Notes on the Integration of the Angular Function in the Parametrization of the Vesica Piscis

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In order to determine α_h, we (overdo it maybe a little bit and) show that

$$\alpha(s) = \int_{\sigma=0}^{s} \sqrt{\frac{a+\cos\sigma}{b+\cos\sigma}\frac{\prod_{i=1}^{n_c}(c_i+\cos\sigma)}{\prod_{i=1}^{n_d}(d_i+\cos\sigma)}}d\sigma, \tag{1}$$

with $n_c \leq n_d$ and d_i being distinct constants[1], can be expressed as a linear combination of elliptic integrals of the third kind[2] with different elliptic characteristics, that is

$$\alpha(s) = \sum_{i=0}^{n_d} \gamma_i \Pi(\delta_i; \phi(s), m) \tag{2}$$

where

$$\phi(s) = \arctan(\zeta \tan \frac{s}{2}). \tag{3}$$

We compute[3] the occurring constants γ_i, δ_i, ζ and m explicitly in terms of the given a, b, c_i and d_i.

Lets therefore assume α to be given by (2). As

$$\frac{d}{ds}\Pi(l; \phi(s), k) = \frac{\phi'}{(1 - l\sin^2\phi)\sqrt{1 - k\sin^2\phi}},$$

the derivative of α reads

$$\alpha' = \frac{\phi'}{\sqrt{1 - m\sin^2\phi}} \sum_{i=0}^{n} \frac{\gamma_i}{1 - \delta_i \sin^2\phi}. \tag{4}$$

[1] Probably, this condition could be generalized, but I'm not (yet) familiar with closed formulas for the coefficients of a partial fraction decomposition with multiple roots.

[2] $\Pi(l; \phi(s), k) = \int_0^{\phi} \frac{1}{(1-l\sin^2\theta)\sqrt{1-k\sin^2\theta}} d\theta$

[3] In order to avoid division by zero, it will turn out that $a \neq 1$, $b \neq -1$ and $d_i \neq -1$ for $i = 1, \ldots, n_d$. In order to avoid lengthy formulations, we therefore start with formal manipulations.

With the introduction of a new constant ζ we make the ansatz (3) for the Jacobi amplitude ϕ of the integrals. Utilizing the trigonometric relations

$$\begin{aligned}\sin\arctan x &= \frac{x}{\sqrt{1+x^2}},\\ \tan\frac{s}{2} &= \frac{\sin s}{\cos s+1},\\ \sin^2 s &= (1-\cos s)(1+\cos s),\end{aligned}$$

$C_k = 1 - k\sin^2\phi$ thus simplifies to

$$\begin{aligned}C_k &= 1-k\frac{\zeta^2\sin^2 s}{\zeta^2\sin^2 s+(1+\cos s)^2}\\ &= 1-k\frac{\zeta^2(1-\cos s)}{(1-\zeta^2)\cos s+1+\zeta^2}\\ &= \frac{(1-\zeta^2(1-k))\cos s+1+\zeta^2(1-k)}{(1-\zeta^2)\cos s+1+\zeta^2}.\end{aligned}$$

With the abbreviation of the, w.r.t. $\cos s$ linear, term

$$D_k = (1-\zeta^2(1-k))\cos s+1+\zeta^2(1-k),$$

we write $C_k = \frac{D_k}{D_0}$.

Furthermore, using the half-argument formula $\cos^2\frac{s}{2} = \frac{1}{2}(\cos s+1)$,

$$\phi' = \frac{\zeta}{(1-\zeta^2)\cos s+1+\zeta^2} = \frac{\zeta}{D_0}.$$

Insertion into (4) therefore yields

$$\begin{aligned}\alpha' &= \frac{\phi'}{\sqrt{C_m}}\sum_{i=0}^{n_d}\frac{\gamma_i}{C_{\delta_i}}\\ &= \sqrt{\frac{D_0}{D_m}}\,\zeta\sum_{i=0}^{n_d}\frac{\gamma_i}{D_{\delta_i}}.\end{aligned}$$

Finally, we introduce a new constant ξ, and make the following two ansatzes

$$\sqrt{\frac{D_0}{D_m}} = \frac{1}{\xi}\sqrt{\frac{\cos s+a}{\cos s+b}}, \tag{5}$$

and

$$\zeta\sum_{i=0}^{n_d}\frac{\gamma_i}{D_{\delta_i}} = \xi\,\frac{\prod_{i=1}^{n_c}(c_i+\cos s)}{\prod_{i=1}^{n_d}(d_i+\cos s)}. \tag{6}$$

This results in feasible solutions, since (5) yields a quadratic polynomial in $\cos s$ and thus three equations for the unknowns m, ζ and ξ. Similarly, (6) yields $n_d + 1$ conditions for the $n_d + 1$ unknowns γ_i, if the δ_i are chosen appropriately.

Before resuming with the computation, we recall the formula for the determination of the coefficients of a partial fraction decomposition[4]:

> Given two polynomials $P(x)$ and $Q(x) = \prod_{i=1}^{n}(x - \alpha_i)$, where the α_i are distinct constants and $\deg P < n$, the partial fraction decomposition reads
>
> $$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)} \frac{1}{(x - \alpha_i)},$$
>
> where Q' is the derivative of the polynomial Q.

Lets first consider (6). In order to be able to compare coefficients, we choose δ_i with $i = 1, \ldots, n_d$, so that D_{δ_i} become multiples of $d_i + \cos s$. This results in

$$d_i = \frac{1 + \zeta^2(1 - \delta_i)}{1 - \zeta^2(1 - \delta_i)},$$

which yields

$$\delta_i = 1 - \frac{d_i - 1}{(d_i + 1)\zeta^2}, \qquad \text{and} \qquad D_{\delta_i} = \frac{2}{d_i + 1}(d_i + \cos s),$$

for $i = 1, \ldots, n_d$. Equation (6) therefore becomes

$$\frac{\zeta}{2\xi}\Big(\frac{\gamma_0}{D_{\delta_0}} + \sum_{i=1}^{n_d} \gamma_i(d_i + 1)\frac{1}{d_i + \cos s}\Big) = \frac{\prod_{i=1}^{n_c}(c_i + \cos s)}{\prod_{i=1}^{n_d}(d_i + \cos s)}.$$

Now we consider the partial fraction decomposition of the ratio of the two polynomials $C(x) = \prod_{i=1}^{n_c}(c_i + x)$ and $D(x) = \prod_{i=1}^{n_d}(d_i + x)$.

In case of $n_c = n_d$, we rewrite

$$\begin{aligned}
\frac{C(x)}{D(x)} &= 1 + \frac{(C - D)(x)}{D(x)} \\
&= 1 + \sum_{i=1}^{n_d} \frac{(C - D)(-d_i)}{D'(-d_i)} \frac{1}{d_i + x} \\
&= 1 + \sum_{i=1}^{n_d} \frac{C(-d_i)}{D'(-d_i)} \frac{1}{d_i + x} \\
&= 1 + \sum_{i=1}^{n_d} \frac{\prod_{j=1}^{n_c}(c_j - d_i)}{\prod_{j \neq i}(d_j - d_i)} \frac{1}{d_i + x}.
\end{aligned}$$

[4] `https://en.wikipedia.org/wiki/Partial_fraction_decomposition`, TODO: find better reference.

In case of $n_c < n_d$, a straight forward computation yields

$$\begin{aligned}\frac{C(x)}{D(x)} &= \sum_{i=1}^{n_d} \frac{C(-d_i)}{D'(-d_i)} \frac{1}{d_i + x} \\ &= \sum_{i=1}^{n_d} \frac{\prod_{j=1}^{n_c}(c_j - d_i)}{\prod_{j\neq i}(d_j - d_i)} \frac{1}{d_i + x}.\end{aligned}$$

It thus follows by comparing coefficients of both modified sides of (6), that

$$\gamma_i = \frac{2\xi}{\zeta} \frac{\prod_{j=1}^{n}(c_j - d_i)}{(1 + d_i)\prod_{j\neq i}(d_j - d_i)} \quad \text{for} \quad i = 1, \ldots, n.$$

Furthermore, we require D_{δ_0} to be constant w.r.t. $\cos s$, and hence define

$$\delta_0 = 1 - \frac{1}{\zeta^2},$$

and consequently

$$\gamma_0 = \begin{cases} \frac{2\xi}{\zeta} & \text{for } n_c = n_d, \\ 0 & \text{for } n_c < n_d. \end{cases}$$

The remaining unknowns are computed from the second ansatz (5) and read

$$\begin{aligned}\zeta &= \pm\frac{\sqrt{a-1}}{\sqrt{a+1}}, \\ m &= \frac{2(a-b)}{(a-1)(b+1)}, \\ \xi &= \pm\frac{\sqrt{a+1}}{\sqrt{b+1}}.\end{aligned}$$

This explains the restrictions for the constants, namely $a \neq 1$, $b \neq -1$ and $d_i \neq -1$ for $i = 1, \ldots, n_d$ mentioned earlier.

Returning to the initial problem of integrating α'_h, we utilize the results from above for $n_c = n_d = 1$ and choose the coefficients

$$a = 2 + 2h, \quad b = c_1 = 2 - 2h, \quad d_1 = \frac{1}{4}(5 - h^2),$$

avoiding $a \neq 1$ and $b \neq -1$.