Geometries on σ-Hermitian matrices

Andrea Blunck Hans Havlicek

1 Introduction: Square Matrices

Ring geometry and the geometry of matrices meet naturally at the ring $R := K^{n\times n}$ of $n \times n$ matrices with entries in a (not necessarily commutative) field K. Our aim is to strengthen the interaction between these disciplines. Below we sketch some results from either side, even though not in their most general form, but in a way which is tailored for our needs.

Let us start with ring geometry, where we follow [7] and [10]: Consider the free left R-module R^2 and the group $\mathrm{GL}_2(R) = \mathrm{GL}_{2n}(K)$ of invertible 2×2-matrices with entries in R. A pair $(A, B) \in R^2$ is called *admissible* if there exists a matrix in $\mathrm{GL}_2(R)$ with (A, B) being its first row. The *projective line over* R, in symbols $\mathbb{P}(R)$, is the set of cyclic submodules $R(A, B)$ for all admissible pairs $(A, B) \in R^2$. Two admissible pairs represent the same point precisely when they are left-proportional by a unit in R, i. e., a matrix from $\mathrm{GL}_n(K)$. Conversely, if $R(A', B') = R(A, B)$ for some pair $(A', B') \in R^2$ and an admissible pair $(A, B) \in R^2$ then (A', B') is admissible too [3, Proposition 2.2]. By [2], the projective line over R allows the following description which is not available for arbitrary rings, as it makes use of the *left row rank* of a matrix X over K (in symbols: $\operatorname{rank} X$):

$$\mathbb{P}(R) = \{R(A, B) \mid A, B \in R,\ \operatorname{rank}(A, B) = n\}. \tag{1}$$

Here $(A, B) \in R^2$ has to be interpreted as the $n \times 2n$ matrix over K arising from A and B by means of horizontal augmentation. Because of (1), the point set of $\mathbb{P}(R)$ is in bijective correspondence with the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ comprising all n-dimensional subspaces of the left K-vector space K^{2n} via

$$\mathbb{P}(R) \to \mathrm{Gr}_{2n,n}(K) : R(A, B) \mapsto \text{left row space of } (A, B). \tag{2}$$

From [13, 2.6], our matrix ring $R = K^{n\times n}$ has *stable rank* 2 [13, § 2]. Viz. for each $(A, B) \in R^2$ which is *unimodular*, i. e., there are $X, Y \in R$ with $AX + BY = I$, there exists $W \in R$ such that $A + BW \in \mathrm{GL}_n(K)$. Consequently, two important results hold: Firstly, any unimodular pair $(A, B) \in R^2$ is admissible [13, 2.11]. Secondly, *Bartolone's parametrisation*

$$R^2 \to \mathbb{P}(R) : (T_1, T_2) \mapsto R(T_2T_1 - I, T_2) \tag{3}$$

is well defined and surjective. This allows us to write the projective line $\mathbb{P}(R)$ in the form

$$\mathbb{P}(R) = \{R(T_2T_1 - I, T_2) \mid T_1, T_2 \in R\}. \tag{4}$$

See [1] and compare with [4] for a generalisation.

Let us now switch to the geometry of matrices, where [14] is our standard reference. By comparing the description of the point set $\mathbb{P}(K^{n\times n}) = \mathbb{P}(R)$ in (1) with the definition of the point set of the *projective space of $m \times n$ matrices over K* in [14, 3.6], one sees immediately that the two definitions coincide for $m = n \geq 2$ up to the immaterial fact that we address a Grassmannian in the vector space K^{2n} rather than in the projective space on K^{2n}. The bijection from (2) turns (3) into a surjective parametric representation of the Grassmannian $\mathrm{Gr}_{2n,n}(K)$, namely

$$R^2 \to \mathrm{Gr}_{2n,n}(K) : (T_1, T_2) \mapsto \text{left row space of } (T_2T_1 - I, T_2). \tag{5}$$

A major difference concerns the *additional structure* which is imposed on $\mathrm{Gr}_{2n,n}(K)$: In the ring-geometric setting the point set $\mathbb{P}(R)$ is endowed with the symmetric and anti-reflexive relation *distant* ($\triangle$) defined by

$$R(A, B) \triangle R(C, D) \quad \Leftrightarrow \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(R).$$

Being distant is equivalent to the complementarity of the n-dimensional subspaces of K^{2n} which correspond via (2). A crucial property of the projective line over our ring R, and more generally over any ring of stable rank 2, is as follows [10, 1.4.2]: Given any two points p and q there exists some point r such that $p \triangle r \triangle q$. In the matrix-geometric setting two n-dimensional subspaces of K^{2n} are called *adjacent* ($\sim$) if, and only if, their intersection has dimension $n - 1$. However, adjacency can be expressed in terms of being distant and vice versa [5, Theorem 3.2]. See also [12, 3.2.4], where complementary subspaces are called *opposite*.

We refer to [6] for several applications of this link between $\mathbb{P}(R)$ and the Grassmannian $\mathrm{Gr}_{2n,n}(K)$, like a unified explicit description of adjacency preserving transformations of $\mathrm{Gr}_{2n,n}(K)$ which avoids the usual distinction between semilinear bijections and non-degenerate sesquilinear forms.

2 σ-Hermitian matrices

Suppose now that the field K admits an *involution*, i. e. an antiautomorphism σ such that $\sigma^2 = \mathrm{id}_K$. As before, we let $R = K^{n\times n}$. The involution σ determines an antiautomorphism of R, namely the *σ-transposition* $M = (m_{ij}) \mapsto (M^\sigma)^\mathrm{T} := (m_{ji}^\sigma)$. The elements of $H_\sigma := \{X \in R \mid X = (X^\sigma)^\mathrm{T}\}$ are the *σ-Hermitian matrices* of R. (In the special case that $\sigma = \mathrm{id}_K$ the field K is commutative, and we obtain the subset of *symmetric matrices* of $K^{n\times n}$.) The set H_σ need not be closed under matrix multiplication. In the terminology of [7, 3.1.5], H_σ is a *Jordan system* of R, where $R = K^{n\times n}$ is considered as an algebra over the centre $Z(K)$ of K. This means that H_σ is a subspace of the $Z(K)$-vector space R which contains I, and which has the property that

$$A^{-1} \in H_\sigma \quad \text{for all} \quad A \in \mathrm{GL}_n(K) \cap H_\sigma. \tag{6}$$

Moreover, H_σ is *Jordan closed*, i. e., it satisfies the condition

$$ABA \in H_\sigma \quad \text{for all} \quad A, B \in H_\sigma. \tag{7}$$

We follow [7, 3.1.14] by defining the *projective line* $\mathbb{P}(H_\sigma)$ over H_σ as

$$\mathbb{P}(H_\sigma) = \{R(T_2T_1 - I, T_2) \mid T_1, T_2 \in H_\sigma\}. \tag{8}$$

From (4), $\mathbb{P}(H_\sigma)$ is a subset of $\mathbb{P}(R)$. *It is important to point out that $\mathbb{P}(H_\sigma)$ is not defined as the set of all $R(A, B)$ with (A, B) admissible and $A, B \in H_\sigma$.* Nevertheless, all points $R(A, I)$ and $R(I, A)$ with $A \in H_\sigma$ belong to $\mathbb{P}(H_\sigma)$.
We now recall the definition of the projective space of σ-Hermitian matrices from [9, III § 3] and [14, 6.8]. Let $\beta : K^{2n} \times K^{2n} \to K$ be the non-degenerate σ-anti-Hermitian sesquilinear form given by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathrm{GL}_{2n}(K). \tag{9}$$

The form β is trace-valued and has Witt index n. The subset of the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ comprising all maximal totally isotropic subspaces is the point set of the *projective space of σ-Hermitian matrices* or, in another terminology, the point set of the *dual polar space* given by β; see [8] or [12, 4.1].
Suppose that $(A, B) \in R^2$ satisfies $\mathrm{rank}(A, B) = n$. By [14, Proposition 6.41], the (n-dimensional) left row space of $(A, B) \in K^{n\times 2n}$ is totally isotropic if, and only if,

$$A(B^\sigma)^\mathrm{T} = B(A^\sigma)^\mathrm{T}. \tag{10}$$

Thus it is easy to decide whether or not an element of the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ is totally isotropic. For example, all pairs (A, I) and (I, A) with $A \in H_\sigma$ give rise to maximal totally isotropic subspaces.
Note that our Jordan system H_σ need not be *strong* (in German: "starkes Jordan-System") in the sense of [7, 3.1.5], as we do not assume any richness conditions. Also, we did not adopt the extra assumptions on σ from [14, p. 306].

By the above, the set of σ-Hermitian matrices gives rise to two subsets of $\mathrm{Gr}_{2n,n}(K)$. The coincidence of these two subsets is not obvious. Indeed, in the ring-geometric setting the subset is given in terms of a *parametric representation*, whereas in the matrix-geometric setting there is a defining *matrix equation*. Our main result states that the two subsets coincide.

Theorem 1 ([6])**.** *Let K be any field admitting an involution σ. The point set of the projective space of σ-Hermitian $n\times n$ matrices over K coincides with the projective line over the Jordan system H_σ of all σ-Hermitian matrices of $R = K^{n\times n}$.*

Our proof of this theorem uses two auxiliary results about dual polar spaces. The first is rather technical.

Lemma 1 ([6])**.** *Let $U = V \oplus W$ be a maximal totally isotropic subspace of (K^{2n}, β) which is given as direct sum of subspaces V and W. Then there exists a maximal totally isotropic subspace, say X, such that $X \cap V^\perp = W$.*

With this result at hand the following can be established:

Lemma 2 ([6])**.** *Let U_1 and U_2 be two maximal totally isotropic subspaces of (K^{2n}, β). Then there exists a maximal totally isotropic subspace X which is a common complement of U_1 and U_2.*

Sketch of the proof of Theorem 1. The proof of one inclusion simply amounts to plugging in representatives of the points from (8) in the matrix equation (10). Conversely, let the left row space of (A, B) be a maximal totally isotropic subspace. By Lemma 2, there exists a maximal totally isotropic subspace of K^{2n} which is a common complement of the left row spaces of $(I, 0)$ and (A, B). In matrix form it can be written as (C, I) with $C \in H_\sigma$. So, in terms of $\mathbb{P}(R)$, we have $R(I, 0) \vartriangle R(C, I) \vartriangle R(A, B)$. Defining $T_1 := C$ and $T_2 := (BC - A)^{-1}B$ gives after some calculations that $R(A, B) = R(T_2T_1 - I, T_2)$ and $R(A, B) \in \mathbb{P}(H_\sigma)$. □

In view of Theorem 1 one may carry over results from $\mathbb{P}(H_\sigma)$ to the projective space of σ-Hermitian matrices; see [6].
Finally, let us mention two open problems:

1. *Is it possible to express the adjacency relation on a projective space of σ-Hermitian matrices in terms of the distant relation on $\mathbb{P}(H_\sigma)$?*
   See [6], [11] and [12, 4.7.1] for further details.
2. *Is it possible to extend the present results from the matrix ring $R = K^{n\times n}$ to other rings which admit an anti-automorphism?*
   An affirmative answer would give, *mutatis mutandis*, an alternative approach to projective lines over certain Jordan systems in terms of a defining equation similar to (10).

References

[1] C. Bartolone. Jordan homomorphisms, chain geometries and the fundamental theorem. *Abh. Math. Sem. Univ. Hamburg*, 59:93–99, 1989.

[2] A. Blunck. Regular spreads and chain geometries. *Bull. Belg. Math. Soc. Simon Stevin*, 6:589–603, 1999.

[3] A. Blunck and H. Havlicek. Projective representations I. Projective lines over rings. *Abh. Math. Sem. Univ. Hamburg*, 70:287–299, 2000.

[4] A. Blunck and H. Havlicek. Jordan homomorphisms and harmonic mappings. *Monatsh. Math.*, 139:111–127, 2003.

[5] A. Blunck and H. Havlicek. On bijections that preserve complementarity of subspaces. *Discrete Math.*, 301:46–56, 2005.

[6] A. Blunck and H. Havlicek. Projective lines over Jordan systems and geometry of Hermitian matrices. *Linear Algebra Appl.*, 433:672–680, 2010.

[7] A. Blunck and A. Herzer. *Kettengeometrien – Eine Einführung*. Shaker Verlag, Aachen, 2005.

[8] P. J. Cameron. Dual polar spaces. *Geom. Dedicata*, 12(1):75–85, 1982.

[9] J. A. Dieudonné. *La Géométrie des Groupes Classiques*. Springer, Berlin Heidelberg New York, 3rd edition, 1971.

[10] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.

[11] M. Kwiatkowski and M. Pankov. Opposite relation on dual polar spaces and half-spin Grassmann spaces. *Results Math.*, 54(3-4):301–308, 2009.

[12] M. Pankov. *Grassmannians of Classical Buildings*, volume 2 of *Algebra and Discrete Mathematics*. World Scientific, Singapore, 2010.

[13] F. D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*, pages 289–350. D. Reidel, Dordrecht, 1985.

[14] Z.-X. Wan. *Geometry of Matrices*. World Scientific, Singapore, 1996.

Andrea Blunck, Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.
andrea.blunck@math.uni-hamburg.de

Hans Havlicek, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10/104, A-1040 Wien, Austria.
havlicek@geometrie.tuwien.ac.at