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Geometry of Architectural Freeform Structures

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This article shows to which extent a particular field of mathematics, namely discrete differential geometry, has recently become relevant in architectural design. It is very interesting that new mathematics has emerged from this cooperation with a branch of knowledge hitherto not known for its use of mathematical methods.

Introduction

Complex freeform structures are one of the most striking trends in contemporary architecture. This direction has been pioneered by architects such as F. Gehry who exploit digital technology originally developed for the automotive and airplane industry for tasks of architectural design and construction. This is not a simple task at all, since the architectural application differs from the original target industries in many ways, including aesthetics, statics, scale and manufacturing technologies.

Whereas metal forming can generate any reasonable shape of a car body, it is much less clear how to actually construct a complicated geometric shape in an architectural design. One has to segment the shape into simpler parts, so-called panels. According to Lars Spuybroek, "panelization is a hugely important issue" [10]. Since available CAD software does not cover this topic, one may have to resort to simpler shapes, to accept higher costs or to try experimental approaches.

Very recent research shows that the use of geometry and computational mathematics bears a great potential to advance the field of freeform architecture. It is a major goal of this paper to sketch these developments and to illustrate them at hand of a few real projects. In fact, it is in place to talk about a new research area, called *Architectural Geometry* [6], which is currently emerging at the border of differential geometry, computational mathematics and architectural design/engineering.

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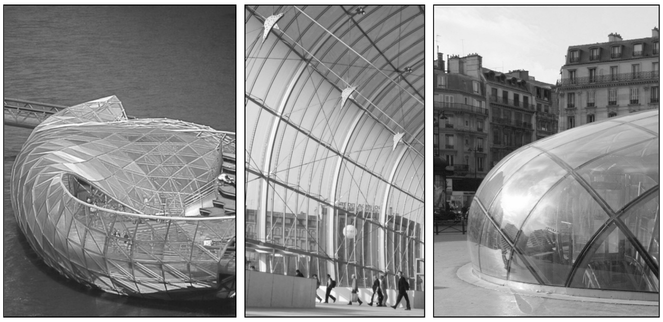


Figure 1: Segmentation of curved surfaces: From left to right: flat panels (Murinsel, Graz), single-curved panels (TGV train station, Strasbourg), and double-curved panels (St. Lazaire metró station, Paris).

Our paper is structured as follows. In Section 1, we discuss the problem of covering freeform shapes with planar quadrilateral panels. The resulting *planar quad (PQ) meshes* possess a number of important advantages over triangular meshes: they have a smaller number of edges, resulting in a smaller number of supporting beams following the edges, less steel and less cost. Quad meshes also have a lower node complexity, which is an important advantage for manufacturing. Panelization with planar quads and an optimized layout of supporting beams can be made accessible with methods from *discrete differential geometry* [5, 6, 7, 4]. Section 2 discusses freeform structures covered by *single curved panels*. It turns out that a basic geometric entity for this purpose, which we call a *developable strip model*, is obtained as a limit shape of a quad mesh with planar faces under a one-directional refinement rule. Developable strip models may be considered as semi-discrete surface representations since they constitute a link between smooth surfaces and discrete surfaces (meshes). In Section 3, we address other types of semi-discrete representations which are suitable for covering negatively curved surface parts with ruled surface panels. Finally, we point to some of the many open problems in architectural geometry.

1 Planar quad meshes and supporting beam layout

Assume that a smooth shape is given ('designed'), and one seeks a way of achieving that design in reality by approximating it by a polyhedron with quadrilateral faces, by subsequently building a steel construction along the edges of that polyhedron, and by realizing the faces as glass panels. For reasons of simplicity, and because this is the typical case anyway, we reduce that problem to the following mathematical abstraction: Given a submanifold $M \subseteq \mathbb{R}^3$, typically with boundary, we ask for a mapping x from $\mathbb{Z}^2$ to $\mathbb{R}^3$ such that all $x(i, j)$ lie close to M and such that each elementary quadrilateral

$$\begin{tikzcd} x(i\,,j+1) \arrow[r, dash] \arrow[d, dash] & x(i+1\,,j+1) \arrow[d, dash] \\ x(i\,,j) \arrow[r, dash] & x(i+1\,,j) \end{tikzcd} \tag{1}$$

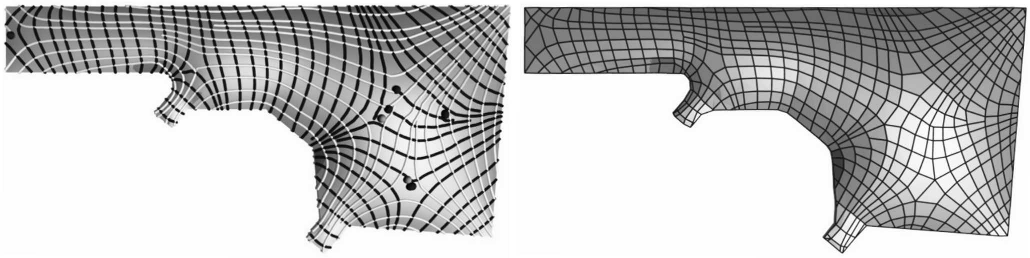


Figure 2: *Left:* Network of conjugate curves on a smooth surface M which away from singularities serve as parameter lines of a conjugate parametrization $f(u,v)$. *Right:* Discrete samples of f yield a mesh $x: V \to M$ with combinatorics (V,E,F) which is mostly that of a regular grid.

is planar (see Figure 3). For practical purposes we also want this face to be convex. A characterization of both properties in elementary terms is that the angle sum in the quadrilateral (1) equals 2π.[1]

In the construction of such discrete surfaces one has a lot of freedom, and it is tempting to solve for $x(i,j)$ in the manner of an initial value problem: Assuming that vertices $x(i,j)$, $x(i+1,j)$, $x(i,j+1)$ lie in the surface M, we consider their affine span U and choose $x(i+1,j+1)$ anywhere on the intersection curve $M \cap U$. This method however does not work in practice, as it does not take aesthetics into account: we have use only for such solutions where each of the polygons $\big(x(i,j)\big)_{i=\text{const.}}$ and $\big(x(i,j)\big)_{j=\text{const.}}$ is visually smooth. It turns out that the right way to approach this problem is to invoke the theory of discrete differential geometry [9, 3], and to consider x as a discrete surface parametrization approximating a smooth one.

In the classical differential geometry of smooth surfaces, there is the notion of *conjugate parametrization* $f(u,v)$ of a surface, which is characterized by linearly dependent vectors $\partial_u f, \partial_v f, \partial_u \partial_v f$. This means that any small quadrilateral

$$\begin{array}{ccc} f(u\,,v+\Delta v) & \text{———} & f(u+\Delta u\,,v+\Delta v) \\ | & & | \\ f(u\,,v) & \text{———} & f(u+\Delta u\,,v) \end{array} \tag{2}$$

whose convex hull's volume has the Taylor polynomial

$$\frac{1}{6}(\Delta u \Delta v)^2 \det(\partial_u f, \partial_v f, \partial_u \partial_v f) + \cdots \tag{3}$$

[1]For an n-gon with vertices $p_0, \ldots, p_{n-1}$, the angle sum s is defined by letting $v_i = p_{i+1} - p_i$ (indices modulo n) and $s = \sum \cos^{-1}(\langle v_i, -v_{i-1}\rangle / (\|v_i\| \cdot \|v_{i-1}\|))$. For general n-gons, the condition of planarity plus convexity reads $s = (n-2)\pi$, by the discrete version of Fenchel's theorem.

is *planar* to a higher degree than in the general case. As it turns out [5], a planar quadrilateral mesh $x(i,j)$ approximating a surface can be effectively found from a conjugate parametrization $f(u,v)$ by letting $x^0(i,j) = f(i\Delta u, j\Delta v)$ and optimizing x^0 towards planarity of quads (see Figure 2).

Computational issues. This optimization procedure is highly nonlinear, as it involves planarity of quadrilaterals as a constraint yet to be achieved, together with target functionals which express smoothness and proximity to a reference surface Φ. Smoothness of a polygon $(x_i)_{i\in\mathbb{Z}}$ is encoded by the quadratic functional $\|\Delta^2 x\|_{l^2} \to \min$ and $\operatorname{dist}(\Phi,\cdot)^2$ is reasonably close to a quadratic function, but planarity of faces contains quite some numerical nastiness. Experiments confirm that it is hopeless to optimize arbitray meshes towards planarity. The reason for this lies also in combinatorial/topological obstructions. However, optimization typically succeeds if initialized from a conjugate parametrization $f(u,v)$. In theory it is easy to find those: one can arbitrarily prescribe the tangent field $(u,v) \mapsto \operatorname{span}(\partial_u f)$. Thus one would expect that the problem of approximate segmentation of a surface into planar quadrilaterals is solved. In practice however, finding $f(u,v)$ is the real crux of the matter, because there are additional constraints such as minimum angles between parameter lines.

Surface layers and beam layout — Offset surfaces. We go one step further and consider not one but *two* discrete surfaces at the same time which we think of as two layers of an actual construction (see Figure 3). A usual condition imposed on them is that they are combinatorially equivalent and located at constant distance from each other (in that case they are called an *offset pair*). Distances make sense only if corresponding edges and faces are parallel, and they can be measured between corresponding faces, or edges, or vertices. The appropriate way of measuring distances depends on the application, one of which is *beam layout* (see Figure 4): We imagine steel beams with a constant rectangular cross section following edge pairs.

For more information on meshes which admit offsets of various kinds, see [5, 7]. Several interesting geometric characterizations of offset properties are known, and the entire theory fits nicely into the *consistence as integrability* paradigm which is

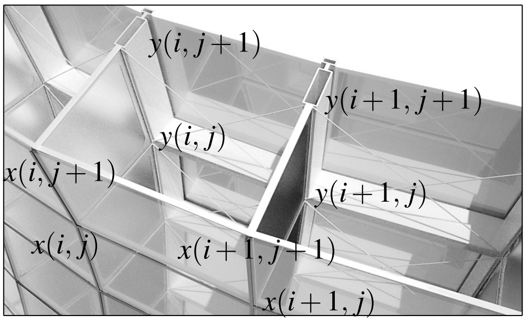


Figure 3: This multilayer construction is based on two discrete surfaces $x:\mathbb{Z}^2\to\mathbb{R}^3$ and $y:\mathbb{Z}^2\to\mathbb{R}^3$ where corresponding faces are planar, and parallel at constant distance. In addition, every pair of corresponding edges lies in a plane *(Image: B. Schneider).*

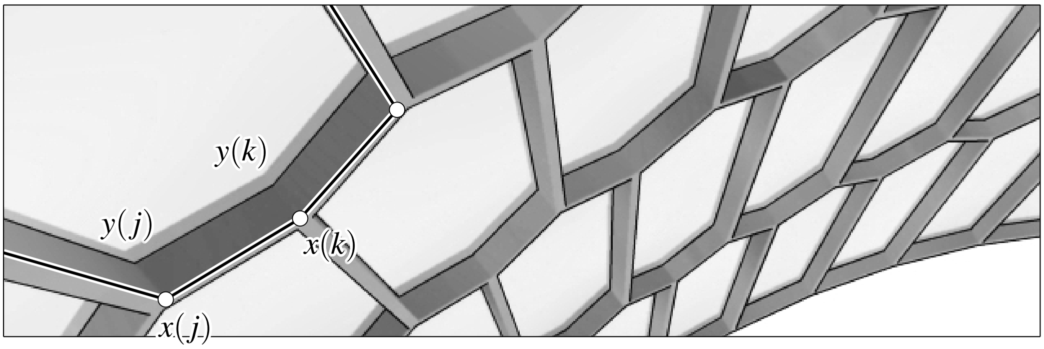


Figure 4: This beam layout is based on a discrete surface $x : V \to \mathbb{R}^3$ which has the combinatorics of a regular hexagonal lattice (V, E, F). Another discrete surface $y : V \to \mathbb{R}^3$ has the property that for all $(k, j) \in E$, corresponding edges $x(k)x(j)$ and $y(k)y(j)$ are parallel at constant distance. In the positively curved areas of the surface, edges of beams with rectangular cross-section have an exact intersection at the nodes *(Image: H. Schmiedhofer).*

the main theme of the monograph [3]. For instance the discrete surface x admits an offset y at constant face-face distance, if and only if for all i, j, in the figure of four edges emanating from the vertex $x(i, j)$,

$$\begin{array}{ccccc} & & x(i,j+1) & & \\ & & | & & \\ x(i-1,j) & \text{———} & x(i,j) & \text{———} & x(i+1,j) \\ & & | & & \\ & & x(i,j-1) & & \end{array}$$

the sums of the two diagonally opposite angle pairs are equal. A further equivalent characterization which extends to non-quadrilateral faces is that for any vertex, the adjacent faces are tangent to a common right circular cone. Such geometric conditions are not difficult to incorporate into optimization procedures and are highly relevant for applications, as still we can approximate 'arbitrary' shapes by discrete surfaces with the face offset property by starting optimization from a principal curvature line parametrization [5].

The case of meshes which admit offsets at constant edge-edge distance behaves in a different way: here the obtainable shapes are still unknown, even for the appropriate smooth analogue (i.e., isothermic surfaces in the sense of Laguerre geometry).

Remark: Discrete curvature theory. We do not want to pass over the fact that the concept of *parallel meshes* developed in [7] naturally leads to a theory of curvatures of discrete surfaces which has its basis in the following: Assume that a

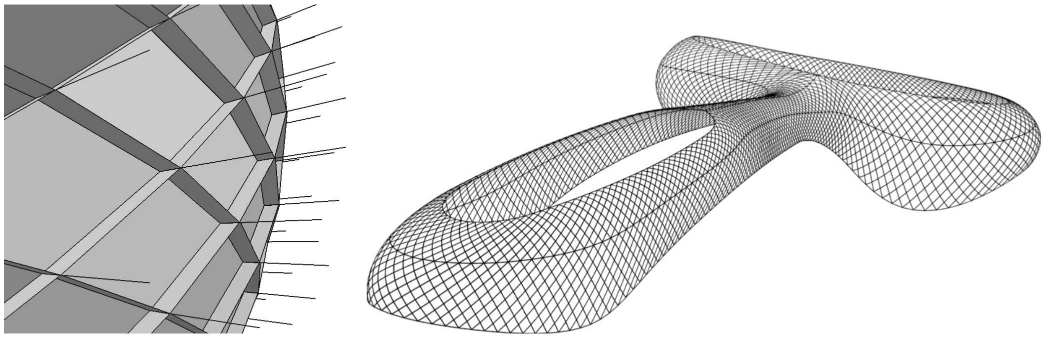
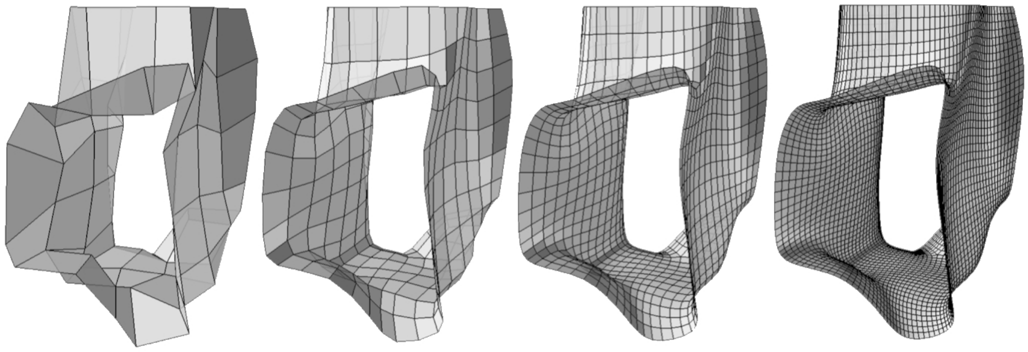


Figure 5: Layout of supporting beams for non-planar quad mesh applied to the project *Yas Island Marina Hotel,* Abu Dhabi, by Asymptote Architecture: The figure on the left shows an optimised solution for the construction of the steel frame aligned with the mesh shown at right.

surface M is equipped with a unit normal vector field, and every point $p \in M$ moves to $p + \delta \cdot n(p)$, where $\delta \in \mathbb{R}$. Then the change in surface area is given by the area integral $\Delta A = \int 1 - 2\delta H + \delta^2 K$, where the functions H, K are mean curvature and Gaussian curvature, respectively. An analogous formula in the discrete category, where movement in orthogonal direction is replaced by passage to an offset mesh leads to the definition of curvatures associated with the faces of discrete surfaces [7, 2]. It is remarkable and was not in the least expected by the authors that the discrete minimal surfaces of [1] occur as a special case.

Supporting beam layout for arbitrary types of meshes. In practice triangular meshes are widely used for covering freeform shapes. A major issue regarding these is the layout of supporting beams: for each node and its adjacent edges one is looking for a configuration of steel beams such that their symmetry planes intersect in a common node axis. Exact solutions to this problem are not feasible for applications in general. This is due to the fact that all meshes parallel to a triangular mesh are scaled copies of the given mesh with respect to some center. Therefore one is looking for approximate solutions which uniformly distribute the error throughout the nodes of the mesh. Similarly such solutions can be applied to types of meshes where parallelity is not defined, e.g. quad meshes with non-planar faces (see Figure 5).

Design of PQ meshes. In order to overcome the great numerical difficulties when optimizing a mesh towards planarity of its faces, we used a strategy common to discrete problems which are in fact discretizations of continuous ones: First solve at a coarse resolution and propagate the result to the next finer resolution, using it as initial values for the next round of optimization. In our case we use the available polyhedral subdivision rules [11] for propagation, and are thus able to greatly facilitate the *design* of polyhedral surfaces with planar quadrilateral faces (see Figure 6).



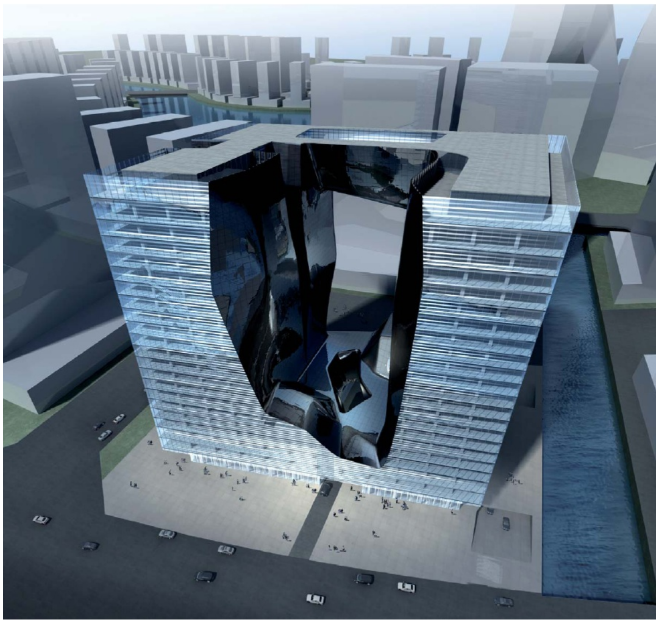


Figure 6: Design of a planar quad mesh via subdivision: Iterated steps of optimization and subdivision (which destroys planarity) lead to a planar quad mesh design useful in practice. The images show an application to the *Opus* project by Zaha Hadid Architects. This work has been performed within the project *MLFS* (grant 813391 funded by the Austrian research council, FFG).

2 Single curved panels

From the design viewpoint it is very desirable to be able to use genuinely curved surfaces without the necessity of segmentation into planar pieces (see e.g. Figure 1, right). Unfortunately this is rather expensive: in order to realize double curved glass panels, a separate mould has to be manufactured for each. The cost of this method has led to the fact that only very few large freeform structures which use double curved panels are in existence. *Kunsthaus Graz* (where the cost of construction is reported to have been considerable) is one, even if its double curved outer surface is only ornamental and does not, for instance, keep out rain.

An elegant compromise which achieves the illusion of true curvedness to a greater extent than polyhedral surfaces are surfaces comprised of *single-curved* panels, each of which is *developable* into the plane and has zero Gaussian curvature. The

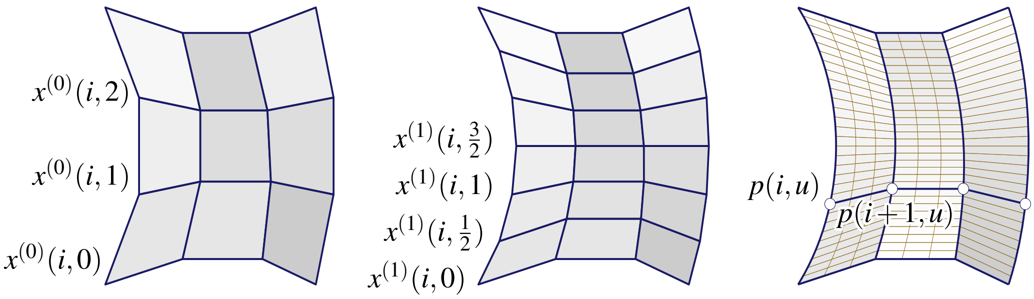


Figure 7: A semidiscrete surface $p : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ as limit of a sequence $(x^{(j)} : \mathbb{Z} \times (2^{-j}\mathbb{Z}) \to \mathbb{R}^3)_{j=0,1,2,\dots}$ of discrete surfaces. Planarity of elementary quadrilaterals implies developability of the limit, assuming smoothness.

manufacturing of such single-curved panels is much easier than that of double-curved ones and basically is the same as bending paper. Today only few buildings which use that idea have been realized, one being the new TGV train station in Strasbourg (see Figure 1, center).

Semidiscrete surface representations. It turned out that an elegant way to describe surfaces consisting of developable strips (*D-strip models*) is a mixture of the discrete and continuous surfaces employed above.

It is a well known theorem of classical differential geometry that the following properties of a surface are essentially equivalent: (i) the surface is developable, i.e., locally isometric to a plane; (ii) the Gaussian curvature equals zero; (iii) the surface locally has a *torsal ruled* parametrization $f(u,v) = (1-u)a(v) + ub(v)$ with $\det(\partial_u a, \partial_u b, b-a) = 0$.

For this reason we consider the *semidiscrete* surface representation $p : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$, where we imagine the actual surface described by p to consist of the union of straight line segments

$$\{\overline{p(i,u)p(i+1,u)} \mid i \in \mathbb{Z}, u \in \mathbb{R}\}, \tag{4}$$

(see Figure 7). The single strips of the semidiscrete surface $p(i,u)$ are developable, if the vectors $\partial_u p$, $\Delta_i p$, $\Delta_i \partial_u p$ are linearly dependent, which means that the elementary quadrilateral

$$\begin{array}{ccc} p(i\,,u+\Delta u) & \text{———} & p(i+1\,,u+\Delta u) \\ | & & | \\ p(i\,,u) & \text{———} & p(i+1\,,u) \end{array} \tag{5}$$

whose convex hull's volume has the Taylor polynomial

$$\frac{1}{6}(\Delta u)^2 \det(\Delta_i p, \partial_u p, \Delta_i \partial_u p) + \cdots$$

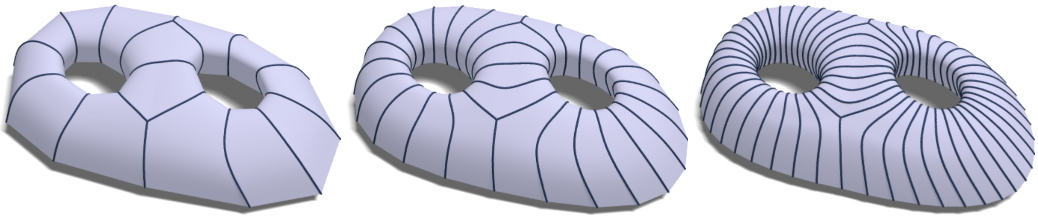


Figure 8: Design of D-strip models via subdivision: A coarse model is optimized so as to become piecewise developable (left). A subdivision rule destroys this property, but yields a good starting point for another round of optimization (center). This procedure is iterated (at right).

is planar to a higher degree than usual. From this property we derive the viewpoint that a D-strip model is a semidiscrete version of a conjugate parametrization and also a semidiscrete version of a PQ mesh. This is exploited by [8], where approximation of surfaces with D-strip models, design of D-strip models, and a geometric theory of D-strip models including offsets is studied.

Computational issues. The basic instrument in computing with D-strip models is an optimization procedure which takes a semidiscrete surface and optimizes it towards developability. It can be used to solve the approximation problem (if initialized from a conjugate surface parametrization, see Figure 9) and for the design problem (if used in an alternating way with a refinement procedure, see Figure 8).

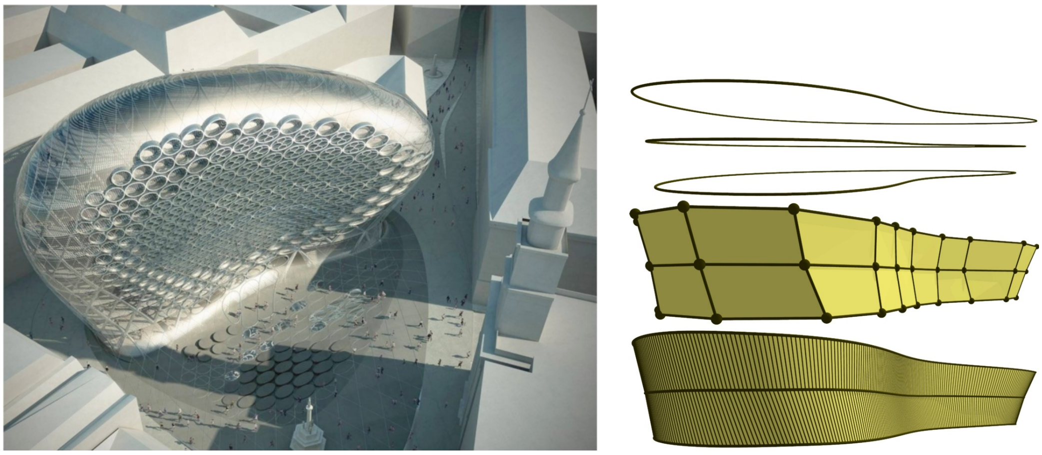
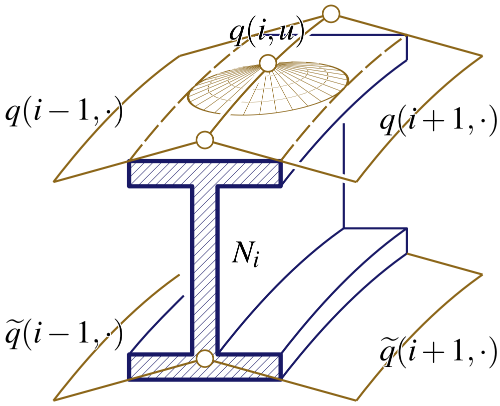


Figure 9: *Szervita Square,* Budapest, a project designed by Zaha Hadid Architects. Example of approximating the outer shell by a D-strip model aligned with planar, parallel sections given by the bottom three floor slabs. Sections, corresponding points used for initialization and the resulting D-strip model are shown at right.

Figure 10: The top and bottom flanges of the I-beam follow an offset pair $p, \widetilde{p}$ of circular D-strip models – shown by dashed lines and one inscribed circle. It can be shown that these circles are contained in families of cones, which are tangent to another offset pair $q, \widetilde{q}$ of D-strip models usable for glass panels, and that the cone axes define a developable strip N_i usable as the vertical web of the I-beam.



Similar to the optimization of meshes with planar faces, optimization towards developability is a numerically challenging task which is bound to fail except for small instances, or for instances initialized with geometric knowlege (using conjugate parametrizations). We used a simple spline model

$$p(i,u) = \sum\nolimits_j b_{ij} N(\gamma u - j),$$

where N is the cubic B-spline basis function and $b_{ij} \in \mathbb{R}^3$ are control points. We formulated all optimization goals, including the developability constraint, as target functions to be minimized. For details, see [8].

Semidiscrete differential geometry. Semidiscrete objects have been considered before in the systematic investigation of k-surface transformations (Jonas, Darboux, Combescure, etc.) as partial limits of $(k+l)$-dimensional discrete surfaces, where k parameters become continuous and l remain discrete [3]. It turns out that discrete integrable systems yield a master theory where many of the classical results, e.g. on permutability of transformations, follow as corollaries. The approach to semidiscrete surfaces described here is basically the case $k = l = 1$. We refrain from systematically discussing the further development of this semidiscrete surface theory, and restrict ourselves to aspects which have applications in architecture.

Offsets. For the purpose of multilayer constructions, we are interested in such D-strip models $p : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ which admit an *offset* at constant distance, which means another D-strip model $\widetilde{p}$ such that either

$$\|p(i,u) - \widetilde{p}(i,u)\| = \text{const.}$$

or alternatively that the distance between developable strips, measured along common normal vectors, is the same. It is also desirable that all ruled surface strips

$$(u,v) \mapsto (1-v)p(i,u) + v\widetilde{p}(i,u) \quad (i \in \mathbb{Z})$$

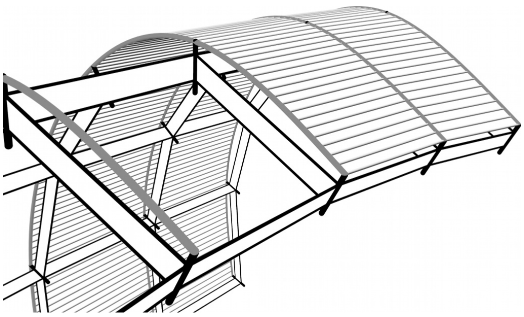


Figure 11: The close connection between PQ meshes and D-strip models can be exploited for *mixed multilayer constructions*. Usually one prefers simple structural elements, which can be achieved using a PQ mesh. An offset of this PQ mesh gives a good initialization for optimization of a D-strip model.

are developable because then we can use them for the definition of curved steel beams (see Figure 10). As it turns out, the *circular* strip models which possess families of inscribed circles, and the *conical* strip models, which possess families of inscribed cones, are the right geometric entity to consider here. Both are semidiscrete versions of principal curvature line parametrizations.

3 Ruled panels and beyond

Different types of segmentation are driven by capabilities of the material used, requirements on the substructure, aesthetics, etc. Up to now we have considered segmentation of surfaces into developable pieces only. These are relevant for materials that can be single curved to a certain extent, like glass, sheet metal or wood. As an example for a material with completely different properties we consider freeform surfaces made from concrete, for which one can not avoid to build freeform moulds or substructures. Therefore the production of moulds must be cheap, which can be achieved in practice e.g. by hot wire cutting of styrofoam. This leads to the necessity of approximating a freeform surface with a sequence of ruled surfaces.

Smooth strip models. Like in section 2, we are naturally led to semidiscrete surface representations, cf. Equation (4). Instead of developability, we aim for smooth transitions between successive ruled strips along their common edge curves. This is the case if the vectors $\Delta_i p, \partial_u p, \Delta_{i+1} p$ are linearly dependent, or, in other words, if the infinitesimal vertex star

$$\begin{array}{ccccc} & & p(i,u+\Delta u) & & \\ & & | & & \\ p(i-1,u) & \text{———} & p(i,u) & \text{———} & p(i+1,u) \qquad (\Delta u \to 0) \\ & & | & & \\ & & p(i,u-\Delta u) & & \end{array} \tag{6}$$

is planar.

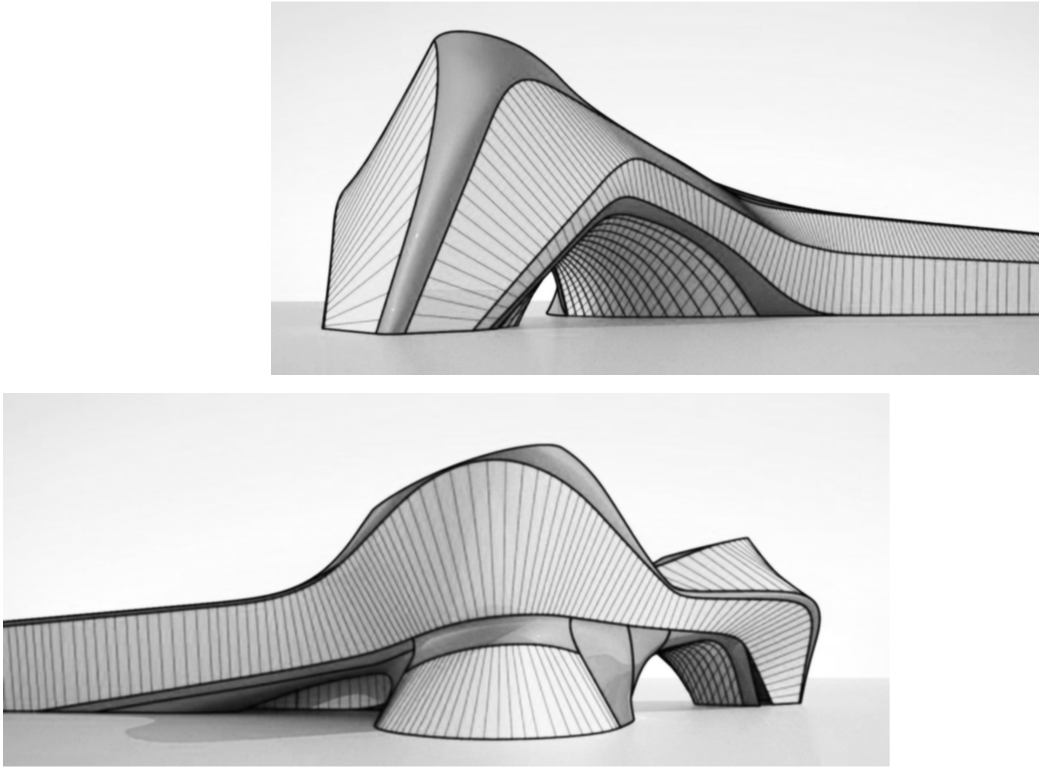
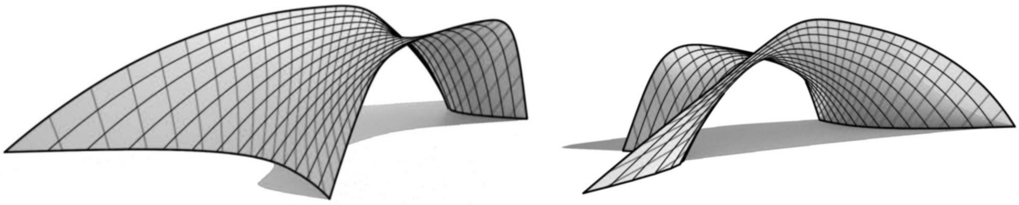


Figure 12: *Above:* Depending on the behavior of the asymptotic curves of the given design surface, negatively curved areas may be approximated by large ruled surface patches, or by a smooth union of ruled surface strips. Here an application to Zaha Hadid Architects' design for the *Nuragic and Contemporary Art Museum* in Cagliari, Italy is shown. *Below:* Details of surface parts which carry a dense sequence of ruled strips (taken from the underside).



Analogous to D-strip models, an optimization procedure is used to compute a smooth ruled strip model. The Gaussian curvature of ruled surfaces is ≤ 0, therefore it only makes sense to approximate negatively curved surfaces with ruled strip models. The initialization of the optimization and the decision on the number of strips one should use can be made by inspecting the asymptotic curves of the given design surface. Rulings should approximately follow the (less curved) asymptotic curves. Hence, small curvature of one family of asymptotic curves implies a small number of ruled strips. Figure 12 shows an example.

Future research. The field of Architectural Geometry is just emerging and thus a large number of problems have not been addressed so far. While solving parts of the existing problems, architects are creating even more complex and challenging shapes and thus provide a steady input to the list of future research topics.

Among the problem areas addressed in this note, the initialization of optimization algorithms with conjugate curve networks is probably the most challenging and important unsolved task. The challenge lies in the incorporation of design intents, while meeting various constraints and dealing with global problems such as the placement of singularities.

There is basically no geometric research on freeform structures from non-ruled double curved panels. Those have to be manufactured with moulds. Depending on the technology being used, moulds may be reusable and then the interesting question arises to cover a freeform surface with panels that can be manufactured with a small number of moulds. If the surface does not exhibit symmetries, precise congruence of panels may not be achievable. However, one can aim at panels that can be cut out from a slightly larger panel produced with the same mould.

The number of tasks being unsolved is also enlarged by the number of different materials being used, since their behavior and production technology has to enter the panel layout computation. Panel layout and the underlying supporting structure may not be totally separated. Therefore, structural aspects have to be incorporated as well.

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