition 8. *Let G, H and I be three distinct points on an external line of $\mathcal{Q}$. Then there exists an external line of $\mathcal{Q}$ that contain G^φ, H^φ, and I^φ.*

Proof. We will establish Proposition 8 by constructing a nontrivial Desargues configuration Z, P_j, Q_j ($j \in \{1,2,3\}$) such that corresponding edges p_j, q_j ($j \in \{1,2,3\}$) meet at G, H and I. The vertices of the triangles P_1, P_2, P_3 and Q_1, Q_2, Q_3 will be external points and the edges p_j, q_j ($j \in \{1,2,3\}$) will be secants of $\mathcal{Q}$ (see Figure 3).

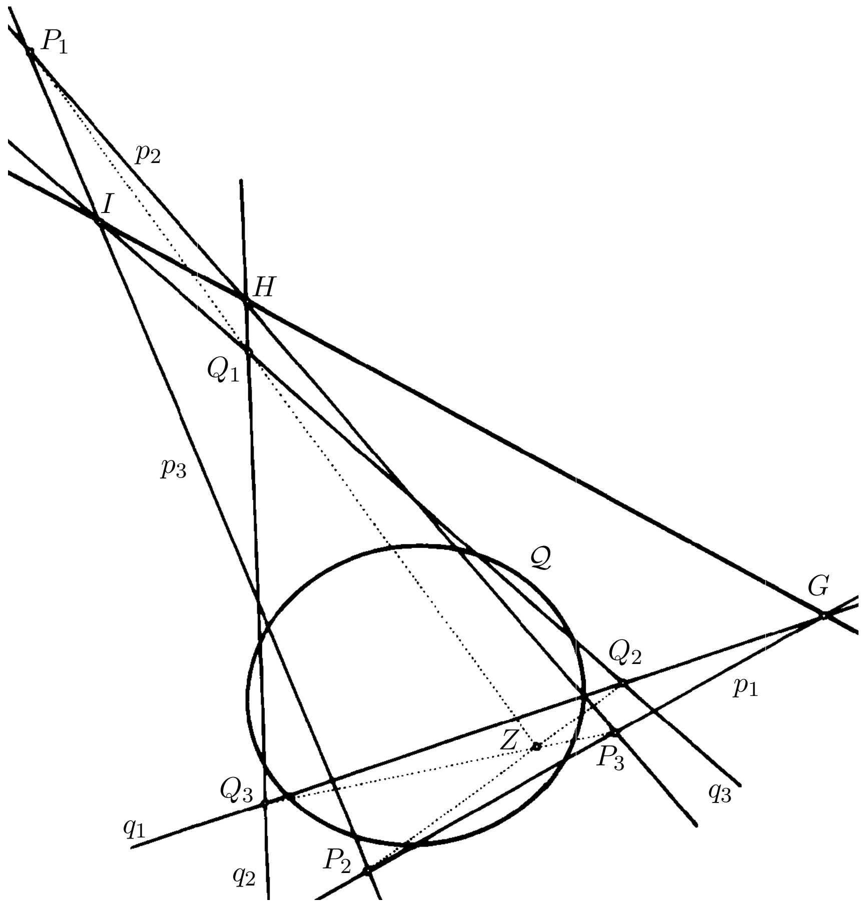

Figure 3: Desargues configuration with P_j, Q_j external points and p_j, q_j secants of $\mathcal{Q}$ ($j \in \{1,2,3\}$)

1. Through each point G, H, and I we choose a tangent line of $\mathcal{Q}$ (t_G, t_H and t_I). Since $u := GH$ does not intersect $\mathcal{Q}$, the points of tangency T_G, T_H, T_I as well as $G_0 := t_H \cap t_I$, $H_0 := t_G \cap t_I$ and $I_0 := t_G \cap t_H$ are mutually distinct and form a triangle[3] (see Figure 4).

   If we choose u as the line at infinity, we get the affine plane $\mathcal{A} := \mathcal{P} \backslash u$.

[3]Just if GH is a tangent line $T_G = T_H$ is possible.

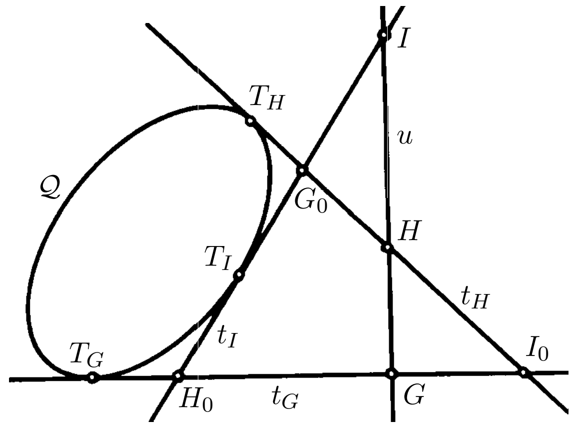

Figure 4: Step 1 of the proof of Proposition 8

We endow $\mathcal{A}$ with a $\mathbb{K}$-*metric*[4]

$$\sigma : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$$

in the sense of S. Gudder [9].

2. Since G_0, H_0, I_0 are external points of $\mathcal{Q}$ and $A_{\mathcal{Q}} \backslash u$ is an open set, there exists a $\rho \in \mathbb{K}$ with $\rho > 0$ such that the open disks $K(G_0, \rho)$, $K(H_0, \rho)$, and $K(I_0, \rho)$ are subsets of $A_{\mathcal{Q}}$. Now we construct, for example, the line p_1: Inside the disk $K(T_G, \frac{\rho}{3})$ we choose the points G_H, G_I such that $G_H \in T_G H$ and $G_I \in T_G I$ are internal points of $\mathcal{Q}$ (see Figure 5).

   Without loss of generality $p_1 := G_H G$ is between t_G and $G_I G$. Therefore the intersection point $p_1 \cap T_G I$ lies between T_G and G_I. Furthermore $p_1 \cap T_G I$ is an internal point of $K(T_G, \frac{\rho}{3})$. But since $\mathbb{K}$ is a Euclidean field and p_1 has at least one internal point, p_1 is a secant of $\mathcal{Q}$. Analogously we are able to construct p_2 and p_3.

3. In the next step we will show that the three points $P_i := p_j \cap p_k$ ($\{i, j, k\} = \{1, 2, 3\}$) are external points of $\mathcal{Q}$ (see Figure 6). With

   $$\sigma(G_0, p_2 \cap G_0 T_I) = \sigma(T_H, H_I) < \frac{\rho}{3}$$

   and

   $$\sigma(p_2 \cap G_0 T_I, P_1) = \sigma(T_I, I_H) < \frac{\rho}{3}$$

   we get

   $$\sigma(G_0, P_1) \leq \sigma(G_0, p_2 \cap G_0 T_I) + \sigma(p_2 \cap G_0 T_I, P_1) < \frac{\rho}{3} + \frac{\rho}{3} < \rho.$$

[4] $\mathcal{A}$ is isomorphic to the affine plane $\mathcal{A}(\mathbb{K}^2)$ over the field $\mathbb{K}$. $\mathcal{A}(\mathbb{K}^2)$ together with $\sigma : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}$, $((x_1, x_2), (y_1, y_2)) \mapsto \sqrt{(y_1 - x_1)^2 + (y_2 - y_2)^2}$ forms a so called $\mathbb{K}$-metric space fulfilling the usual conditions of a metric space. It turns out [9] that for a $\mathbb{K}$-metric space there exists a cardinal α, such that the intersection of a family, with the cardinality less than α, of open sets is open. Such *α-topological spaces* over $\mathbb{K}$-metric spaces have a lot of properties with topological spaces over metric spaces in common. For example they are Hausdorff, and they are even normal. For a detailed description see [9].

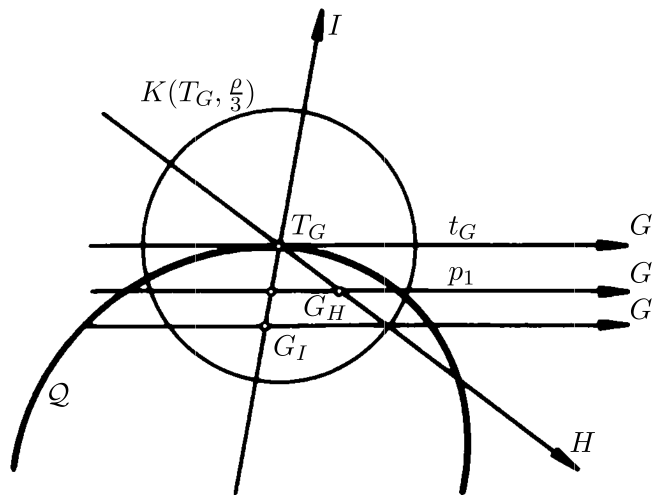

Figure 5: Step 2 of the proof of Proposition 8

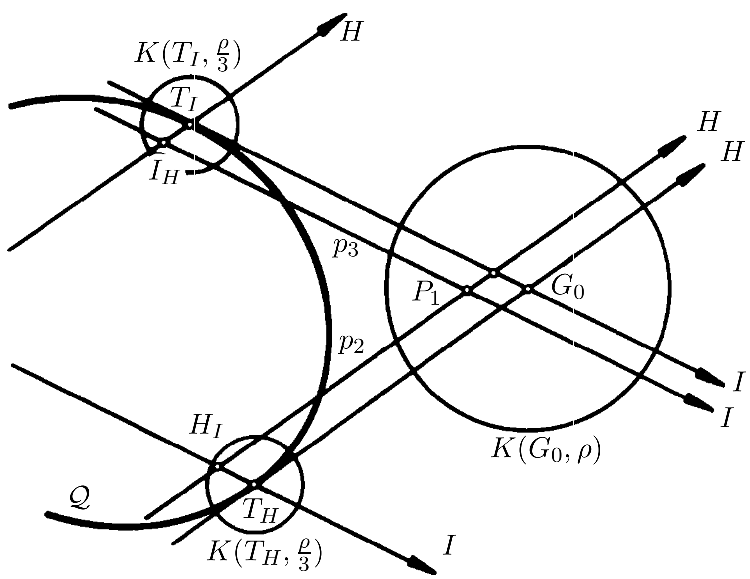

Figure 6: Step 3 of the proof of Proposition 8

Therefore $P_1 \in K(G_0, \rho)$ is an external point of $\mathcal{Q}$. Analogously this can be shown for P_2 and P_3.

4. Let us choose $\widehat{\rho} \in \mathbb{K}$ with $0 < \widehat{\rho} < \frac{\rho}{3}$ such that there exist points $\widehat{M_i} \in p_i$ ($i \in \{1, 2, 3\}$) with $K(\widehat{M_i}, \widehat{\rho}) \subset \mathcal{H}$. Furthermore let $Q_1 \in K(P_1, \widehat{\rho})$ neither be on P_1H nor on P_1I and let $\tau : \mathcal{A} \to \mathcal{A}$ be the translation with $\tau(P_1) = Q_1$. Then Q_1 is again an external point of $\mathcal{Q}$ since

$$\sigma(G_0, Q_1) \leq \sigma(G_0, P_1) + \sigma(P_1, Q_1) < \frac{2\rho}{3} + \frac{\rho}{3} = \rho.$$

Likewise we are able to see that $Q_2 := P_2^\tau$ and $Q_3 := P_3^\tau$ are external points. The lines $q_j := p_j^\tau$ ($j \in \{1,2,3\}$) are secants of $\mathcal{Q}$, since they carry internal points.

Now we have found a non-trivial Desargues configuration with the required properties.

The property of being a secant line of $\mathcal{Q}$ does not change when we apply φ. Furthermore, φ maps external points $A \in g$ to external points $A^\varphi \in g^\varphi$ (see (4)). Therefore φ maps the Desargues configuration from above onto a Desargues configuration with the same properties. But $\mathcal{P}$ is a Desarguesian plane, so that G^φ, H^φ, and I^φ are again collinear.

It remains to be shown that $G^\varphi H^\varphi$ is an external line of $\mathcal{Q}$. Every two orthogonally intersecting lines determine an external line. On every external line there is a pair of points A, B such that A^π intersects B^π orthogonally. Since φ maps orthogonally intersecting lines onto orthogonally intersecting lines, the φ-images of external lines are again external lines. □

Now we achieved the aim of this paper. Using Proposition 7, we can show Proposition 4 and 5 in exactly the same way as we did above. Altogether we get an extension of Proposition 5:

Theorem 1. *Let φ be a bijection satisfying*

$$\overline{a} \sim \overline{b} \Longrightarrow \overline{a}^\varphi \sim \overline{b}^\varphi \qquad \forall \overline{a}, \overline{b} \in \overline{\mathcal{G}}. \tag{5}$$

in a hyperbolic space Π_h with $\dim \Pi_h \neq 3$. Then φ is induced by a $\mathcal{Q}$-collineation ψ of Π. Moreover, φ is a Plücker transformation.

Remark. Plücker transformations in hyperbolic spaces with $\dim \Pi_h = 3$ cannot be investigated with the methods introduced in this paper. In Proposition 3 the crucial property of $\mathcal{H}$ is $\dim \Pi_h \geq 4$. In section 4.1 and 4.2 we use more than once that hyperbolic spaces with $\dim \Pi_h = 2$ are the only ones in which no skew lines exist. Moreover we use that two intersecting lines do not have a common orthogonal line. Therefore we will have to use completely different methods for the 3-dimensional case, which will be discussed in a forthcoming paper.

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