Convolution surfaces of quadratic triangular Bézier surfaces

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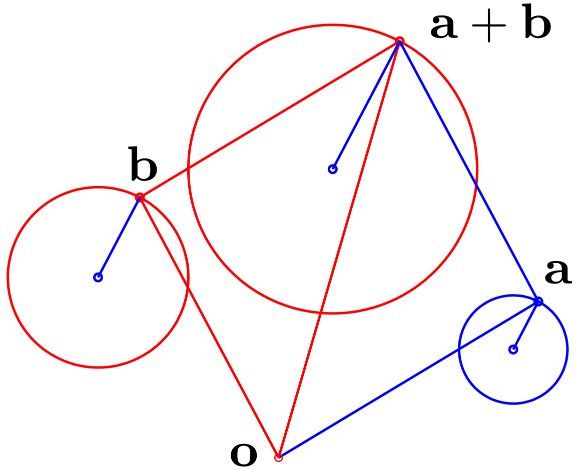
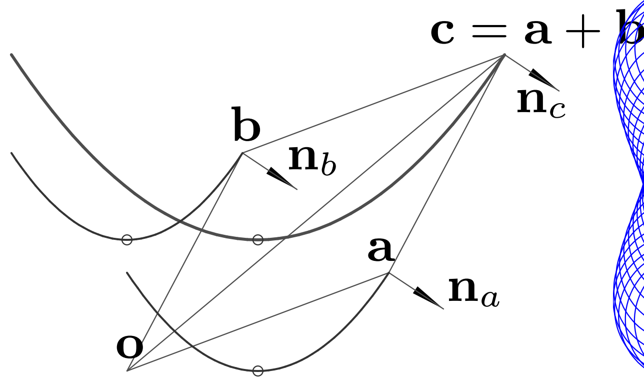
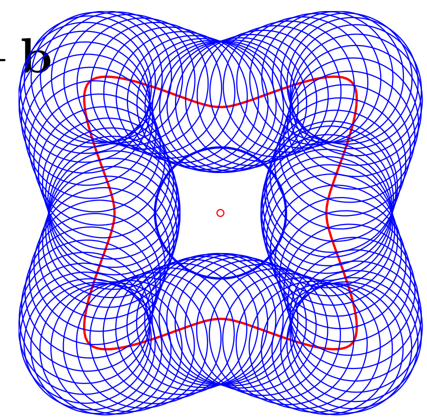
Convolution surfaces

- parametrized surface $F : \mathbf{f}(u,v)$, normal vector field $\mathbf{n}_f(u,v)$
- parametrized surface $G : \mathbf{g}(s,t)$, normal vector field $\mathbf{n}_g(u,v)$
- convolution surface

$$F \star G = \{\mathbf{f} + \mathbf{g} | \mathbf{f} \in F, \mathbf{g} \in G, \mathbf{n}_f \parallel \mathbf{n}_g\}. \quad (1)$$







LN surfaces

Jüttler, 1998; Jüttler and Sampoli, 2000.

- S is called *LN-surface* if there exists a *rational* parameterization $\mathbf{s}(u, v)$ with normal vector field

$$\mathbf{n}(u, v) = (u, v, 1).$$

- Tangent planes of S are $T(u, v):\; h(u, v) + ux + vy + z = 0.$
- *Dual affine equation:* $W = h(U, V) = \frac{a(U,V)}{b(U,V)}$.
- The set of tangent planes of an LN-surface is a *graph of a rational function.*

LN surfaces 2

- Homog. coord. Y_0, Y_1, Y_2, Y_3 with $U = Y_1/Y_3$, $V = Y_2/Y_3$, $W = Y_0/Y_3$,

$$S: \ Y_0 Y_3^{n-l-1}\, b(Y_1, Y_2, Y_3) - Y_3^{n-k} a(Y_1, Y_2, Y_3) = 0,$$

  where $\deg(a) = k$ and $\deg(b) = l$.

- *Characterization of LN surfaces*: The plane at infinity $\omega = \mathbb{R}(1, 0, 0, 0)$ is an $n-1$-fold plane of S, since all $n-2$-th partial derivatives vanish at $(1, 0, 0, 0)$.

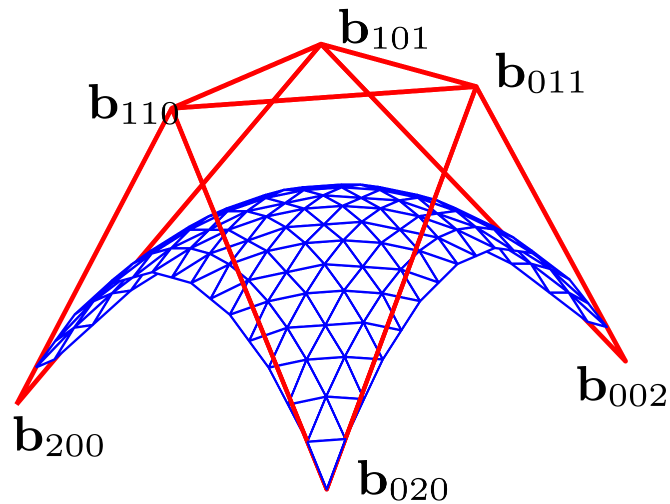
- Convolution surfaces $F \star G$ of arbitrary rational surfaces F and LN surfaces G are rational (Sampoli, Peternell, Jüttler, 2006).

Quadratic triangular Bézier surfaces

S is called a quadratic triangular Bézier surface if it admits a parametrization

$$\mathbf{s}(u,v,w) = \sum_{i+j+k=2} B^2_{ijk}(u,v,w)\mathbf{b}_{ijk},$$

with respect to the barycentric coordinates (u,v,w). Sederberg, Anderson, 1985.



- Control points $\mathbf{b}_{ijk}$,
- Bernstein polynomials $B^2_{ijk} = \frac{2!}{i!j!k!}u^i v^j w^k$.

With respect to affine parameters u, v, $(w = 1 - u - v)$ the monomial form of the parametrization of S looks like

$$\mathbf{s}(u,v) = \frac{1}{2}\mathbf{a}u^2 + \mathbf{b}uv + \frac{1}{2}\mathbf{c}v^2 + \mathbf{d}u + \mathbf{e}v + \mathbf{f},$$

with $\mathbf{a}, \ldots \mathbf{f} \in \mathbb{R}^3$.

Quadratic triangular Bézier surfaces 2

Degen, 1994; Coffmann, et. al. 1996. Albrecht, 2002.

- S is of order < 4 and class 3.
- S carries a 2-par. family of parabolas (possibly degenerate).
- The Veronese $V_2^2 \in \mathbb{P}^5$ is parametrized by

$$\mathbf{v}(u_0, u_1, u_2) = (u_0^2, u_0u_1, u_0u_2, u_1^2, u_1u_2, u_2^2)\mathbb{R} \in \mathbb{P}^5,$$

- S is the projection of V_2^2 by

$$\pi : \begin{bmatrix} u_0^2 \\ \mathbf{s}(u_0, u_1, u_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{f} & \mathbf{d} & \mathbf{e} & \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} \cdot \mathbf{v}^T.$$

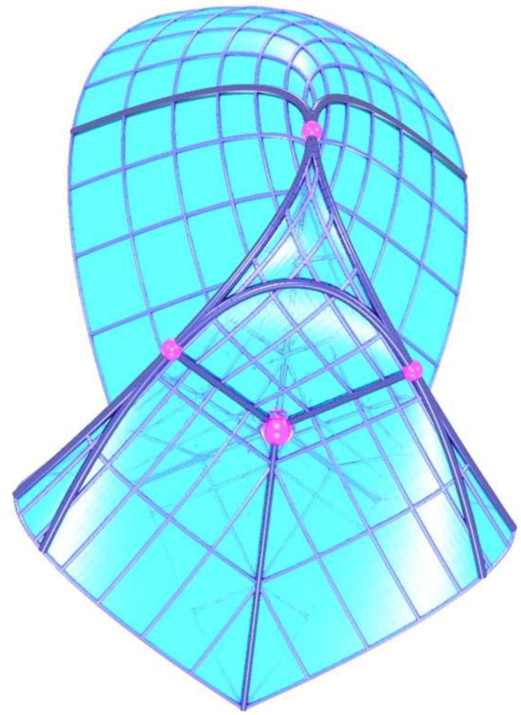
- S is called *Steiner roman surface*, Steiner, 1882.

Quadratic triangular Bézier surface 3

- Let $C = S \cap \omega$ be the curve at infinity of S.
- C is obtained for $u_0^2 = 0$ and thus a double curve of S. It is parameterized by

$$(\mathbf{a}u_1^2 + 2\mathbf{b}u_1u_2 + \mathbf{c}u_1^2)\mathbb{R} \ \in \ \omega.$$

  C is a conic if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are linearly independent, otherwise C is degenerate.
- Quadratic triangular Bézier surfaces are LN-surfaces. (Lávička, Bastl 2006.)



Polynomial *Steiner surface* with three double lines meeting at the triple point and conics touching at the pinch points.

Dual parameterization

- Partial derivatives of $\mathbf{s}$

$$\mathbf{s}_u(u,v) = \mathbf{a}u + \mathbf{b}v + \mathbf{d}, \quad \mathbf{s}_v(u,v) = \mathbf{b}u + \mathbf{c}v + \mathbf{e}.$$

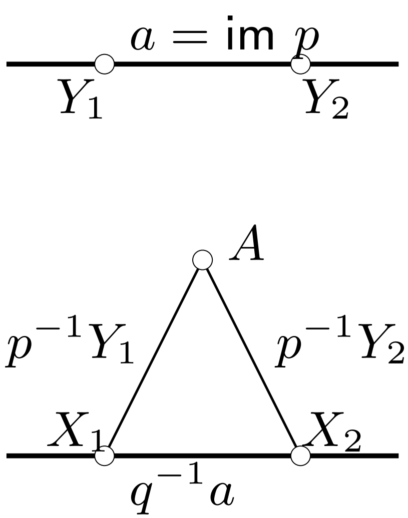
- Tangent planes $T(u,v): \ (\mathbf{x} - \mathbf{s}(u,v))^T \cdot (\mathbf{s}_u \times \mathbf{s}_v)(u,v) = 0$ and coefficient vector $\mathbf{t}(\mathbf{u}) = (-\mathsf{det}(\mathbf{s}, \mathbf{s}_u, \mathbf{s}_v), \mathbf{s}_u \times \mathbf{s}_v)$.
- Define the matrices $P := (\mathbf{d}, \mathbf{a}, \mathbf{b})$ and $Q := (\mathbf{e}, \mathbf{b}, \mathbf{c})$,
- Projective mappings $p, q : \mathbb{P}^2 \to \mathbb{P}^2$.

$$p: \ \mathbf{u}\mathbb{R} \mapsto (P\mathbf{u})\mathbb{R}, \quad q: \ \mathbf{u}\mathbb{R} \mapsto (Q\mathbf{u})\mathbb{R}, \text{ with } \mathbf{u} = (u_0, u_1, u_2).$$

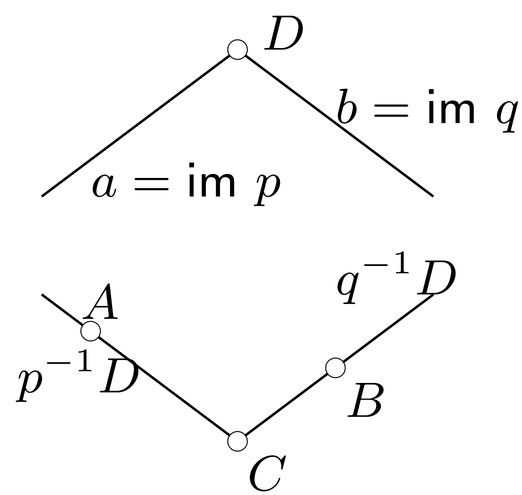
- The parameterization $\mathbf{t}(\mathbf{u})$ (or $\mathbf{n}$) has a *base point* at $\mathbf{u}^\star\mathbb{R}$ if and only if $\mathsf{rk}\ (P\mathbf{u}^\star, Q\mathbf{u}^\star) = 1$. This implies $\mathbf{n}(\mathbf{u}^\star) = (0,0,0)^T$ and $\mathbf{t}(\mathbf{u}^\star) = (0,0,0,0)$.

Base points of the dual parameterization

- rk P=3,rk Q=3: Base points are given by the eigenvectors of $Q^{-1}P$.
- rk $P = 2$, rk $Q = 3$: Base point $A =$ ker (P). The further (0,1,2) base points X_1, X_2 are contained in q^{-1}(im (p)).
- rk $P = 2$, rk $Q = 2$: Base points $A =$ ker (P) and $B =$ ker (Q). The further base point is $C = p^{-1}(D) \cap q^{-1}(D)$ with $D =$ im $(P) \cap$ im (Q). ($C = A$ or $C = B$ or even $A = B$ is possible.)



rk $P = 2$, rk $Q = 3$



rk $P =$ rk $Q = 2$

Quadratic Cremona transformations

- A mapping $\phi: \ \mathbb{P}^2 \to \mathbb{P}^2$ with $\mathbf{u}\mathbb{R} \mapsto \mathbf{v}\mathbb{R}$ is called a quadratic *Cremona transformation* if

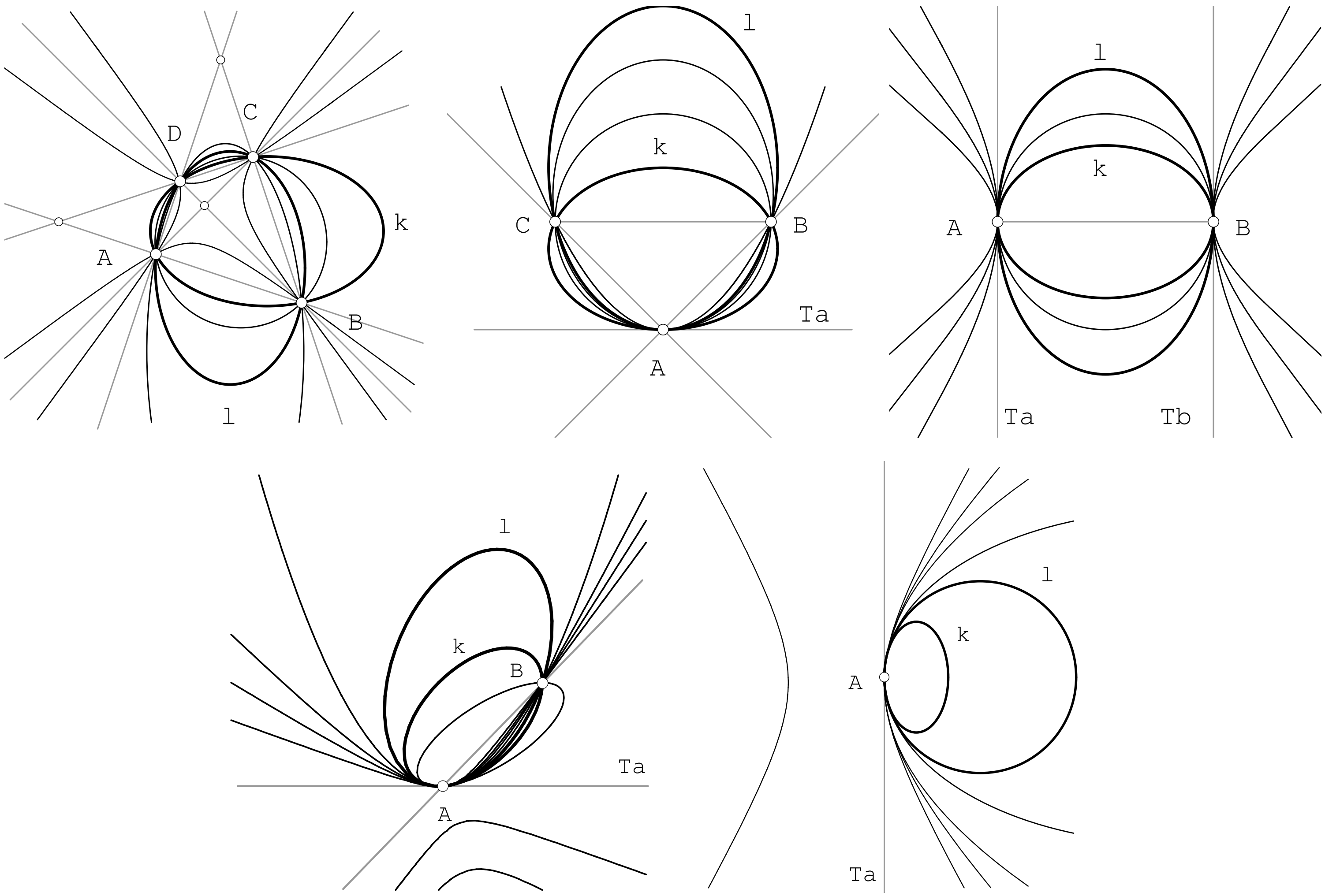
$$\mathbf{v}\mathbb{R} = (q_0(\mathbf{u}) : q_1(\mathbf{u}) : q_2(\mathbf{u}))$$

  where q_i are homogeneous quadratic polynomials and ϕ^{-1} is of the same form (birational).

- A quadratic Cremona transformation $\phi: \ \mathbb{P}^2 \to \mathbb{P}^2$ possesses an associated net of conics in $\mathbb{P}^2$.

- A net of conics is a special 2-par. family of conics, obtained from pencils of conics by removing one interpolation condition.

Pencils of concis



Nets of conics

- Conics through three points $A = (1:0:0)$, $B = (0:1:0)$ and $C = (0:0:1)$. The net is spanned by $x_1x_2 = 0$, $x_2x_0 = 0$, and $x_0x_1 = 0$ and a general conic c is $\alpha x_1x_2 + \beta x_2x_0 + \gamma x_0x_1 = 0$.

$$\begin{aligned} \phi: \quad & (x_0 : x_1 : x_2) \mapsto (x_0' : x_1' : x_2') = (x_1x_2 : x_0x_2 : x_0x_1), \\ \phi^{-1}: \quad & (x_0' : x_1' : x_2') \mapsto (x_0 : x_1 : x_2) = (x_1'x_2' : x_0'x_2' : x_0'x_1'). \end{aligned}$$

- Conics through $B = (0:0:1)$ and line element $A = (1:0:0)$ and $a: \ x_2 = 0$ The net is spanned by $x_1x_2 = 0$, $x_0x_2 = 0$, and $x_1^2 = 0$.

$$\begin{aligned} \phi: \quad & (x_0 : x_1 : x_2) \mapsto (x_0' : x_1' : x_2') = (x_1x_2 : x_0x_2 : x_1^2), \\ \phi^{-1}: \quad & (x_0' : x_1' : x_2') \mapsto (x_0 : x_1 : x_2) = (x_1'x_2' : x_0'x_2' : x_1'^2). \end{aligned}$$

- Osculating element $k: \ x_1^2 - x_0x_2 = 0$ and point $A = (1:0:0)$. The net is spanned by k and $x_1x_2 = 0$ and $x_2^2 = 0$.

$$\begin{aligned} \phi: \quad & (x_0 : x_1 : x_2) \mapsto (x_0' : x_1' : x_2') = (x_1x_2 : x_1^2 - x_0x_2 : x_2^2), \\ \phi^{-1}: \quad & (x_0' : x_1' : x_2') \mapsto (x_0 : x_1 : x_2) = (x_0'^2 - x_1'x_2' : x_0'x_2' : x_2'^2). \end{aligned}$$

LN property of quadratic Bézier triangles

- S is a quadratic triangular Bézier surface if it has a parametrization

$$\mathbf{s}(u,v) = \frac{1}{2}\mathbf{a}u^2 + \mathbf{b}uv + \frac{1}{2}\mathbf{c}v^2 + \mathbf{d}u + \mathbf{e}v + \mathbf{f}, \text{ with } \mathbf{a}, \ldots \mathbf{f} \in \mathbb{R}^3.$$

- Normal vector $\mathbf{n} = (n_1, n_2, n_3)$

$$\mathbf{n}(u,v) = (\mathbf{a}\times\mathbf{b})u^2 + (\mathbf{a}\times\mathbf{c})uv + (\mathbf{b}\times\mathbf{c})v^2 + (\mathbf{a}\times\mathbf{e}+\mathbf{d}\times\mathbf{b})u + (\mathbf{b}\times\mathbf{e}+\mathbf{d}\times\mathbf{c})v + \mathbf{d}\times\mathbf{e}.$$

- Resultants of the coordinate funcitons

$$\mathsf{Res}(\mathbf{n}_1, \mathbf{n}_2, v) = (b_3e_3 - c_3d_3 + (b_3^2 - a_3c_3)u)p(u),$$
$$\mathsf{Res}(\mathbf{n}_2, \mathbf{n}_3, v) = (b_1e_1 - c_1d_1 + (b_1^2 - a_1c_1)u)p(u),$$
$$\mathsf{Res}(\mathbf{n}_1, \mathbf{n}_3, v) = (b_2e_2 - d_2c_2 + (b_2^2 - a_2c_2)u)p(u).$$

- The zeros of the cubic polynomial $p(u)$ determine the base points of a Cremona transformation. This implies that
- The conics $\mathbf{n}_i(u,v) = 0$, $i = 1, 2, 3$ form a net.

LN property of quadratic Bézier triangles 2

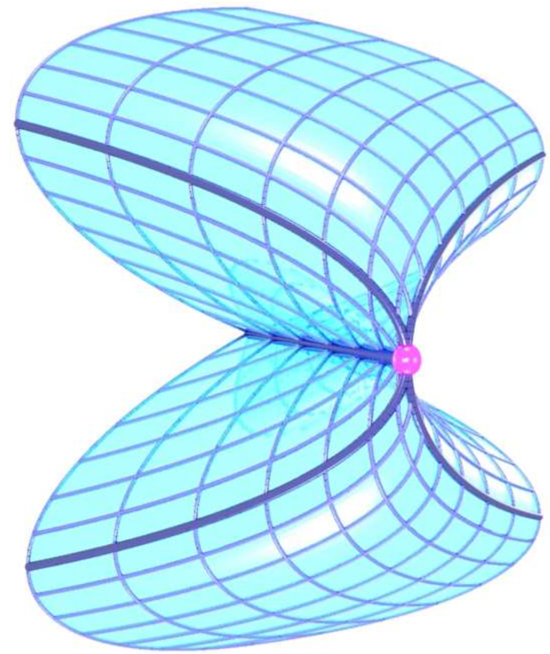
- The conics $\mathbf{n}_i(u, v) = 0$, $i = 1, 2, 3$ form a net.
- Associated with the net there exists a Cremona transformation which maps the net of conics to the set of lines of $\mathbb{P}^2$.
- Thus the normal vector $\mathbf{n}(u, v)$ of S can be linearly parameterized.
- This proves the LN-property of quadratic triangular Bézier surfaces.
- All convolution surfaces of S with arbitrary rational surfaces as well as the offsets of S are rational.

Affine normal forms

Peters, Reif, 1998.

	Parameterization	Transformation	Dual homogeneous equation
1	$\mathbf{f} = (u^2, v^2, u+v)$	$u = \frac{-1}{2s},\ v = \frac{-1}{2t}$	$4Y_0Y_1Y_2 - Y_3^2(Y_1+Y_2) = 0$
2	$\mathbf{f} = (u^2, v^2+u, v)$	$u = \frac{-t}{2s},\ v = \frac{-1}{2t}$	$4Y_0Y_1Y_2 - Y_2^3 - Y_1Y_3^2 = 0$
3	$\mathbf{f} = (u^2, uv, v)$	$u = \frac{-1}{t},\ v = \frac{2s}{t^2}$	$Y_0Y_2^2 + Y_1Y_3^2 = 0$
4	$\mathbf{f} = (u^2+v, uv, u)$	$u = \frac{-s}{t},\ v = \frac{2s^2-t}{t^2}$	$Y_0Y_2^2 + Y_1^3 - Y_1Y_2Y_3 = 0$
5	$\mathbf{f} = (u^2-v^2, uv, u)$	$u = \frac{-2s}{4s^2+t^2},\ v = \frac{-t}{4s^2+t^2}$	$Y_0(Y_2^2+4Y_1^2) - Y_1Y_3^2 = 0$
6	$\mathbf{f} = (u^2, v^2, uv+u)$	$u = \frac{2t}{1-4st},\ v = \frac{-1}{1-4st}$	$Y_0(4Y_1Y_2 - Y_3^2) - Y_2Y_3^2 = 0$
7	$\mathbf{f} = (u^2, v^2, uv+u+v)$	$u = \frac{2t-1}{1-4st},\ v = \frac{2s-1}{1-4st}$	$Y_0(4Y_1Y_2 - Y_3^2) + Y_3^3$ $-Y_3^2(Y_1+Y_2) = 0$
8	$\mathbf{f} = (u^2, v^2+u, uv)$	$u = \frac{2t^2}{1-4st},\ v = \frac{-t}{1-4st}$	$Y_0(4Y_1Y_2 - Y_3^2) + Y_2^3 = 0$
9	$\mathbf{f} = (u^2, v^2+u, uv-v)$	$u = \frac{1+2t^2}{1-4st},\ v = \frac{-t-2s}{1-4st}$	$Y_0(4Y_1Y_2 - Y_3^2) - Y_2^3$ $-Y_3^2(Y_1+Y_2) = 0$

Example 1



- Consider $\mathbf{s} = (u^2, v^2, u+v)^T$ with normal vector $\mathbf{n} = (-2v, -2u, 4uv)^T$.
- The projective mappings p and q are both singular $\mathsf{rk}\ P = \mathsf{rk}\ Q = 2$.
- The Cremona transform of type 1 is

$$\phi: \ u = -\frac{1}{2s}, \quad v = -\frac{1}{2t}$$

and yields $\mathbf{n} = (s, t, 1)^T$.

- LN-parameterization and tangent planes

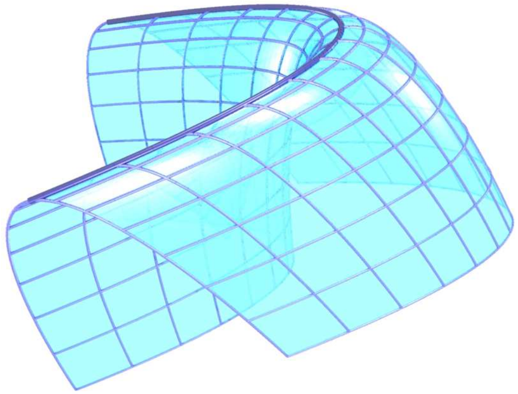
$$\mathbf{s}(s,t) = \frac{1}{4s^2t^2}(t^2, s^2, -2st(s+t))^T, \quad T(s,t): \ \frac{s+t}{4st} + sx + ty + z = 0.$$

- Dual equation of the surface S as $4Y_0Y_1Y_2 - Y_3^2(Y_1+Y_2) = 0$.

Example 2

- Consider $\mathbf{s} = (u^2, v^2 + u, v)^T$ with normal vector $\mathbf{n} = (1, -2u, 4uv)^T$.
- Base points are given by $\ker Q$ and $\ker P$.
- The Cremona transform of type 2 is

$$\phi: \ u = -\frac{t}{2s}, \quad v = -\frac{1}{2t}.$$

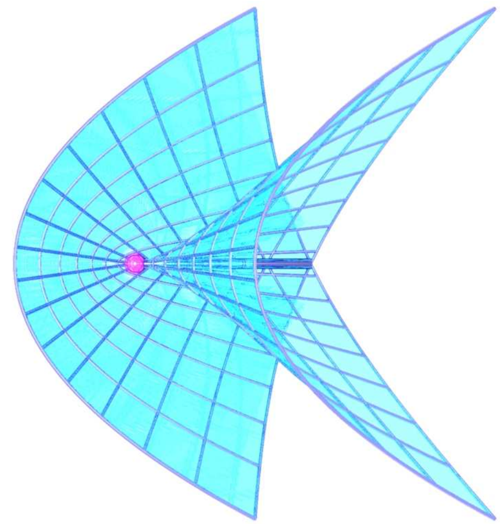


- LN-parameterization and tangent planes

$$\mathbf{s}(s,t) = \frac{1}{4s^2t^2}(t^4, s(s-2t^3), -2s^2t)^T, \quad T(s,t): \ \frac{s+t^3}{4st} + sx + ty + z = 0.$$

- Dual equation of the surface S as $4Y_0Y_1Y_2 - Y_2^3 - Y_3^2Y_1 = 0$.

Example 3



- Consider $\mathbf{s} = (u^2, v^2 + u, uv)^T$ with normal vector $\mathbf{n} = (u - 2v^2, -2u^2, 4uv)^T$.
- Only one base point determined by $\ker Q$.
- The Cremona transform of type 3 is

$$\phi : \ u = \frac{2t^2}{1 - 4st}, \quad v = \frac{-t}{1 - 4st}.$$

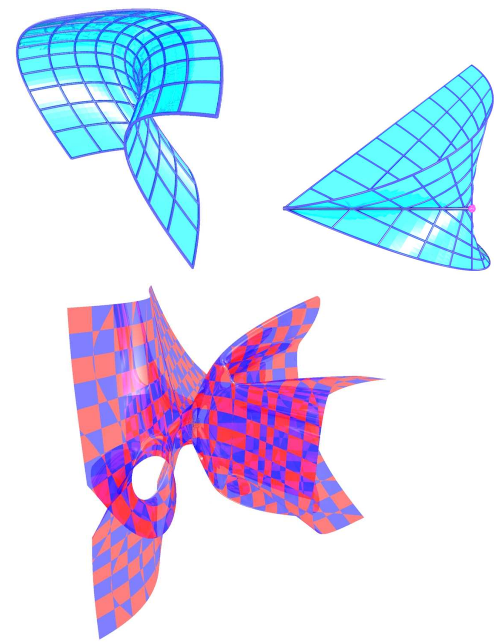
- LN-parameterization and tangent planes

$$\mathbf{s}(s,t) = \frac{1}{(4st-1)^2}(4t^4, -t^2(8st-3), -2t^3)^T,$$

$$T(s,t) : \ \frac{s + t - 1}{4st - 1} + sx + ty + z = 0.$$

- Dual equation of the surface S as $Y_0(4Y_1Y_2 - Y_3^2) - Y_2^3 = 0$.

Convolution surface example 1



- F and G are given by $\mathbf{f} = (u^2, v^2 + u, v)^T$ and $\mathbf{g} = (st, s^2, s+t)^T$, with the normal vectors $\mathbf{n}_f = (1, -2u, 4uv)^T$ and $\mathbf{n}_g = (2s, s - t, -2s^2)^T$.

- Cremona transforms

$$\phi_F \quad : \quad u = -\frac{y}{2x}, \quad v = -\frac{1}{2y},$$

$$\phi_G \quad : \quad s = -\frac{1}{x}, \quad t = \frac{2y - x}{x^2}.$$

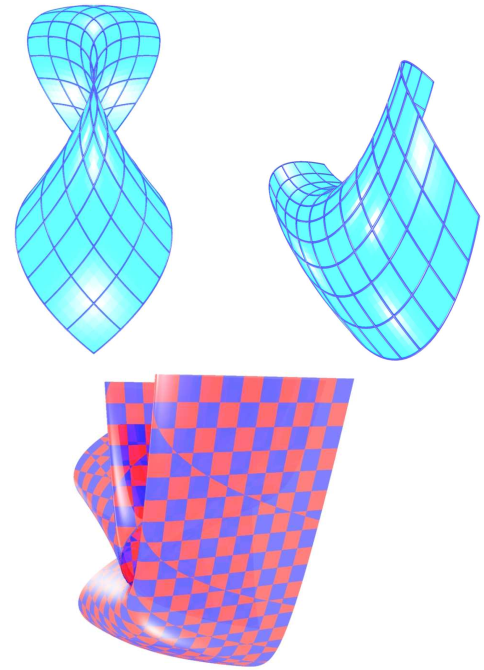
- results in

$$\mathbf{f}(x, y) = \frac{1}{4x^2y^2}(y^4, x(x - 2y^3), -2x^2y)^T, \quad \mathbf{g}(x, y) = \frac{1}{x^3}(x - 2y, x, 2x(y - x))^T.$$

- Convolution surface $C = F \star G$ is parametrized by $\mathbf{f}(x, y) + \mathbf{g}(x, y)$,

$$\mathbf{c}(x, y) = \frac{1}{4x^2y^2}(xy^2 + 4x - 8y, x^2 - 2xy^3 + 4y^2, 4y^2 - x^2 - 4xy)^T.$$

Convolution surface example 2



- F and G are given by $\mathbf{f} = (2u^2, 2v^2, 2(u+v))^T$ and $\mathbf{g} = (s^2, t, t^2+s)^T$, with the normal vectors $\mathbf{n}_f = (-8v, -8u, 16uv)^T$ and $\mathbf{n}_g = (-1, -4st, 2s)^T$.
- Cremona transforms

$$\phi_F \quad : \quad u = -\frac{1}{2y}, \quad v = -\frac{1}{2x},$$

$$\phi_G \quad : \quad s = -\frac{y}{2}, \quad t = -\frac{1}{2x}.$$

- results in

$$\mathbf{f}(x,y) = \frac{1}{2x^2y^2}(y^2, x^2, -2xy(x+y))^T, \ \mathbf{g}(x,y) = \frac{1}{4x^2}(1, -2xy, x(xy^2-2))^T.$$

- Convolution surface $C = F \star G$ is parametrized by $\mathbf{f}(x,y) + \mathbf{g}(x,y)$,

$$\mathbf{c}(x,y) = \frac{1}{4x^2y^2}(3y^2, 2x^2(1-y^3), xy(xy^3-4x-6y))^T.$$

Summary

- Quadratic triangular Bézier surfaces S are LN-surfaces,
- Reparametrization using planar quadratic Cremona transformations,
- Explicit construction of the rational convolution surfaces with any other rational surface.

Thank you!