Cayley's surface revisited

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Abstract

Cayley's (ruled cubic) surface carries a three-parameter family of twisted cubics. We describe the contact of higher order and the dual contact of higher order for these curves and show that there are three exceptional cases.

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1 Introduction

1.1. The geometry on Cayley's surface and the geometry in the ambient space of Cayley's surface has been investigated by many authors from various points of view. See, among others, [5], [9], [10], [11], and [14]. In these papers the reader will also find a lot of further references.

As a by-product of a recent publication [8], it turned out that the Cayley surface (in the real projective 3-space) carries a one-parameter family of twisted cubics which have mutually contact of order four. These curves belong to a well-known three-parameter family of twisted cubics $c_{\alpha,\beta,\gamma}$ on Cayley's surface; cf. formula (2) below. All of them share a common point U with a common tangent t, and a common osculating plane ω, say. However, according to [2, pp. 96–97] such a one-parameter family of twisted cubics with contact of order four should not exist: "*Zwei Kubiken dieser Art, die einander in U mindestens fünfpunktig berühren, sind identisch.*"

The aim of the present communication is to give a complete description of the order of contact (at U) for the twisted cubics mentioned above. In particular, it will be shown in Theorem 1 that the twisted cubics with parameter $\beta = \frac{3}{2}$ play a distinguished role, a result that seems to be missing in the literature. Furthermore, since the order of contact is not a self-dual notion, we also investigate the order of dual contact for twisted cubics $c_{\alpha,\beta,\gamma}$. Somewhat surprisingly, in the dual setting the parameters $\beta = \frac{5}{2}$ and $\beta = \frac{7}{3}$ are exceptional; see Theorem 3.

In Section 2.5 we show that certain results of Theorem 1 have a natural interpretation in terms of the *twofold isotropic geometry* which is based on the absolute flag (U, t, ω), and in terms of the *isotropic geometry* in the plane ω which is given by the flag (U, t). Section 3.3 is devoted to the interplay between Theorem 1 and Theorem 3.

1.2. The calculations which are presented in this paper are long but straightforward. Hence a computer algebra system (Maple V) was used in order to accomplish this otherwise tedious job. Nevertheless, we tried to write down all major steps of the calculations in such a form that the reader may verify them without using a computer.

2 Contact of higher order

2.1. Throughout this paper we consider the three-dimensional real projective space $\mathbb{P}_3(\mathbb{R})$. Hence a point is of the form $\mathbb{R}\boldsymbol{x}$ with $\boldsymbol{x} = (x_0, x_1, x_2, x_3)^{\mathrm{T}}$ being a non-zero vector in $\mathbb{R}^{4\times 1}$. We choose the plane ω with equation $x_0 = 0$ as *plane at infinity*, and we regard $\mathbb{P}_3(\mathbb{R})$ as a projectively closed affine space. For the basic concepts of projective differential geometry we refer to [1] and [7].

2.2. The following is taken from [2], although our notation will be slightly different. *Cayley's* (*ruled cubic*) *surface* is, to within collineations of $\mathbb{P}_3(\mathbb{R})$, the surface F with equation

$$3x_0x_1x_2 - x_1^3 - 3x_3x_0^2 = 0. \tag{1}$$

The line $t : x_0 = x_1 = 0$ is on F. More precisely, it is a torsal generator of second order and a directrix for all other generators of F. The point $U = \mathbb{R}(0, 0, 0, 1)^{\mathrm{T}}$ is the cuspidal point on t. In Figure 1 a part of the surface F is displayed in an affine neighbourhood of the point U. In contrast to our general setting, $x_3 = 0$ plays the role of the plane at infinity in this illustration.

On the surface F there is a three-parameter family of cubic parabolas which can be described as follows: Each triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ with $\beta \neq 0$ gives rise to a function

$$\Phi_{\alpha,\beta,\gamma} : \mathbb{R}^{2\times 1} \to \mathbb{R}^{4\times 1} : \boldsymbol{u} = (u_0, u_1)^{\mathrm{T}} \mapsto$$
$$\left(u_0^3, u_0^2(u_1 - \gamma u_0), \frac{u_0(u_1^2 + \alpha u_0^2)}{\beta}, \frac{(u_1 - \gamma u_0)}{3\beta}\left(3(u_1^2 + \alpha u_0^2) - \beta(u_1 - \gamma u_0)^2\right)\right)^{\mathrm{T}}.$$

If moreover $\beta \neq 3$ then $\Phi_{\alpha,\beta,\gamma}$ yields the mapping

$$\mathbb{P}_1(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}) : \mathbb{R}\boldsymbol{u} \mapsto \mathbb{R}(\Phi_{\alpha,\beta,\gamma}(\boldsymbol{u})); \tag{2}$$

its image is a *cubic parabola* $c_{\alpha,\beta,\gamma} \subset F$. All these cubic parabolas have the common point U, the common tangent t and the common osculating plane ω. We add in passing that for $\beta = 3$ we have $\Phi_{\alpha,3,\gamma}\left((0, u_1)^{\mathrm{T}}\right) = \boldsymbol{o}$ for all $u_1 \in \mathbb{R}$, whereas the points of the form $\mathbb{R}\left(\Phi_{\alpha,3,\gamma}((1,u_1)^{\mathrm{T}})\right)$ comprise the affine part of a *parabola*, $c_{\alpha,3,\gamma}$ say, lying on F. Each curve $c_{\alpha,\beta,\gamma}$ ($\beta \neq 0$) is on the *parabolic cylinder* with equation

$$\alpha x_0^2 - \beta x_0 x_2 + (x_1 + \gamma x_0)^2 = 0. \tag{3}$$

The mapping $(\alpha,\beta,\gamma) \mapsto c_{\alpha,\beta,\gamma}$ is injective, since different triples (α,β,γ) yield different parabolic cylinders (3).

Figure 2 shows some generators of F, and five cubic parabolas $c_{\alpha,\beta,0}$ together with their corresponding parbolic cylinders, where α ranges in $\{-\frac{3}{2}, -\frac{3}{4}, 0, \frac{3}{4}, \frac{3}{2}\}$ and $\beta = \frac{3}{2}$.

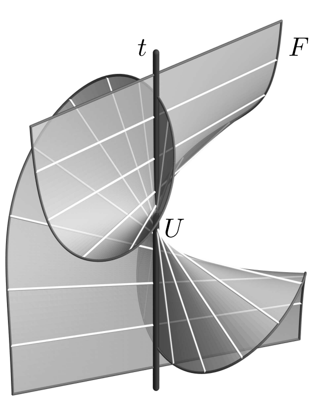

Figure 1.

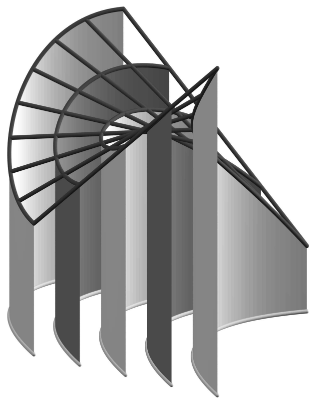

Figure 2.

2.3. Our first goal is to describe the order of contact at U of cubic parabolas given by (2). Since twisted cubics with contact of order five are identical [1, pp. 147–148], we may assume without loss of generality that the curves are distinct, and that the order of contact is less or equal four.

Theorem 1 *Distinct cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ on Cayley's ruled surface have*

(a) *second order contact at U if, and only if, $\beta = \overline{\beta}$ or $\beta = 3 - \overline{\beta}$;*

(b) *third order contact at U if, and only if, $\beta = \overline{\beta}$ and $\gamma = \overline{\gamma}$, or $\beta = \overline{\beta} = \frac{3}{2}$;*

(c) *fourth order contact at U if, and only if, $\beta = \overline{\beta} = \frac{3}{2}$ and $\gamma = \overline{\gamma}$.*

Proof. We proceed in two steps:

(i) First, we consider the quadratic forms

$$Q_1 : \mathbb{R}^{4\times 1} \to \mathbb{R} : \boldsymbol{x} \mapsto 6x_0x_3 - 2x_1x_2, \quad Q_2 : \mathbb{R}^{4\times 1} \to \mathbb{R} : \boldsymbol{x} \mapsto 4x_2^2 - 6x_1x_3$$

which determine a hyperbolic paraboloid and a quadratic cone, respectively. Their intersection is the cubic parabola $c_{0,2,0}$, given by

$$\mathbb{R}(u_0,u_1)^{\mathrm{T}} \mapsto \mathbb{R}\left(u_0^3, u_0^2u_1, \frac{u_0u_1^2}{2}, \frac{u_1^3}{6}\right)^{\mathrm{T}},$$

and the line $x_2 = x_3 = 0$. The tangent planes of the two surfaces at U are different.

Next, let $G := (g_{ij})_{0\leq i,j\leq 3} \in \mathrm{GL}_4(\mathbb{R})$ be a lower triangular matrix, i.e., $g_{ij} = 0$ for all $j > i$. The collineation which is induced by such a matrix G fixes the point U, the line t, and the plane ω; it takes $c_{0,2,0}$ to a cubic parabola, say c'. In order to determine the order of contact of $c_{0,2,0}$ and c' we follow [1, p. 147]. As $U = \mathbb{R}\left(\Phi_{0,2,0}((0,1)^{\mathrm{T}})\right)$, so we expand for $n = 1, 2$ the functions[1]

$$H_n : \mathbb{R} \to \mathbb{R} : u_0 \mapsto (Q_n \circ G \circ \Phi_{0,2,0})\left((u_0,1)^{\mathrm{T}}\right) =: \sum_{m=0}^{6} h_{nm} u_0^m \tag{4}$$

in terms of powers of u_0 and obtain

$$\begin{aligned}
&h_{10} = h_{11} = h_{12} = 0, && h_{13} = g_{00}g_{33} - g_{11}g_{22},\\
&h_{14} = 3g_{00}g_{32} - g_{10}g_{22} - 2g_{11}g_{21}, && h_{20} = h_{21} = 0,\\
&h_{22} = g_{22}^2 - g_{11}g_{33}, && h_{23} = -g_{10}g_{33} - 3g_{11}g_{32} + 4g_{21}g_{22},\\
&h_{24} = -6g_{11}g_{31} - 3g_{10}g_{32} &&\\
&\qquad\quad + 4g_{20}g_{22} + 4g_{21}^2; &&
\end{aligned} \tag{5}$$

the remaining coefficients $h_{15}, h_{16}, h_{25}, h_{26}$ will not be needed. Note that the matrix entry g_{30} does not appear in (5).

(ii) We consider the collineation of $\mathbb{P}_3(\mathbb{R})$ which is induced by the regular matrix

$$M_{\alpha,\beta,\gamma} := \frac{1}{18\beta(\beta-3)}\begin{pmatrix} 3\beta & 0 & 0 & 0\\ -3\beta\gamma & 3\beta & 0 & 0\\ 3\alpha & 0 & 6 & 0\\ \gamma(-3\alpha+\beta\gamma^2) & 3(\alpha-\beta\gamma^2) & 6\gamma(\beta-1) & -6(\beta-3)\end{pmatrix},$$

where $(\alpha,\beta,\gamma) \in \mathbb{R}^3$ and $\beta \neq 0, 3$. Obviously, it fixes the point U and takes $c_{0,2,0}$ to $c_{\alpha,\beta,\gamma}$, since

$$\Phi_{\alpha,\beta,\gamma} = 6(\beta-3)M_{\alpha,\beta,\gamma} \circ \Phi_{0,2,0}.$$

The (irrelevant) scalar factor in the definition of $M_{\alpha,\beta,\gamma}$ enables us to avoid fractions in the matrix

$$M_{\alpha,\beta,\gamma}^{-1} = \begin{pmatrix} 6(\beta-3) & 0 & 0 & 0\\ 6\gamma(\beta-3) & 6(\beta-3) & 0 & 0\\ -3\alpha(\beta-3) & 0 & 3\beta(\beta-3) & 0\\ \gamma(3\alpha-3\alpha\beta-2\beta\gamma^2) & 3(\alpha-\beta\gamma^2) & 3\beta\gamma(\beta-1) & -3\beta\end{pmatrix}.$$

The order of contact at U of the cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ coincides with the order of contact at U of $c_{0,2,0}$ and that cubic parabola which arises from $c_{0,2,0}$ under the action

[1] Observe that sometimes we do not distinguish between a linear mapping and its canonical matrix.

of the matrix

$$
\begin{aligned}
&2\overline{\beta}(\overline{\beta}-3)\, M^{-1}_{\alpha,\beta,\gamma}\cdot M_{\overline{\alpha},\overline{\beta},\overline{\gamma}} = \\
&=\begin{pmatrix}
2\overline{\beta}(\beta-3) & 0 & 0 & 0\\
2\overline{\beta}(\beta-3)(\gamma-\overline{\gamma}) & 2\overline{\beta}(\beta-3) & 0 & 0\\
(\beta-3)(\overline{\alpha}\beta-\alpha\overline{\beta}) & 0 & 2\beta(\beta-3) & 0\\
* & \overline{\beta}(\alpha-\beta\gamma^2)-\beta(\overline{\alpha}-\overline{\beta}\,\overline{\gamma}^2) & 2\beta(\beta\gamma-\overline{\beta}\,\overline{\gamma}-\gamma+\overline{\gamma}) & 2\beta(\overline{\beta}-3)
\end{pmatrix}.
\end{aligned}
$$

This matrix takes over the role of the matrix G from the first part of the proof. (Its entry in the south-west corner has a rather complicated form and will not be needed). Therefore $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ have contact of order k at U if, and only if, in (4) the coefficients h_{n0}, $h_{n1},\ldots h_{nk}$ vanish for $n=1,2$.

By (5), this leads for $k=2$ to the single condition

$$h_{22} = 4\beta(\beta-3)(3-\beta-\overline{\beta})(\overline{\beta}-\beta) = 0$$

which proves the assertion in (a). By virtue of (a), for $k=3$ there are two cases. If $\beta=\overline{\beta}$ then h_{13} vanishes and we obtain the condition

$$h_{23} = 8\beta^2(\beta-3)(2\beta-3)(\overline{\gamma}-\gamma) = 0,$$

whereas $\beta = 3-\overline{\beta}$ yields

$$h_{13} = 4\beta(\beta-3)^2(2\beta-3) = 0, \quad h_{23} = 4\beta(\beta-3)^2(2\beta-3)(\gamma+2\overline{\gamma}) = 0.$$

Altogether this proves (b). Finally, for $k=4$ there again are two possibilities: If $\beta=\overline{\beta}$ and $\gamma=\overline{\gamma}$ then h_{14} vanishes, whence we get

$$h_{24} = 4\beta^2(\beta-3)(2\beta-3)(\overline{\alpha}-\alpha) = 0.$$

Note that here $\alpha\neq\overline{\alpha}$, since $c_{\alpha,\beta,\gamma}\neq c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$. On the other hand, if $\beta=\overline{\beta}=\frac{3}{2}$ then the conditions read

$$h_{14} = \frac{81}{2}\,(\overline{\gamma}-\gamma) = 0, \quad h_{24} = \frac{81}{2}(2\overline{\gamma}+\gamma)(\overline{\gamma}-\gamma) = 0.$$

This completes the proof. $\square$

Alternatively, the preceding results could be derived from [6, Theorem 1] which describes contact of higher order between curves in d-dimensional real projective space.

2.4. In the following pictures we adopt once more the same alternative point of view like in Figure 1, i.e., the plane with equation $x_3=0$ is at infinity.

In Figure 3 two curves $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ are displayed. As $(\alpha,\beta,\gamma) = (0,\frac{1}{10},0)$ and $(\,\overline{\alpha},\overline{\beta},\overline{\gamma}\,) = (1, 3-\frac{1}{10},\frac{1}{10})$, they have contact of second order at U.

A family of curves $c_{\alpha,\beta,0}$ with $\alpha = -3, -2, \ldots, 3$ and $\beta = \frac{3}{2}$ is shown in Figure 4. All of them have mutually contact of order four at U. These curves are, with respect to the chosen affine chart ($x_3 \neq 0$), cubic hyperbolas for $\alpha < 0$, a cubic parabola for $\alpha = 0$, and cubic ellipses for $\alpha > 0$; the corresponding values of α are written next to the images of the curves. See also Figure 2 for another picture of this family, although with different values for α and $x_0 = 0$ as plane at infinity.

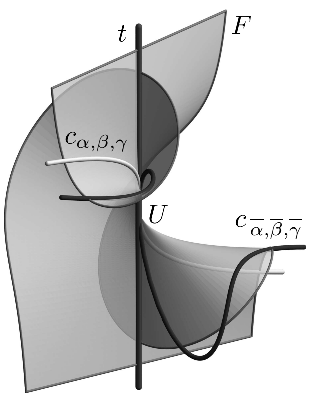

Figure 3.

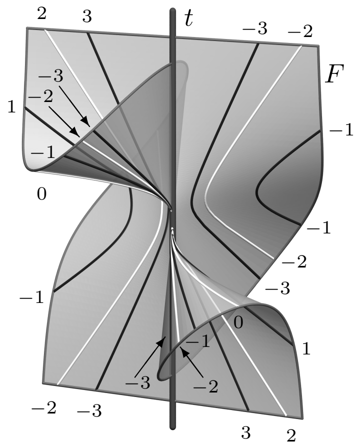

Figure 4.

2.5. It follows from Theorem 1 that cubic parabolas $c_{\alpha,\beta,\gamma}$ with $\beta = \frac{3}{2}$ play a special role. In order to explain this from a geometric point of view we consider the *tangent surface* of a cubic parabola $c_{\alpha,\beta,\gamma}$ and, in particular, its intersection with the plane at infinity. It is well known that this is a conic $p_{\alpha,\beta,\gamma}$ together with the line t. In fact, via the first derivative of the local parametrization $\mathbb{R} \to \mathbb{P}_3(\mathbb{R}) : u_1 \mapsto \mathbb{R}\big(\Phi_{\alpha,\beta,\gamma}((1,u_1)^{\mathrm{T}})\big)$ of $c_{\alpha,\beta,\gamma}$ we see that $p_{\alpha,\beta,\gamma} \setminus \{U\}$ is given by

$$u_1 \mapsto \mathbb{R}\left(0, 1, \frac{2u_1}{\beta}, \frac{3-\beta}{\beta}\, u_1^2 + \frac{2\gamma(\beta-1)}{\beta}\, u_1 + \frac{\alpha}{\beta} - \gamma^2\right)^{\mathrm{T}}. \tag{6}$$

The plane at infinity carries in a natural way the structure of an *isotropic* (or *Galileian*) *plane* with the absolute flag (U,t). Each point $\mathbb{R}(0,1,x_1,x_2)^{\mathrm{T}} \in \omega \setminus t$ can be identified with the point $(x_1,x_2)^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$. In this way the standard basis of $\mathbb{R}^{2\times 1}$ determines a unit length and a unit angle in the isotropic plane [12, pp. 11–16].

From this point of view each $p_{\alpha,\beta,\gamma}$ is an *isotropic circle*. By (6), its *isotropic curvature* [12, p. 112] equals $\frac{1}{2}\,\beta(3-\beta) \leq \frac{9}{8}$; this bound is attained for $\beta = \frac{3}{2}$.

It is well known that two isotropic circles $p_{\alpha,\beta,\gamma}$ and $p_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ have second order contact at the point U if, and only if, their isotropic curvatures are the same [12, pp. 41–42], i.e. for $\beta = \overline{\beta}$ or for $\beta = 3 - \overline{\beta}$. From this observation one could also derive the assertion in

Theorem 1 (a) as follows: We introduce an auxiliary euclidean metric in a neighbourhood of U, and we take into account that the ratio of the euclidean curvatures at U of the curves $c_{\alpha,\beta,\gamma}$ and $p_{\alpha,\beta,\gamma}$ (the curves $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ and $p_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$) equals $4:3$; see [13, p. 212] for this theorem of E. Beltrami.

The flag (U,t,ω) turns $\mathbb{P}_3(\mathbb{R})$ into a *twofold isotropic* (or *flag*) *space*. The definition of metric notions in this space is based upon the identification of $\mathbb{R}(1,x_1,x_2,x_3)^{\mathrm{T}} \in \mathbb{P}_3(\mathbb{R}) \setminus \omega$ with $(x_1,x_2,x_3)^{\mathrm{T}} \in \mathbb{R}^{3\times 1}$, and the canonical basis of $\mathbb{R}^{3\times 1}$; see [3].

By [4, p. 137], each cubic parabola $c_{\alpha,\beta,\gamma}$ has the *twofold isotropic conical curvature* $\frac{1}{2}\,\beta(3-\beta) \leq \frac{9}{8}$. Hence the following characterization follows.

Theorem 2 *Among all cubic parabolas $c_{\alpha,\beta,\gamma}$ on the Cayley surface F, the cubic parabolas with $\beta = \frac{3}{2}$ are precisely those with maximal twofold isotropic conical curvature.*

Yet another interpretation is as follows: The regular matrix

$$B_\beta := \operatorname{diag}\left(1, \frac{3-\beta}{\beta}, \frac{3-\beta}{\beta}, \frac{3-\beta}{\beta}\right), \text{ where } \beta \in \mathbb{R} \setminus \{0,3\},$$

yields a *homothetic transformation* of $\mathbb{P}_3(\mathbb{R})$ which maps the cubic parabola $c_{0,\beta,0}$ to $c_{0,3-\beta,0}$, since

$$(B_\beta \circ \Phi_{0,\beta,0})\left((u_0,u_1)^{\mathrm{T}}\right) = \Phi_{0,3-\beta,0}\left(\left(u_0, \frac{3-\beta}{\beta}\,u_1\right)^{\mathrm{T}}\right) \text{ for all } (u_0,u_1)^{\mathrm{T}} \in \mathbb{R}^{2\times 1}.$$

As all points at infinity are invariant, the corresponding isotropic circles $p_{0,\beta,0}$ and $p_{0,3-\beta,0}$ coincide. This homothetic transformation is identical if, and only if, $\beta = \frac{3}{2}$.

The Cayley surface F admits a 3-parameter collineation group; see [2, p. 96] formula (9). The action of this group on the family of all cubic parabolas $c_{\alpha,\beta,\gamma}$ is described in [2, p. 97], formula (12). (In the last part of that formula some signs have been misprinted. The text there should read $\overline{\alpha} = -a_0^2\frac{\beta^2}{4} - a_0a_1\beta\gamma + a_1^2\alpha + b_0\beta$). By virtue of this action, our previous result on homothetic transformations can be generalized to other cubic parabolas on F.

3 Dual contact of higher order

3.1. The question remains how to distinguish between cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ satisfying the first condition ($\beta = \overline{\beta}$) in Theorem 1 (a), and those which meet the second condition ($\beta = 3 - \overline{\beta}$). A similar question arises for the two conditions in Theorem 1 (b). We shall see that such a distinction is possible if we consider the *dual curves* which are formed by the osculating planes (i.e. cubic developables). Recall that $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ have, by definition, *dual contact of order k* at a common osculating plane σ, if their dual curves have contact of order k at the "point" σ of the dual projective space.

We shall identify the dual of $\mathbb{R}^{4\times 1}$ with the vector space $\mathbb{R}^{1\times 4}$ in the usual way; so planes (i.e. points of the dual projective space) are given by non-zero *row vectors*. Thus, for example, a plane $\mathbb{R}(y_0, y_1, y_2, y_3)$ is tangent to the Cayley surface (1) if, and only if,

$$3y_0y_3^2 - 3y_1y_2y_3 + y_2^3 = 0. \tag{7}$$

We note that all these tangent planes comprise a Cayley surface in the dual space.

For each twisted cubic there exists a unique null polarity (symplectic polarity) which takes each point of the twisted cubic to its osculating plane. In particular, the null polarity of the cubic parabola $c_{0,2,0}$ is induced by the linear bijection

$$\mathbb{R}^{4\times 1} \to \mathbb{R}^{1\times 4} : \boldsymbol{x} \mapsto (N_{0,2,0}\cdot \boldsymbol{x})^{\mathrm{T}} \text{ with } N_{0,2,0} := \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

We are now in a position to prove the following result.

Theorem 3 *Distinct cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ on Cayley's ruled surface have*

(a) *second order dual contact at ω if, and only if, $\beta = \overline{\beta}$;*

(b) *third order dual contact at ω if, and only if, $\beta = \overline{\beta}$ and $\gamma = \overline{\gamma}$, or $\beta = \overline{\beta} = \frac{5}{2}$;*

(c) *fourth order dual contact at ω if, and only if, $\beta = \overline{\beta} = \frac{7}{3}$ and $\gamma = \overline{\gamma}$.*

Proof. The matrix $(M^{\mathrm{T}}_{\alpha,\beta,\gamma})^{-1}\cdot N_{0,2,0}$ determines a duality of $\mathbb{P}_3(\mathbb{R})$ which maps the set of points of $c_{0,2,0}$ onto the set of osculating planes of $c_{\alpha,\beta,\gamma}$. Since the product of a duality and the inverse of a duality is a collineation, we obtain the following:

The order of dual contact at ω of the given curves $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ coincides with the order of contact at U of the cubic parabola $c_{0,2,0}$ and that cubic parabola which arises from $c_{0,2,0}$ under the collineation given by the matrix

$$\begin{aligned} &2\beta(\beta-3)\,N^{-1}_{0,2,0}\cdot M^{\mathrm{T}}_{\alpha,\beta,\gamma}\cdot(M^{\mathrm{T}}_{\overline{\alpha},\overline{\beta},\overline{\gamma}})^{-1}\cdot N_{0,2,0} = \\ &= \begin{pmatrix} 2(\beta-3)\overline{\beta} & 0 & 0 & 0\\ 2\overline{\beta}(\beta\gamma-\overline{\beta}\,\overline{\gamma}-\gamma+\overline{\gamma}) & 2\overline{\beta}(\overline{\beta}-3) & 0 & 0\\ \overline{\alpha}\beta-\alpha\overline{\beta}+\beta\overline{\beta}(\gamma^2-\overline{\gamma}^2) & 0 & 2\beta(\overline{\beta}-3) & 0\\ * & (\overline{\beta}-3)(\alpha\overline{\beta}-\overline{\alpha}\beta) & 2\beta(\overline{\beta}-3)(\gamma-\overline{\gamma}) & 2\beta(\overline{\beta}-3) \end{pmatrix}. \end{aligned}$$

Here $*$ denotes an entry that will not be needed.

We now proceed as in the proof of Theorem 1. By substituting the entries of the matrix above into (5), we read off necessary and sufficient conditions for dual contact of order k at the plane ω of $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$.

For $k = 2$ we get the single condition

$$h_{22} = 4\beta(\overline{\beta} - 3)^2(\beta - \overline{\beta}) = 0$$

which proves the assertion in (a). By (a), we let $\beta = \overline{\beta}$ for the discussion of $k = 3$. Then h_{13} vanishes and we arrive at the condition

$$h_{23} = 8\beta^2(\beta - 3)(2\beta - 5)(\overline{\gamma} - \gamma) = 0,$$

from which (b) is immediate. Finally, for $k = 4$ we distinguish two cases: If $\beta = \overline{\beta}$ and $\gamma = \overline{\gamma}$ then h_{14} vanishes and we are lead to the condition

$$h_{24} = 4\beta^2(\beta - 3)(3\beta - 7)(\overline{\alpha} - \alpha) = 0.$$

Note that here $\alpha \neq \overline{\alpha}$, since $c_{\alpha,\beta,\gamma} \neq c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$. The proof of (c) will be finished by showing that the case $\beta = \overline{\beta} = \frac{5}{2}$ does not occur. From the assumption $\beta = \overline{\beta} = \frac{5}{2}$ follows the first condition

$$h_{14} = \frac{75}{2}\,(\gamma - \overline{\gamma}) = 0.$$

Now, letting $\gamma = \overline{\gamma}$, the second condition

$$h_{24} = \frac{25}{4}\,(\alpha - \overline{\alpha}) = 0$$

is obtained. However, both conditions cannot be satisfied simultaneously, since the first condition and $c_{\alpha,\beta,\gamma} \neq c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ together imply that $\alpha \neq \overline{\alpha}$. □

3.2. By combining the results of Theorem 1 and Theorem 3, it is an immediate task to decide whether or not two (not necessarily distinct) cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\overline{\alpha},\overline{\beta},\overline{\gamma}}$ have contact at U and at the same time dual contact at ω of prescribed orders. In particular, we infer that two cubic parabolas of this kind, with fourth order contact at U and fourth order dual contact at ω, are identical.

3.3. In this section we aim at explaining how the results of Theorems 1 and 3 are related to each other.

Let us choose a *fixed* real number $\beta \neq 0, 3$. We consider the local parametrization

$$\Psi_\beta : \mathbb{R}^2 \to \mathbb{P}_3(\mathbb{R}) : (\alpha, u) \mapsto \mathbb{R}\left(\Phi_{\alpha,\beta,0}((1,u)^{\mathrm{T}})\right)$$

of F; its image is $F \setminus t$, i.e. the affine part of F. For our fixed β and $\gamma = 0$ the affine parts of the parabolic cylinders (3) form a partition of $\mathbb{P}_3(\mathbb{R}) \setminus \omega$; see Figure 2. Hence Ψ_β is injective so that through each point $P \in F \setminus t$ there passes a unique curve $c_{\alpha,\beta,0}$. Consequently, we can define a mapping Σ of $F \setminus t$ into the dual projective space by

$$P \in c_{\alpha,\beta,0} \setminus \{U\} \overset{\Sigma}{\longmapsto} \text{ osculating plane of } c_{\alpha,\beta,0} \text{ at } P. \tag{9}$$

Theorem 4 *The image of the affine part of the Cayley surface F under the mapping Σ described in* (9) *consists of tangent planes of a Cayley surface for $\beta \neq 0, 3, \frac{8}{3}$, and of tangent planes of a hyperbolic paraboloid for $\beta = \frac{8}{3}$.*

Proof. As the null polarity of $c_{\alpha,\beta,0}$ arises from the matrix

$$\begin{aligned} N_{\alpha,\beta,0} &:= (M^{-1}_{\alpha,\beta,0})^{\mathrm{T}} \cdot N_{0,2,0} \cdot M^{-1}_{\alpha,\beta,0} \\ &= 18(\beta-3)\begin{pmatrix} 0 & -\alpha(\beta-4) & 0 & -\beta \\ \alpha(\beta-4) & 0 & -\beta(\beta-3) & 0 \\ 0 & \beta(\beta-3) & 0 & 0 \\ \beta & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{10}$$

so the Σ-image of a point $P = \mathbb{R}\left(\Phi_{\alpha,\beta,0}((1,u)^{\mathrm{T}})\right)$ is the plane which is described by the non-zero row vector

$$6(\beta-3)\Big((\beta-3)(u^2-3\alpha)u, -3(\beta-3)u^2-3\alpha, 3\beta(\beta-3)u, 3\beta\Big). \tag{11}$$

In discussing $\Sigma(F \setminus t)$ there are two cases:
(i) Suppose that $\beta \neq \frac{8}{3}$. Then a duality of $\mathbb{P}_3(\mathbb{R})$ is determined by the regular matrix

$$D_\beta := \frac{18}{\beta-3}\begin{pmatrix} 0 & 0 & 0 & -(3\beta-8) \\ 0 & 0 & -(3\beta-8) & 0 \\ 0 & \beta(\beta-3)^2 & 0 & 0 \\ \beta(\beta-3)^2 & 0 & 0 & 0 \end{pmatrix}.$$

Letting

$$\alpha' := \alpha(\beta-3) \text{ and } \beta' := \frac{3\beta-8}{\beta-3}, \tag{12}$$

the transpose of $(D_\beta \circ \Phi_{\alpha',\beta',0})\left((1,(\beta-3)u)^{\mathrm{T}}\right)$ is easily seen to equal the row vector in (11). Hence $\Sigma(F \setminus t)$ is part of a Cayley surface in the dual space which in turn, by (7), is the set of tangent planes of a Cayley surface in $\mathbb{P}_3(\mathbb{R})$.
(ii) If $\beta = \frac{8}{3}$ then the row vector (11) simplifies to

$$-2\left(\frac{-(u^2-3\alpha)u}{3}, u^2-3\alpha, -\frac{8u}{3}, 8\right)$$

Thus the set $\Sigma(F \setminus t)$ is part of the non-degenerate ruled quadric in the dual space with equation $y_0y_3 - y_1y_2 = 0$ (in terms of dual coordinates). In other words, $\Sigma(F \setminus t)$ consists of tangent planes of a hyperbolic paraboloid in $\mathbb{P}_3(\mathbb{R})$. □

Let us add the following remark. The linear fractional transformation

$$\Lambda : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\} : \xi \mapsto \frac{3\xi-8}{\xi-3}$$

is an involution such that our fixed $\beta \neq 0, 3, \frac{8}{3}$ goes over to β', as defined in (12), whereas $\Lambda(\frac{8}{3}) = 0$. In particular, if $\beta = \frac{7}{3}$ then $\beta' = \Lambda(\beta) = \frac{3}{2}$. This explains the relation between Theorem 1 (c) and Theorem 3 (c). Also the fixed values of Λ are noteworthy:

For $\beta = \Lambda(\beta) = 2$ the curves $c_{\alpha,2,0}$ are *asymptotic curves* of F, i.e., the osculating plane of $c_{\alpha,2,0}$ at each point $P \neq U$ is the tangent plane of F at P. This means that the planes of the set $\Sigma(F \setminus t)$ are tangent planes of F rather than tangent planes of another Cayley surface.

For $\beta = \Lambda(\beta) = 4$ it is immediate form (10) that the matrix $N_{\alpha,4,0}$ does not depend on the parameter $\alpha \in \mathbb{R}$, whence in this particular case the mapping Σ is merely the restriction of a null polarity of $\mathbb{P}_3(\mathbb{R})$ to the affine part of the Cayley surface F.

3.4. There remains the problem to find a geometric interpretation of the value $\beta = \frac{5}{2}$ which appears in Theorem 3 (b).

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References

[1] G. Bol. *Projektive Differentialgeometrie I.* Vandenhoek & Ruprecht, Göttingen, 1950.

[2] H. Brauner. Geometrie auf der Cayleyschen Fläche. *Sb. österr. Akad. Wiss., Abt. II* **173** (1964), 93–128.

[3] H. Brauner. Geometrie des zweifach isotropen Raumes I. *J. reine angew. Math.* **224** (1966), 118–146.

[4] H. Brauner. Geometrie des zweifach isotropen Raumes II. *J. reine angew. Math.* **226** (1967), 132–158.

[5] H. Brauner. Neuere Untersuchungen über windschiefe Flächen. *Jahresber. Deutsch. Math.-Ver.* **70** (1967), 61–85.

[6] W. Degen. Some remarks on Bézier curves. *Comput. Aided Geom. Design* **5** (1988), 259–268.

[7] W. Degen. Projektive Differentialgeometrie. In O. Giering and J. Hoschek, editors, *Geometrie und ihre Anwendungen*, pages 319–374, Carl Hanser Verlag, München, Wien, 1994.

[8] H. Havlicek and K. List. A three-dimensional Laguerre geometry and its visualization. In G. Weiß, editor, *Proceedings – Dresden Symposium Geometry: constructive & kinematic (DSG.CK)*, pages 122–129, Institut für Geometrie, Technische Universität Dresden, Dresden, 2003.

[9] M. Husty. Zur Schraubung des Flaggenraumes $J_3^{(2)}$. *Ber. Math.-Stat. Sekt. Forschungszent. Graz* **217** (1984).

[10] R. Koch. *Geometrien mit einer CAYLEYschen Fläche 3. Grades als absolutem Gebilde*. Thesis, Universität Stuttgart, 1968.

[11] M. Oehler. *Axiomatisierung der Geometrie auf der Cayleyschen Fläche*. Thesis, Universität Stuttgart, 1969.

[12] H. Sachs. *Ebene isotrope Geometrie*. Vieweg, Braunschweig, Wiesbaden, 1987.

[13] K. Strubecker. *Differentialgeometrie. I. Kurventheorie der Ebene und des Raumes*. de Gruyter, Berlin, 2nd edition, 1964.

[14] A. Wiman. Über die Cayleysche Regelfläche dritten Grades. *Math. Ann.* **113** (1936), 161–198.

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