Affine Circle Geometry over Quaternion Skew Fields

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Abstract

We investigate the affine circle geometry arising from a quaternion skew field and one of its maximal commutative subfields.

1 Introduction

1.1

The present paper is concerned with the chain geometry $\Sigma(K, L)$ (cf. [1]) on a field extension L/K, where K is a maximal commutative subfield of a quaternion skew field L. Thus L is not a K–algebra. This has many geometric consequences. Best known is probably that three distinct points do not determine a unique chain. As in ordinary Möbius–geometry, it is possible to obtain an affine plane by deleting one point, but a more sophisticated technique is necessary in order to define the lines of this plane. We take a closer look on this construction from two different points of view, starting either from a spread of lines associated to $\Sigma(K, L)$ or the point model of this spread on the Klein quadric. The chains of $\Sigma(K, L)$ yield the lines, degenerate circles and non–degenerate circles of such an affine plane. We establish some properties of these circles and show that degenerate circles are affine Baer subplanes. If K is Galois over the centre of L then each non–degenerate circle can be written as intersection of two affine Hermitian varieties.

We encourage the reader to compare our results with the survey article [9] on chain geometry over an algebra and [10]. There is an extensive literature on the real quaternions. A lot of references can be found, e.g., in [1], [2], [5], [15], [16].

1.2

Throughout this paper L will denote a quaternion skew field with centre Z and K will be a maximal commutative subfield of L. The following exposition follows [4], [8], [13, pp.168–171].

Choose any element $a \in K \setminus Z$ with minimal equation, say

$$a^2 + a\lambda_1 + \mu_1 = 0 \quad (\lambda_1, \mu_1 \in Z).$$

If K/Z is Galois then

$$(\overline{}) : K \to K, \quad u = \xi + a\eta \mapsto \overline{u} := \xi - (\lambda_1 + a)\eta \quad (\xi, \eta \in Z)$$

is an automorphism of order 2 fixing Z elementwise[1]. There exists an element $i \in L \setminus K$ such that

$$i^{-1}ui = \overline{u} \quad \text{for all } u \in K,$$

[1] By an appropriate choice of a it would be possible to have $\lambda_1 = 0$ (Char$K \neq 2$) or $\lambda_1 = 1$ (Char$K = 2$).

whence

$$ui = i\overline{u} \quad \text{for all } u \in K. \tag{1}$$

If K/Z is not Galois then, obviously, $\mathrm{Char} K = 2$ and $\lambda_1 = 0$. The mapping

$$D : K \to K, \quad u = \xi + a\eta \mapsto u^D := a\eta \quad (\xi, \eta \in Z)$$

is additive and satisfies $(uu')^D = u^D u' + uu'^D$ for all $u, u' \in K$, i.e., D is a derivation of K. There exists an $i \in L \setminus K$ such that

$$a^{-1} i a = i + 1$$

which leads to the rule

$$ui = iu + u^D \quad \text{for all } u \in K. \tag{2}$$

In every case the element i has a minimal equation over Z, say

$$i^2 + i\lambda_2 + \mu_2 = 0 \quad (\lambda_2, \mu_2 \in Z).$$

If K/Z is Galois then $i^2 \in Z$, whence $\lambda_2 = 0$. If K/Z is not Galois then i and $i+1$ have the same minimal equation. This implies $\lambda_2 = 1$. The mapping

$$A : L \to L, \quad u + iv \mapsto \left\{ \begin{array}{ll} \overline{u} - iv & : K/Z \text{ Galois}, \\ u + v + vi & : K/Z \text{ not Galois}, \end{array} \right\} \quad (u, v \in K) \tag{3}$$

is an involutory antiautomorphism of L fixing K. The norm of $x \in L$ is given by $N(x) := x^A x$.

1.3

The mappings $(\overline{})$ and D allow, respectively, the following geometric interpretations:

Let $\mathbf{V}$ be a right vector space over Z, $\dim \mathbf{V} \geq 2$. We are extending $\mathbf{V}$ to $\mathbf{V} \otimes_Z K$ with $\mathbf{v} \in \mathbf{V}$ to be identified with $\mathbf{v} \otimes 1$. Then define a mapping $\mathbf{V} \otimes_Z K \to \mathbf{V} \otimes_Z K$ by

$$\sum_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes k_{\mathbf{v}} \mapsto \left\{ \begin{array}{ll} \sum\limits_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes \overline{k}_{\mathbf{v}} & : K/Z \text{ Galois}, \\ \sum\limits_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes k_{\mathbf{v}}^D & : K/Z \text{ not Galois}, \end{array} \right\} \quad (k_{\mathbf{v}} \in K).$$

By abuse of notation, this mapping will also be written as $(\overline{})$ and D, respectively.

In terms of the projective spaces $\mathcal{P}_Z(\mathbf{V})$ and $\mathcal{P}_K(\mathbf{V} \otimes_Z K)$ the first projective space is being embedded in the second one as a Baer subspace. If $\mathbf{x}K$ is a point of $\mathcal{P}_K(\mathbf{V} \otimes_Z K) \setminus \mathcal{P}_Z(\mathbf{V})$ then through this point there is a unique line of $\mathcal{P}_K(\mathbf{V} \otimes_Z K)$ containing more than one point of $\mathcal{P}_Z(\mathbf{V})$. That line is given by

$$\mathbf{x}K \vee \overline{\mathbf{x}}K \quad \text{and} \quad \mathbf{x}K \vee (\mathbf{x}^D)K,$$

respectively[2]. Note that defining a mapping by setting $\mathbf{x}K \mapsto (\mathbf{x}^D)K$ is ambiguous, since

$$(\mathbf{x}u)^D = \mathbf{x}^D u + \mathbf{x}u^D \quad \text{for all } \mathbf{x} \in \mathbf{V} \otimes_Z K,\ u \in K.$$

We give a second interpretation in terms of affine planes[3]:

Lemma 1 *Let $\mathbf{W}$ be a right vector space over K, $\dim \mathbf{W} = 2$, and let $\{\mathbf{u}, \mathbf{v}\}$ be a basis of $\mathbf{W}$. Then*

$$\begin{array}{ll} \{\mathbf{u}k + \mathbf{v}\overline{k} \mid k \in K\} & : K/Z \text{ Galois}, \\ \{\mathbf{u}k + \mathbf{v}k^D \mid k \in K\} & : K/Z \text{ not Galois}, \end{array} \tag{4}$$

is an affine Baer subplane (over Z) of the affine plane on $\mathbf{W}$.

[2]At least in the first case this is very well known.

[3]Cf. the concept of 'Minimalkoordinaten' described, e.g., in [17, p.35]

Proof. If $\mathbf{u}'$ and $\mathbf{v}'$ are linearly independent vectors of $\mathbf{W}$ then the set of all linear combinations of $\mathbf{u}'$ and $\mathbf{v}'$ with coefficients in Z is an affine Baer subplane over Z. Write $k = \xi + a\eta$ with $\xi, \eta \in Z$.

If K/Z is Galois then

$$\mathbf{u}k + \mathbf{v}\overline{k} = (\mathbf{u} + \mathbf{v})\xi + (\mathbf{u}a - \mathbf{v}(\lambda_1 + a))\eta.$$

The vectors $\mathbf{u} + \mathbf{v}$ and $\mathbf{u}a - \mathbf{v}(\lambda_1 + a)$ are linearly independent, since otherwise we would have the contradiction $\overline{a} = -\lambda_1 - a = a$.

If K/Z is not Galois then

$$\mathbf{u}k + \mathbf{v}k^D = \mathbf{u}\xi + (\mathbf{u} + \mathbf{v})a\eta.$$

The vectors $\mathbf{u}$ and $(\mathbf{u} + \mathbf{v})a$ are linearly independent. ■

2 Projective Chain Geometry on L/K

2.1

Let L/K be given as before. Following [1, p.320ff.] we obtain an incidence structure $\Sigma(K, L)$ as follows: The points of $\Sigma(K, L)$ are the points of the projective line over L, viz. $\mathcal{P}_L(L^2)$, the blocks, now called chains, are the K-sublines of $\mathcal{P}_L(L^2)$. However, in contrast to [1], we shall regard L^2 as right vector space over K rather than L. Each s–dimensional subspace of L^2 (over L) is $2s$–dimensional over K, whence $\mathcal{P}_K(L^2) =: \mathcal{P}_K$ is 3–dimensional. The points of $\Sigma(K, L)$ now appear as lines of a spread of $\mathcal{P}_K$, say $\mathcal{S}_{L/K}$; cf. [6], [7]. If t is a line of $\mathcal{P}_K$ not contained in $\mathcal{S}_{L/K}$ then through each point of t there goes exactly one line of $\mathcal{S}_{L/K}$. The subset $\mathcal{C}$ of $\mathcal{S}_{L/K}$ arising in this way is a chain of $\Sigma(K, L)$. We call t a transversal line of the chain $\mathcal{C}$. If L/K is not Galois then each chain has exactly one transversal line, otherwise exactly two transversal lines that are interchanged under the non–projective collineation

$$\iota : \mathcal{P}_K \to \mathcal{P}_K, \quad (l_0, l_1)K \mapsto (l_0 i, l_1 i)K. \tag{5}$$

Cf. [8, Theorem 2], [11].

2.2

Write $\mathcal{L}$ for the set of lines of $\mathcal{P}_K$ and $\gamma : \mathcal{L} \to \widehat{\mathcal{P}}_K$ for the Klein mapping. Here $\widehat{\mathcal{P}}_K$ is the ambient space of the Klein quadric $\mathcal{Q} := \mathcal{L}^\gamma$. The underlying vector space of $\widehat{\mathcal{P}}_K$ is $L^2 \wedge L^2$ (over K). In [8, Theorem 1] it is shown that there is a unique 5–dimensional Baer subspace Π_Z (over Z) of $\widehat{\mathcal{P}}_K$ such that

$${\mathcal{S}_{L/K}}^\gamma = \Pi_Z \cap \mathcal{Q}.$$

With respect to Π_Z the set ${\mathcal{S}_{L/K}}^\gamma$ is an oval quadric, i.e. a quadric without lines. A subset $\mathcal{C}$ of $\mathcal{S}_{L/K}$ is a chain if, and only if, there exists a 3–dimensional subspace $\mathcal{X}$ of $\widehat{\mathcal{P}}_K$ such that[4]

$$\mathcal{X} \cap \Pi_Z \text{ is a 3–dimensional subspace of } \Pi_Z, \tag{6}$$

$$\mathcal{C}^\gamma = \mathcal{X} \cap {\mathcal{S}_{L/K}}^\gamma \text{ is an elliptic quadric of } \mathcal{X} \cap \Pi_Z \text{ (over } Z), \tag{7}$$

$$\mathcal{X} \cap \mathcal{Q} \text{ contains a line of } \widehat{\mathcal{P}}_K; \tag{8}$$

cf. [8, Theorem 1].

[4] If L is the skew field of real quaternions then K is a field of complex numbers and Z the field of real numbers. Here conditions (6) and (7) are already sufficient to characterize the γ–images of chains.

2.3

The automorphism group of $\Sigma(K,L)$ is formed by all bijections of $\mathcal{S}_{L/K}$ taking chains to chains in both directions. If κ is a collineation or a duality of $\mathcal{P}_K$ with ${\mathcal{S}_{L/K}}^{\kappa} = \mathcal{S}_{L/K}$ then κ is yielding an automorphism of $\Sigma(K,L)$. Conversely, according to [12] and [8, Theorem 4], each automorphism of $\Sigma(K,L)$ can be induced by an automorphic collineation or duality of $\mathcal{S}_{L/K}$, say κ. This κ is uniquely determined for K/Z not being Galois, otherwise the product of ι (cf. formula (5)) and κ is the only other solution.

Transferring these results to $\widehat{\mathcal{P}}_K$ establishes that an automorphic collineation μ of the Klein quadric is the γ–transform of an automorphic collineation or duality of $\mathcal{S}_{L/K}$ if, and only if, Π_Z is invariant under μ. If K/Z is Galois, then the γ–transform of the collineation ι (cf. (5)) is the Baer involution of $\widehat{\mathcal{P}}_K$ fixing Π_Z pointwise. See [8, Theorem 4].

2.4

Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two chains with a common element, say $p \in \mathcal{S}_{L/K}$. We say that $\mathcal{C}_0$ is **tangent** to $\mathcal{C}_1$ at p if there exist transversal lines t_i of $\mathcal{C}_i$ $(i = 0, 1)$ such that p, t_0, t_1 are in one pencil of lines. This is a reflexive and symmetric relation.

If K/Z is Galois then there is also an orthogonality relation on the set of chains: If $\mathcal{C}_i$ $(i = 0, 1)$ are chains with transversal lines t_i, ${t_i}^{\iota}$, respectively, then $\mathcal{C}_0$ is said to be **orthogonal** to $\mathcal{C}_1$ if t_0 intersects both t_1 and ${t_1}^{\iota}$. This relation is symmetric, since ι is an involution. Given two orthogonal chains their transversal lines form a skew quadrilateral.

The two definitions above are not given in an intrinsic way. However, both relations are invariant under automorphic collineations and dualities of $\mathcal{S}_{L/K}$ and hence invariant under automorphisms of $\Sigma(K,L)$.

The proofs of the following results are left to the reader: Chains $\mathcal{C}_0$, $\mathcal{C}_1$ are tangent at $p \in \mathcal{C}_0 \cap \mathcal{C}_1$ if, and only if, their images under the Klein mapping are quadrics with the same tangent plane at the point p^{γ}. A chain $\mathcal{C}_0$ is orthogonal to a chain $\mathcal{C}_1$ if, and only if, the subspace of $\widehat{\mathcal{P}}_K$ spanned by ${\mathcal{C}_0}^{\gamma}$ contains the orthogonal subspace (with respect to the Klein quadric) of ${\mathcal{C}_1}^{\gamma}$.

3 Affine Circle Geometry on L/K

3.1

With the notations introduced in section 2, select one line of $\mathcal{S}_{L/K}$ and label it ∞. Let $\widetilde{\mathcal{A}}$ be a (projective) plane of $\mathcal{P}_K$ through ∞ and write $\mathcal{A} := \widetilde{\mathcal{A}} \setminus \infty$. Then $\mathcal{A}$ can be viewed as an affine plane with ∞ as line at infinity. The mapping

$$\rho : \mathcal{S}_{L/K} \setminus \{\infty\} \to \mathcal{A}, \quad s \mapsto \mathcal{A} \cap s \tag{9}$$

is well–defined and bijective. A chain $\mathcal{C}$ containing ∞ yields an affine line $(\mathcal{C} \setminus \{\infty\})^{\rho}$ if, and only if, $\mathcal{C}$ has a transversal line in $\widetilde{\mathcal{A}}$. Two chains with transversal lines in $\widetilde{\mathcal{A}}$ yield parallel lines if, and only if, the chains are tangent at ∞.

If $\widetilde{\mathcal{A}}'$ is any plane through ∞ then, with $\mathcal{A}' := \widetilde{\mathcal{A}}' \setminus \infty$, the mapping

$$\beta : \mathcal{A} \to \mathcal{A}', \quad \mathcal{A} \cap s \mapsto \mathcal{A}' \cap s \quad (s \in \mathcal{S}_{L/K} \setminus \{\infty\})$$

is a well–defined bijection[5]. This β is an affinity if either $\widetilde{\mathcal{A}}' = \widetilde{\mathcal{A}}$ or $\widetilde{\mathcal{A}}' = \widetilde{\mathcal{A}}^{\iota}$ [6, Theorem 5]; the second alternative is only possible when K/Z is Galois.

[5] One could also select some point $A \in \infty$ and then obtain an affine plane by a dual construction.

3.2

The group of automorphic collineations of $\mathcal{S}_{L/K}$ operates 3–fold transitively on the lines of $\mathcal{S}_{L/K}$ [1, p.322]. Thus we may transfer ∞ to the line given by $(0,1)L$. Moreover, for all $c \in L$, $c \neq 0$

$$(l_0, l_1)K \mapsto (cl_0, cl_1)K \quad ((0,0) \neq (l_0, l_1) \in L^2)$$

is an automorphic collineation of $\mathcal{S}_{L/K}$ fixing ∞. Hence, without loss of generality, we may assume in the sequel that

$$\infty = \mathcal{P}_K((0,1)L) \text{ and } \widetilde{\mathcal{A}} = (1,0)K \vee \infty.$$

Then the mapping (9) becomes

$$\mathcal{P}_K((l_0, l_1)L) \mapsto (1, l_1 l_0^{-1})K. \tag{10}$$

We shall identify $\mathcal{A}$ with L via[6] $(1,l)K \equiv l$. Thus L gets the structure of an affine plane over K. We shall emphasize this by writing $\mathcal{A}_K(L)$ rather than L.

Theorem 1 *Let κ be an automorphic collineation or duality of $\mathcal{S}_{L/K}$ fixing ∞. Then there exist elements $m_0, m_1, m \in L$, $m_0, m_1 \neq 0$ and an automorphism or antiautomorphism J of L with $K^J = K$ such that*

$$x^{\rho^{-1}\kappa\rho} = m_1 x^J m_0 + m \quad \textit{for all } x \in L. \tag{11}$$

The additional conditions

$$J \textit{ is an automorphism of } L, \tag{12}$$
$$m_0 \in K \textit{ or, only if } K/Z \textit{ is Galois, } m_0 i^{-1} \in K \tag{13}$$

together are necessary and sufficient for $\rho^{-1}\kappa\rho$ to be an affinity of $\mathcal{A}_K(L)$.

Proof. The assertion in formula (11) is obviously true.

Now suppose that $\rho^{-1}\kappa\rho$ is an affinity of $\mathcal{A}_K(L)$. Then κ has to take each chain with a transversal line in $\widetilde{\mathcal{A}}$ to a chain with a transversal line in $\widetilde{\mathcal{A}}$. Hence $\widetilde{\mathcal{A}}^\kappa = \widetilde{\mathcal{A}}$ or, only if K/Z is Galois, $\widetilde{\mathcal{A}}^\kappa = \widetilde{\mathcal{A}}^\iota$. Therefore κ cannot be a duality, so that J cannot be an antiautomorphism [8, Theorem 4]. Consequently, $g : x \mapsto m_1 x^J m_0$ has to be a semilinear mapping of the right vector space L over K. We infer from

$$xk \overset{g}{\mapsto} (m_1 x^J m_0)(m_0^{-1} k^J m_0) \quad \text{for all } x \in L,\ k \in K$$

that $m_0^{-1} K m_0 = K$. There are two possibilities: If

$$m_0^{-1} k m_0 = k \quad \text{for all } k \in K$$

then m_0 is a non–zero element of K, since K is a maximal commutative subfield of L. On the other hand, however only if K/Z is Galois, also

$$m_0^{-1} k m_0 = \overline{k} \quad \text{for all } k \in K$$

is possible. Now, again using that K is maximal commutative, it follows from (1) that $m_0 i^{-1} \in K$.

The proof of the converse is a straightforward calculation. ■

[6] This is accordance with the inhomogeneous notation used in [1].

3.3

If $\mathcal{C}$ is a chain such that $(\mathcal{C} \setminus \{\infty\})^\rho$ is not a line of $\mathcal{A}_K(L)$ then $(\mathcal{C} \setminus \{\infty\})^\rho$ will be named a **circle**. There are two kinds of circles: If $\infty \in \mathcal{C}$ then the circle is called **degenerate**, otherwise **non–degenerate**. The following Lemma shows that distinct chains cannot define the same circle. In addition it establishes that a circle cannot be degenerate and non–degenerate at the same time:

Lemma 2 *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two chains such that $\mathcal{C}_0 \setminus \{\infty\} = \mathcal{C}_1 \setminus \{\infty\}$. Then $\mathcal{C}_0 = \mathcal{C}_1$.*

Proof. According to (6), (7), (8) there exists a 3–dimensional subspace $\mathcal{X}_0$ of $\widehat{\mathcal{P}}_K$ with

$$\mathcal{C}_0{}^\gamma = \mathcal{X}_0 \cap \Pi_Z \cap \mathcal{Q}.$$

Since $\mathcal{C}_0{}^\gamma$ is an oval quadric of $\mathcal{X}_0 \cap \Pi_Z$ and Z is infinite, $(\mathcal{C}_0 \setminus \{\infty\})^\gamma$ is still spanning $\mathcal{X}_0$. Repeating this, mutatis mutandis, for $\mathcal{C}_1$ gives $\mathcal{X}_0 = \mathcal{X}_1$, whence $\mathcal{C}_0 = \mathcal{C}_1$, as required. ■

3.4

By Lemma 2, we may unambiguously speak of a line being **tangent** to a circle at some point $P \in \mathcal{A}_K(L)$ or of circles **touching** at P if they arise from chains that are tangent at $P^{\rho^{-1}}$.

A degenerate circle has no tangent lines. A point P of a non–degenerate circle is called **regular** if there exists a tangent line of that circle at P. If such a circle is given as $\mathcal{C}^\rho$, $\mathcal{C}$ a chain, then $P \in \mathcal{C}^\rho$ is regular if, and only if, P (regarded as point of $\mathcal{A}$) is incident with a transversal line of $\mathcal{C}$. Thus a non–degenerate circle has either one or two regular points.

3.5

If K/Z is Galois then call two lines, or a circle and a line, or two circles of $\mathcal{A}_K(L)$ **orthogonal** if they arise from orthogonal chains.

By virtue of the collineation ι (cf. formula (5)), a line $lK + m$ $(l, m \in L,\ l \neq 0)$ is orthogonal to all lines being parallel to liK.

We introduce a unitary scalar product $*$ on the right vector space L over K by setting

$$(u + iv) * (u' + iv') := \overline{u}u' + \mu_2 \overline{v}v' \quad \text{for all } u, u', v, v' \in K. \tag{14}$$

This scalar product is describing the orthogonality relation on lines from above. Moreover, $(u + iv) * (u + iv) = N(u + iv)$, whence the norm is a Hermitian form[7] on L over K.

It is easily seen that there exists no line orthogonal to a degenerate circle. The join of the two regular points of a non–degenerate circle is the only line being orthogonal to that circle. It will be called the **midline** of the circle. The midline is orthogonal to both tangent lines.

All affinities described in Theorem 1 are preserving orthogonality.

3.6

Let $\mathcal{C}$ be a chain such that $\Delta := (\mathcal{C} \setminus \{\infty\})^\rho$ is a degenerate circle. Then either there are two points or there is one point on the line ∞ incident with transversal lines of

[7] If K/Z is not Galois then the norm does not seem to be a quadratic or Hermitian form on L over K.

$\mathcal{C}$. We call these points at infinity of $\mathcal{A}_K(L)$ the **absolute points** or the **absolute directions** of Δ. This terminology will be motivated in 3.10.

The group $\mathrm{AGL}(1,L)$ of all transformations (11) with $m_0 = 1$ operates sharply 2-fold transitively on $\mathcal{A}_K(L)$. Thus each degenerate circle can be transferred under $\mathrm{AGL}(1,L)$ to a degenerate circle through 0 and 1. Write

$$L^\circ := \begin{cases} L \setminus (K \cup Ki) : K/Z \text{ Galois}, \\ L^\circ := L \setminus K \quad : K/Z \text{ not Galois}. \end{cases}$$

Then, by [1, p.329] and (13), the degenerate circles through 0 and 1 are exactly the sets

$$cKc^{-1} \quad \text{with } c \in L^\circ. \tag{15}$$

From now on assume that a degenerate circle Δ is given by (15). Let $\mathcal{C}$ be the chain with transversal line $(c,0)K \vee (0,c)K$. Then $\Delta = (\mathcal{C} \setminus \{\infty\})^\rho$, whence cK is an absolute direction of Δ. Each affinity of $\mathrm{AGL}(1,L)$ (cf. formula (11)) with $m_1, m \in cKc^{-1}$ ($m_1 \neq 0$, $m_0 = 1$ as before) takes Δ onto Δ.

Theorem 2 *Each degenerate circle of $\mathcal{A}_K(L)$ is an affine Baer subplane of $\mathcal{A}_K(L)$ with the centre of L as underlying field.*

Proof. It is sufficient to show this for a degenerate circle given by (15). Set $c^{-1} =: d + ie$ with $d, e \in K$. Then, by (1) and (2),

$$cKc^{-1} = \begin{cases} \{(cd)k + (cie)\overline{k} \mid k \in K\} : K/Z \text{ Galois}, \\ \{k + (ce)k^D \mid k \in K\} \quad : K/Z \text{ not Galois}. \end{cases}$$

Now the assertion follows by Lemma 1. ■

3.7

Next we turn to non–degenerate circles.

Theorem 3 *All non–degenerate circles of the affine plane $\mathcal{A}_K(L)$ are in one orbit of* $\mathrm{AGL}(1,L)$.

Proof. Let $\mathcal{C}_0$ be the chain with transversal line

$$(1,0)K \vee (i,i)K. \tag{16}$$

Then $\Gamma_0 := {\mathcal{C}_0}^\rho$ is a non–degenerate circle with regular point 0.

Let K/Z be Galois. Then 1 is the other regular point of Γ_0. If Γ_1 is a non–degenerate circle then there exists an affinity $\alpha \in \mathrm{AGL}(1,L)$ taking the regular points of Γ_1 to 0 and 1, respectively. Hence ${\Gamma_1}^{\alpha\rho^{-1}}$ is a chain with one transversal line through $(1,0)K$ and the other transversal line through $(1,1)K$. Applying the collineation ι on $(1,1)K$ establishes that (16) is a transversal line of this chain, whence $\Gamma_0 = {\Gamma_1}^\alpha$.

Now assume that K/Z is not Galois. If Γ_1 is a non–degenerate circle then there exists an affinity $\alpha \in \mathrm{AGL}(1,L)$ taking the only regular point of Γ_1 to 0. The chain ${\Gamma_1}^{\alpha\rho^{-1}}$ has a unique transversal line through $(1,0)K$ and some point of the plane $(i,0)K \vee \infty$, say

$$(id, e + if)K \quad \text{with } d, e, f \in K,\ d, e + if \neq 0.$$

There exists an element $m_1 \in L \setminus \{0\}$ such that $m_1(e + if) = id$. The collineation κ of $\mathcal{P}_K$ given by $(l_0, l_1)K \mapsto (l_0, m_1 l_1)K$ leaves $\mathcal{S}_{L/K}$ invariant, fixes the point $(1,0)K$ as well as the line ∞ and takes $(id, e + if)K$ to $(i,i)K$. Hence the induced affinity $\rho^{-1}\kappa\rho$ of $\mathcal{A}_K(L)$ carries ${\Gamma_1}^\alpha$ over to Γ_0. ■

3.8

The non–degenerate circle Γ_0 arising from the chain $\mathcal{C}_0$ with transversal line (16) has the parametric representation

$$\{ik_1(k_0+ik_1)^{-1} \mid (0,0)\neq(k_0,k_1)\in K^2\}; \tag{17}$$

cf. also [1, Satz 3.2]. Next we establish an equation for Γ_0:

Theorem 4 *The non–degenerate circle Γ_0 given by (17) equals the set of all points $u+iv$ $(u,v\in K)$ satisfying*[8]

$$u = N(u+iv). \tag{18}$$

Proof. The term $ik_1(k_0+ik_1)^{-1}$ in formula (17) can be rewritten as follows: If K/Z is Galois then

$$\begin{aligned} ik_1(k_0+ik_1)^{-1} &= ik_1(\overline{k_0}-ik_1)\left((k_0+ik_1)(\overline{k_0}-ik_1)\right)^{-1} \\ &= (\mu_2k_1\overline{k_1}+i\overline{k_0}k_1)(k_0\overline{k_0}+\mu_2k_1\overline{k_1})^{-1}, \end{aligned}$$

otherwise

$$\begin{aligned} ik_1(k_0+ik_1)^{-1} &= ik_1(k_0+k_1+k_1i)\left((k_0+ik_1)(k_0+k_1+k_1i)\right)^{-1} \\ &= (\mu_2k_1^2+ik_0k_1)\left(k_0^2+k_0k_1+(k_0k_1)^D+\mu_2k_1^2\right)^{-1}. \end{aligned}$$

Now, since

$$N(u+iv) = \begin{cases} u\overline{u}+\mu_2v\overline{v} & : K/Z \text{ Galois,} \\ u^2+uv+(uv)^D+\mu_2v^2 & : K/Z \text{ not Galois,} \end{cases}$$

it is easily seen that all points of Γ_0 are satisfying equation (18).

Conversely, let $q+ir$ $(q,r\in K)$ be a solution of (18). If $q=0$ then $r=0$, whence we have a point of Γ_0. Otherwise set

$$k_0 := \left\{ \begin{array}{ll} \mu_2\overline{rq^{-1}} & : K/Z \text{ Galois,} \\ \mu_2rq^{-1} & : K/Z \text{ not Galois,} \end{array} \right\} \text{ and } k_1 := 1.$$

The point of Γ_0 with these parameters equals $q+ir$. ■

3.9

We are able to say a little bit more about non–degenerate circles provided that K/Z is Galois. Formula (18) becomes

$$N(u+iv)-u = (u-1+iv)*(u+iv) = 0. \tag{19}$$

Thus, if we intersect each line through 0 with its orthogonal line through 1 then the set of all such points of intersection equals Γ_0. This is a nice analogon to a well–known property of opposite points on a Euclidean circle[9].

Theorem 5 *Let K/Z be Galois. Write $E:=\{y\in K \mid y+\overline{y}=1\}$ and $\mathcal{H}_e$ $(e\in E)$ for the affine Hermitian variety formed by all points $u+iv$ $(u,v\in K)$ subject to the equation*

$$N(u+iv) = eu+\overline{eu}.$$

Then the non–degenerate circle Γ_0 given by (17) can be written as

$$\Gamma_0 = \mathcal{H}_e\cap\mathcal{H}_f \quad \textit{for all } e,f\in E \textit{ with } e\neq f. \tag{20}$$

[8] In the elementary plane of complex numbers the same kind of equation gives a circle through 0 and 1.

[9] The points 0 and 1 are, however, the only points of Γ_0 with this property.

Proof. A straightforward calculation yields

$$E = \begin{cases} \frac{1}{2} + (\lambda_1 + 2a)Z & : \mathrm{Char} K \neq 2, \\ a\lambda_1^{-1} + Z & : \mathrm{Char} K = 2, \end{cases}$$

whence E is infinite. Given $q + ir \in \Gamma_0$ $(q, r \in K)$ then $q \in Z$ implies

$$\Gamma_0 \subset \bigcap_{e \in E} \mathcal{H}_e.$$

Choose distinct elements $e, f \in E$ and $q + ir \in \mathcal{H}_e \cap \mathcal{H}_f$ $(q, r \in K)$. Then

$$N(q + ir) - N(q + ir) = eq + \overline{eq} - fq - \overline{fq} = 0.$$

But

$$\frac{e - f}{\overline{f} - \overline{e}} = 1,$$

so that $q = \overline{q}$ and therefore $q + ir \in \Gamma_0$. ■

3.10

There is an alternative approach to $\mathcal{A}_K(L)$ via the point model of $\Sigma(K, L)$ on the Klein quadric $\mathcal{Q}$.

Write $I := \infty^\gamma$ and $\mathcal{Z}$ for the γ–image of the ruled plane on $\widetilde{\mathcal{A}}$; this $\mathcal{Z}$ is a plane on the Klein quadric. Furthermore let $\widetilde{\mathcal{F}}$ be any plane of $\widehat{\mathcal{P}}_K$ skew to $\mathcal{Z}$ and write

$$\pi : \widehat{\mathcal{P}}_K \setminus \mathcal{Z} \to \widetilde{\mathcal{F}} \tag{21}$$

for the projection with centre $\mathcal{Z}$ onto the plane $\widetilde{\mathcal{F}}$. It is well known from descriptive line geometry that there exists a collineation ψ of $\widetilde{\mathcal{A}}$ onto $\widetilde{\mathcal{F}}$ such that

$$(p \cap \widetilde{\mathcal{A}})^\psi = p^{\gamma\pi}$$

for all lines p of $\mathcal{P}_K$ not contained in $\widetilde{\mathcal{A}}$. Cf., e.g., [3]. We turn $\widetilde{\mathcal{F}}$ into an affine plane $\mathcal{F}$, say, by regarding $\widetilde{\mathcal{F}} \cap I^\perp$ as its line at infinity; here $I^\perp$ denotes the tangent hyperplane of the Klein quadric at I. Then $\infty^\psi = \mathcal{F} \cap I^\perp$.

The bijectivity of ρ implies that ${\mathcal{S}_{L/K}}^\gamma \setminus \{I\}$ is mapped bijectively under π onto the affine plane $\mathcal{F}$. The restriction

$$\pi \mid {\mathcal{S}_{L/K}}^\gamma \setminus \{I\}$$

can be seen as a **generalized stereographic projection** of the oval quadric ${\mathcal{S}_{L/K}}^\gamma$ of Π_Z onto the affine plane[10] $\mathcal{F}$.

Let $\mathcal{C}$ be a chain. Then $\mathcal{C}^\gamma = \mathcal{X} \cap \mathcal{Q} \cap \Pi_Z$ for some 3-dimensional subspace $\mathcal{X}$ of $\widehat{\mathcal{P}}_K$. We leave it to the reader to show that $(\mathcal{C} \setminus \{\infty\})^{\gamma\pi}$ is an affine line if $\mathcal{X} \cap \mathcal{Z}$ is a line through I, a degenerate circle if $\mathcal{X} \cap \mathcal{Z} = \{I\}$ and a non–degenerate circle if $\mathcal{X} \cap \mathcal{Z}$ is some point other than I.

Using the mapping $\gamma\pi\psi^{-1}$ instead of ρ is very convenient to establish results on the images of traces [1, p.327], since their γ–images are just the regular conics on ${\mathcal{S}_{L/K}}^\gamma$ [8, 3.4]. We sketch just one result without proof:

Let $\mathcal{C}$ be a chain through ∞ such that $(\mathcal{C} \setminus \{\infty\})^\rho =: \Delta$ is a degenerate circle of $\mathcal{A}_K(L)$. Then the ρ–images of traces in $\mathcal{C}$ are on one hand the lines of the affine plane Δ and on the other hand certain ellipses of Δ. If these ellipses are extended to conics of $\mathcal{A}_K(L)$ then the absolute directions of Δ determine their points at infinity[11]. This is the well–known concept of absolute circular points. Δ is a Euclidean plane representing the extension K/Z. Cf. [14].

[10] A 'usual' stereographic projection would map onto a 4–dimensional affine space over Z rather than an affine plane over K.

[11] There is only one such point if K/Z is not Galois.

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