A CHARACTERISTIC PROPERTY OF ELLIPTIC PLÜCKER TRANSFORMATIONS

Dedicated to Walter Benz on the occasion of his 65th birthday

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We discuss elliptic Plücker transformations of three-dimensional elliptic spaces. These are permutations on the set of lines such that any two related (orthogonally intersecting or identical) lines go over to related lines in both directions. It will be shown that for "classical" elliptic 3-spaces a bijection of its lines is already a Plücker transformation, if related lines go over to related lines. Moreover, if the ground field admits only surjective monomorphisms, then "bijection" can be replaced by "injection".

1 INTRODUCTION AND MAIN RESULTS

Let $(\mathcal{P}, \mathcal{L}, \pi)$ be a 3-dimensional *elliptic space*, i.e. a projective space $(\mathcal{P}, \mathcal{L}) = \mathrm{PG}(3, F)$ endowed with an elliptic absolute polarity π. Points $X, Y \in \mathcal{P}$ are π-conjugate (orthogonal) if $X \in Y^\pi$ or, equivalently, if $Y \in X^\pi$. Given a subspace $\mathcal{T} \subset \mathcal{P}$ we denote by $\mathcal{T}^\pi$ its π-polar subspace, i.e. the set of points $Y \in \mathcal{P}$ which are π-conjugate to all points of $\mathcal{T}$. Due to the absence of π-self-conjugate (π-absolute) points, $\mathcal{T}$ and $\mathcal{T}^\pi$ are complementary subspaces. Moreover, the field F is necessarily infinite [10, 5.3]. Let us recall the terminology and some results of [8]: Given $a, b \in \mathcal{L}$ we put

$$\begin{array}{rcll} a \approx b & :\Longleftrightarrow & a \cap b^\pi \neq \emptyset \text{ and } a \cap b \neq \emptyset & \text{(orthogonally intersecting lines)},\\ a \sim b & :\Longleftrightarrow & a \approx b \text{ or } a = b & \text{(related lines)},\\ a \parallel b & :\Longleftrightarrow & \#\{x \mid x \approx a \text{ and } x \approx b\} \geq 3 & \text{(Clifford parallel lines)}. \end{array}$$

The relation $\approx$ is symmetric and the pair $(\mathcal{L}, \sim)$ is a Plücker space in the sense of W. Benz [2]. An *elliptic Plücker transformation* is a bijection $\varphi : \mathcal{L} \to \mathcal{L}$ such that

$$a \sim b \Longleftrightarrow a^\varphi \sim b^\varphi. \tag{1}$$

In the sequel we shall assume that $(\mathcal{P}, \mathcal{L}, \pi)$ is *classical*, i.e., the following conditions hold true:

1. The underlying field F is commutative and $\operatorname{Char} F \neq 2$.
2. π is a projective polarity.
3. There exist Clifford parallel lines $a, b \in \mathcal{L}$ with $b \notin \{a, a^\pi\}$.

For example, the real elliptic 3-space fits into this concept. We shall comment on the third condition in Section 2.
In [8] all Plücker transformations of a classical elliptic 3-space have been described in the realm of the ambient space of the Klein quadric representing the lines of $(\mathcal{P}, \mathcal{L})$.
The main results of this paper are:

THEOREM 1 *Let $(\mathcal{P}, \mathcal{L}, \pi)$ be a 3-dimensional classical elliptic space. If $\varphi : \mathcal{L} \to \mathcal{L}$ is a bijection satisfying*

$$a \sim b \Longrightarrow a^\varphi \sim b^\varphi, \tag{2}$$

then φ is an elliptic Plücker transformation.

THEOREM 2 *Let $(\mathcal{P}, \mathcal{L}, \pi)$ be a 3-dimensional classical elliptic space with underlying field F. Suppose that there are only surjective monomorphisms $F \to F$. If $\varphi : \mathcal{L} \to \mathcal{L}$ is an injection satisfying (2), then φ is an elliptic Plücker transformation.*

We shall establish a series of Propositions in Section 3 that end up in proofs for Theorem 1 and Theorem 2.
Let us remark that [8] contains results on generalized elliptic spaces of dimensions 2 and ≥ 4, namely a description of their Plücker transformations and characterization in the spirit of Theorem 1. The corresponding proofs are short and straightforward, whereas the 3-dimensional case seems to be much more involved.
For results and references on other groups of Plücker transformations see, among others, [2], [3], [8], [9] and [16]. Finally, we refer to [5], [7], [11, p. 75], [12], [13], [14], [15] and [17] for an axiomatic descriptions of polarities, elliptic spaces and Clifford parallelism as well as a connection with quaternion skew fields.

2 CLIFFORD PARALLELISM

In [8] we have aimed at understanding the line geometry of a classical elliptic 3-space $(\mathcal{P}, \mathcal{L}, \pi)$ via the ambient space $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}}) = \mathrm{PG}(5, F)$ of the Klein quadric. Write

$$\gamma : \mathcal{L} \to \widehat{\mathcal{P}}, \; a \mapsto a^\gamma$$

for the Klein mapping and put $\Gamma := \mathcal{L}^\gamma = \operatorname{im} \gamma$ for the Klein quadric. The projective polarity associated with the Klein quadric is named κ.
The absolute polarity π gives rise to a projective collineation $\alpha : \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}$ characterized by $a^{\gamma\alpha} = a^{\pi\gamma}$ for all $a \in \mathcal{L}$. Since π has no self-polar lines, there is no α-invariant point on the

Klein quadric. However, since $(\mathcal{P},\mathcal{L},\pi)$ is classical, all α-invariant points[1] form two skew planes of $(\widehat{\mathcal{P}},\widehat{\mathcal{L}})$, say $\mathcal{E}_L$ and $\mathcal{E}_R$, with $\mathcal{E}_L^\kappa=\mathcal{E}_R$ [8, p. 45]. We remark that in an appropriate quadratic extension of $\mathrm{PG}(3,F)$ the absolute polarity π becomes the polarity of a ruled quadric[2]. The two reguli on this quadric go over to distinct conics on the Klein quadric spanning the planes $\mathcal{E}_L$ and $\mathcal{E}_R$, respectively. Cf. part IV of the fundamental paper by G. Weiß [19] on real metric line geometry.

We infer from $\mathcal{E}_L\cap\Gamma=\emptyset$ that the polarity of the Klein quadric induces an elliptic projective polarity in $\mathcal{E}_L$, say κ_L, thus turning $\mathcal{E}_L$ into an elliptic plane. By symmetry of $L=$ "left" and $R=$ "right", this carries over to $\mathcal{E}_R$. The planes $\mathcal{E}_L$ and $\mathcal{E}_R$ give rise to projections

$$\lambda : \widehat{\mathcal{P}}\setminus\mathcal{E}_L\to\mathcal{E}_R,\ X\mapsto(X\vee\mathcal{E}_L)\cap\mathcal{E}_R,$$
$$\rho : \widehat{\mathcal{P}}\setminus\mathcal{E}_R\to\mathcal{E}_L,\ X\mapsto(X\vee\mathcal{E}_R)\cap\mathcal{E}_L,$$

with the property that $(a^{\gamma\lambda},a^{\gamma\rho},a^\gamma,a^{\pi\gamma})$ is a harmonic range of points for each $a\in\mathcal{L}$. Let $a,b\in\mathcal{L}$. We define

$$a\parallel_L b \quad :\Longleftrightarrow\quad a^{\gamma\lambda}=b^{\gamma\lambda}\quad\text{(left parallel lines)},$$
$$a\parallel_R b \quad :\Longleftrightarrow\quad a^{\gamma\rho}=b^{\gamma\rho}\quad\text{(right parallel lines)}.$$

Moreover, by [8, pp. 44–46],

$$a\parallel b\Longleftrightarrow a\parallel_L b\text{ or }a\parallel_R b, \tag{3}$$
$$b\in\{a,a^\pi\}\Longleftrightarrow a\parallel_L b\text{ and }a\parallel_R b, \tag{4}$$
$$a\approx b\Longleftrightarrow a^{\gamma\lambda},b^{\gamma\lambda}\ \kappa_R\text{-conjugate and }a^{\gamma\rho},b^{\gamma\rho}\ \kappa_L\text{-conjugate}. \tag{5}$$

Left parallelism $\parallel_L$ is an equivalence relation. The equivalence class of $a\in\mathcal{L}$ is an elliptic linear congruence of lines (regular spread) [8, Lemma 3]; this spread is denoted by

$$\mathcal{S}_L(a):=\{x\mid x\parallel_L a\}.$$

Given a line $p\in\mathcal{L}\setminus\mathcal{S}_L(a)$ then, by the regularity of the spread $\mathcal{S}_L(a)$,

$$\mathcal{R}_L(a|p):=\{x\mid x\parallel_L a\text{ and }x\cap p\neq\emptyset\}$$

is a regulus. These results carry over to $\parallel_R$ in an obvious way.

Let $a\approx b$. If $Q\in\mathcal{P}$, then there exist lines $x_Q\in\mathcal{S}_L(a)$ and $y_Q\in\mathcal{S}_L(b)$, concurrent at Q, since $\mathcal{S}_L(a)$ and $\mathcal{S}_L(b)$ are spreads. According to [8, Lemma 5, I], we obtain $x_Q\approx y_Q$, whence

$$\mathcal{S}_L(a)=\{x\in\mathcal{S}_L(a)\mid\exists\, y\in\mathcal{S}_L(b)\text{ with }x\approx y\}; \tag{6}$$

[1]The existence of an α-invariant point is equivalent to the existence of a π-invariant general linear complex of lines $\subset\mathcal{L}$ or, in other words, is equivalent to the existence of a symplectic polarity of $(\mathcal{P},\mathcal{L})$ commuting with π. If there would be no α-invariant points, then the lines a and a^π would be the only Clifford parallel lines for any $a\in\mathcal{L}$ [8, Lemma 2].

[2]If we are given an elliptic projective polarity of $\mathrm{PG}(3,F)$ (F commutative), then it is possible to obtain self-conjugate points by a single quadratic extension of $\mathrm{PG}(3,F)$, but in general it takes two consecutive quadratic extensions to get self-polar lines.

$\mathcal{S}_L(b)$ can be described likewise. On the other hand, by [8, Lemma 5, II],

$$\mathcal{R}_L(a|b) = \{x \in \mathcal{S}_L(a) \mid \exists\, y \in \mathcal{S}_R(b) \text{ with } x \approx y\}, \tag{7}$$
$$\mathcal{R}_R(b|a) = \{y \in \mathcal{S}_R(b) \mid \exists\, x \in \mathcal{S}_L(a) \text{ with } y \approx x\}. \tag{8}$$

Consequently, $\mathcal{R}_L(a|b)$ and $\mathcal{R}_R(b|a)$ are mutually opposite reguli[3] and

$$x \approx y \text{ for all } x \in \mathcal{R}_L(a|b) \text{ and all } y \in \mathcal{R}_R(b|a). \tag{9}$$

These results remain true if the terms "left" and "right" are interchanged.

3 PROOFS

In the subsequent Propositions let $(\mathcal{P}, \mathcal{L}, \pi)$ be a 3-dimensional classical elliptic space. Suppose, furthermore, that $\varphi : \mathcal{L} \to \mathcal{L}$ is an injection satisfying (2).

PROPOSITION 1 *For all $a, b \in \mathcal{L}$ the following properties hold true:*

$$a \approx b \Longrightarrow a^\varphi \approx b^\varphi, \tag{10}$$
$$a \parallel b \Longrightarrow a^\varphi \parallel b^\varphi, \tag{11}$$
$$a^{\pi\varphi} = a^{\varphi\pi}, \tag{12}$$
$$a \parallel b \text{ and } b \notin \{a, a^\pi\} \Longrightarrow \text{ either } a^\varphi \parallel_L b^\varphi \text{ or } a^\varphi \parallel_R b^\varphi. \tag{13}$$

Proof. We deduce (10) from the injectivity of φ and (2). Now (11) is immediate from the definition of $\parallel$, the injectivity of φ and (10). In order to establish (12) choose points $A_0 \in a$ and $A_1 \in a^\pi$. Setting $A_2 := A_1^\pi \cap a^\pi$, $a_1 := A_0 \vee A_1$ and $a_2 := A_0 \vee A_2$ yields that

$$a \approx a_1 \approx a_2 \approx a, \quad a^\pi \approx a_1 \approx a_2 \approx a^\pi.$$

Hence a_1^φ and a_2^φ are concurrent and distinct by (10). Therefore

$$(a_1^\varphi \cap a_2^\varphi)^\pi \cap (a_1^\varphi \vee a_2^\varphi) =: a' \text{ and } a'^\pi \neq a'$$

are the only two lines that are intersecting both a_1^φ and a_2^φ orthogonally. Thus $\{a', a'^\pi\} = \{a^\varphi, a^{\pi\varphi}\}$. Now (13) follows from $b^\varphi \notin \{a^\varphi, a^{\varphi\pi}\}$ and (4). □

PROPOSITION 2 *If $a \in \mathcal{L}$, then either $\mathcal{S}_L(a)^\varphi \subset \mathcal{S}_L(a^\varphi)$ or $\mathcal{S}_L(a)^\varphi \subset \mathcal{S}_R(a^\varphi)$.*

Proof. Assume to the contrary that our assertion does not hold. We infer from (4), (10), $\#\mathcal{S}_L(a) = \#F = \infty$ and the injectivity of φ that $\mathcal{S}_L(a^\varphi) \cap \mathcal{S}_R(a^\varphi) = \{a^\varphi, a^{\varphi\pi}\} = \{a^\varphi, a^{\pi\varphi}\}$ is a proper subset of $\mathcal{S}_L(a)^\varphi$. Therefore $\mathcal{S}_L(a)^\varphi$ cannot be a subset of both $\mathcal{S}_L(a^\varphi)$ and $\mathcal{S}_R(a^\varphi)$. Hence there exist distinct lines $x, y \in \mathcal{S}_L(a) \setminus \{a, a^\pi\}$ such that $a^\varphi \parallel_L x^\varphi$ and $a^\varphi \parallel_R y^\varphi$. Moreover, $x^\varphi \parallel y^\varphi$ by (3) and (11).
If $x^\varphi \parallel_L y^\varphi$, then $a^\varphi \parallel_L x^\varphi \parallel_L y^\varphi$. It follows that y^φ is both left and right parallel to a^φ. Hence, by (4) and (12), we obtain $y^\varphi \in \{a^\varphi, a^{\varphi\pi}\} = \{a^\varphi, a^{\pi\varphi}\}$. This is contradicting the injectivity of φ.
Likewise, $x^\varphi \parallel_R y^\varphi$ yields a contradiction. □

[3] In a real elliptic 3-space these two reguli are on the well-known *Clifford surface*. Formula (9) reflects the fact that this quadric admits locally Cartesian coordinates.

PROPOSITION 3 *If $\mathcal{S}_L(a)^\varphi \subset \mathcal{S}_L(a^\varphi)$ for at least one line $a \in \mathcal{L}$, then*

$$\mathcal{S}_L(b)^\varphi \subset \mathcal{S}_L(b^\varphi) \text{ for all } b \in \mathcal{L}. \tag{14}$$

Proof. At first let $a \approx b$. Assume to the contrary that $\mathcal{S}_L(b)^\varphi \not\subset \mathcal{S}_L(b^\varphi)$, whence Proposition 2 gives $\mathcal{S}_L(b)^\varphi \subset \mathcal{S}_R(b^\varphi)$. If $\mathcal{S}_L(a)$ is written down according to (6), then (10) gives that for each $x^\varphi \in \mathcal{S}_L(a)^\varphi \subset \mathcal{S}_L(a^\varphi)$ there exists a $y^\varphi \in \mathcal{S}_L(b)^\varphi \subset \mathcal{S}_R(b^\varphi)$ such that $x^\varphi \approx y^\varphi$. We infer from (7) and (8), applied to $a^\varphi \approx b^\varphi$, that

$$\mathcal{S}_L(a)^\varphi \subset \mathcal{R}_L(a^\varphi|b^\varphi) \text{ and } \mathcal{S}_L(b)^\varphi \subset \mathcal{R}_R(b^\varphi|a^\varphi).$$

There exists a line c such that $a \approx c \approx b$. Application of (9) yields (in an obvious shorthand notation) $c \approx \mathcal{R}_L(a|c)$ and $c \approx \mathcal{R}_L(b|c)$. Therefore, by (10),

$$c^\varphi \approx \mathcal{R}_L(a|c)^\varphi \subset \mathcal{R}_L(a^\varphi|b^\varphi) \text{ and } c^\varphi \approx \mathcal{R}_L(b|c)^\varphi \subset \mathcal{R}_R(b^\varphi|a^\varphi).$$

Due to the injectivity of φ, c^φ has to be a transversal line of two infinite sets of lines contained in opposite reguli, respectively. This is an absurdity.
Next assume $a \not\approx b$. Then there is a finite sequence $a \approx a_1 \approx \cdots \approx a_n \approx b$, whence the proof from above carries over to all lines. □

PROPOSITION 4 *If $\mathcal{S}_L(a)^\varphi \subset \mathcal{S}_L(a^\varphi)$ for all $a \in \mathcal{L}$, then*

$$\mathcal{S}_R(b)^\varphi \subset \mathcal{S}_R(b^\varphi) \text{ for all } b \in \mathcal{L}. \tag{15}$$

Proof. Given a line $b \in \mathcal{L}$ there exists a line $a \in \mathcal{L}$ with $b \approx a$. By (9) and by the definition of the relation $\approx$, $\mathcal{R}_R(b|a)$ and $\mathcal{R}_L(a|b)$ are mutually opposite reguli containing $\{b, b^\pi\}$ and $\{a, a^\pi\}$, respectively. There exist lines

$$b_1 \in \mathcal{R}_R(b|a) \setminus \{b, b^\pi\} \text{ and } a_1 \in \mathcal{R}_L(a|b) \setminus \{a, a^\pi\}.$$

Analogously, the distinct related lines b^φ and a^φ give rise to mutually opposite reguli $\mathcal{R}_R(b^\varphi|a^\varphi)$ and $\mathcal{R}_L(a^\varphi|b^\varphi)$ containing $\{b^\varphi, b^{\pi\varphi}\} = \{b^\varphi, b^{\varphi\pi}\}$ and $\{a^\varphi, a^{\pi\varphi}\} = \{a^\varphi, a^{\varphi\pi}\}$, respectively; cf. (12). Now $a_1 \parallel_L a$, $a_1 \approx b$, the present assumption on $\mathcal{S}_L(a)^\varphi$ and (10) yield

$$a_1^\varphi \in \mathcal{R}_L(a^\varphi|b^\varphi) \setminus \{a^\varphi, a^{\varphi\pi}\}.$$

Since $b_1 \approx \{a, a^\pi, a_1\}$, we infer from (10) that $b_1^\varphi \approx \{a^\varphi, a^{\varphi\pi}, a_1^\varphi\}$. Therefore b_1^φ is a transversal line of the regulus $\mathcal{R}_L(a^\varphi|b^\varphi)$. So

$$b_1^\varphi \in (\mathcal{R}_R(b^\varphi|a^\varphi) \setminus \{b^\varphi, b^{\varphi\pi}\}) \subset (\mathcal{S}_R(b^\varphi) \setminus \{b^\varphi, b^{\varphi\pi}\})$$

and $b_1^\varphi \nparallel_L b^\varphi$ by (4). Proposition 2 extends to the φ-images of right parallel classes in an obvious way. Hence the assertion follows. □

Propositions 3 and 4 hold true, mutatis mutandis, if $\mathcal{S}_L(a)^\varphi \subset \mathcal{S}_R(a^\varphi)$. Therefore, the injection φ is either preserving or interchanging left and right parallelism. According to these possibilities φ will be called *direct* or *opposite*.
In the sequel we shall confine our attention on a direct mapping φ. The subsequent Propositions remain true when the terms “left” and “right” are interchanged.

PROPOSITION 5 *If φ is direct, then*

$$a \nparallel_R b \text{ and } a \parallel_L b \Longrightarrow a^\varphi \nparallel_R b^\varphi \text{ for all } a, b \in \mathcal{L}. \tag{16}$$

Proof. We infer from $a \nparallel_R b$, $a \parallel_L b$ and (4) that $b \notin \{a, a^\pi\}$. By the injectivity of φ and (12), we obtain $b^\varphi \notin \{a^\varphi, a^{\varphi\pi}\}$. Now $a^\varphi \parallel_L b^\varphi$ and (4) yield $a^\varphi \nparallel_R b^\varphi$. □

PROPOSITION 6 *If φ is direct, then*

$$\varphi_L : \mathcal{E}_L \to \mathcal{E}_L, \ a^{\gamma\rho} \mapsto a^{\varphi\gamma\rho} \quad (a \in \mathcal{L})$$

is a well-defined mapping.

Proof. By (15), the definition of φ_L is unambiguous for all points in $\operatorname{im}\gamma\rho$. The restriction of ρ to the Klein quadric Γ is surjective, since Γ contains a plane. Thus the assertion follows. □

PROPOSITION 7 *Let $a \in \mathcal{L}$. Then $\varphi_L | \mathcal{S}_L(a)^{\gamma\rho}$ is injective.*

Proof. Two distinct points of $\mathcal{S}_L(a)^{\gamma\rho}$ can be written as $b_1^{\gamma\rho} \neq b_2^{\gamma\rho}$ with $b_1, b_2 \in \mathcal{S}_L(a)$. Hence $b_1 \parallel_L b_2$, but $b_1 \nparallel_R b_2$. By (16), $b_1^\varphi \nparallel_R b_2^\varphi$. Now $b_1^{\varphi\gamma\rho} \neq b_2^{\varphi\gamma\rho}$ follows from the definition of $\parallel_R$. □

We remark that $\mathcal{S}_L(a)^{\gamma\rho}$ is in general a proper subset of $\mathcal{E}_L$: The image of $\mathcal{S}_L(a)$ under the Klein mapping γ is an oval quadric in the 3-dimensional subspace $\mathcal{T} := \mathcal{E}_L \vee a^\gamma$ of $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}})$; cf. the proof of [8, Lemma 3]. The restriction of the projection ρ to $\mathcal{S}_L(a)^\gamma$ may be seen as a *gnomonic projection*, since $\mathcal{E}_R \cap \mathcal{T}$ (the centre of the projection) is an interior point of that oval quadric. The surjectivity of this gnomonic projection is equivalent to the fact that every right parallel class has non-empty intersection with $\mathcal{S}_L(a)$. We state two sufficient conditions for $\mathcal{S}_L(a)^{\gamma\rho} = \mathcal{E}_L$: If F is a Euclidean field, then any line through an interior point of an oval quadric is a secant; cf., for example, [4, vol. II, p. 54]. Hence any gnomonic projection is surjective. If F is a Pythagorean field and if the absolute polarity π can be described by the standard bilinear form on F^4, then a line joining any point of $\mathcal{E}_L$ with any point of $\mathcal{E}_R$ is a secant of the Klein quadric; cf. [8, Remark 4]. Thus the gnomonic projections arising from ρ are then surjective.

PROPOSITION 8 *The mapping $\varphi_L : \mathcal{E}_L \to \mathcal{E}_L$ takes κ_L-conjugate points to κ_L-conjugate points. Moreover, φ_L is a full lineation.*

Proof. Let $\mathcal{F} \subset \mathcal{L}$ be a ruled plane. The restriction of $\gamma\rho$ to $\mathcal{F}$ is a collineation of $\mathcal{F}$ (regarded as dual projective plane) onto $\mathcal{E}_L$. Given points $U, V \in \mathcal{E}_L$ there exist lines $u, v \in \mathcal{F}$ with $u^{\gamma\rho} = U$ and $v^{\gamma\rho} = V$. The set $\{x \mid x \in \mathcal{F}, x \approx u\}$ is a pencil of lines. Its image under $\gamma\rho$ is a line. More precisely, this is the polar line of U with respect to κ_L; see (5). It follows from (10) and (5) that

$$U, V \ \kappa_L\text{-conjugate} \iff u \approx v \Longrightarrow a^\varphi \approx v^\varphi \Longrightarrow U^{\varphi_L}, V^{\varphi_L} \ \kappa_L\text{-conjugate}. \tag{17}$$

If $g \subset \mathcal{E}_L$ is a line, then (17) implies $g^{\varphi_L} \subset g^{\kappa_L \varphi_L \kappa_L}$. From this observation it is immediate that φ_L is a lineation, that is, a collinearity-preserving mapping.
Finally, choose lines $a, b, c \in \mathcal{F}$ with $a \approx b \approx c \approx a$. By (5), $\{a^{\gamma\rho}, b^{\gamma\rho}, c^{\gamma\rho}\} \subset \mathcal{E}_L$ is a self-polar triangle with respect to κ_L. We form the reguli

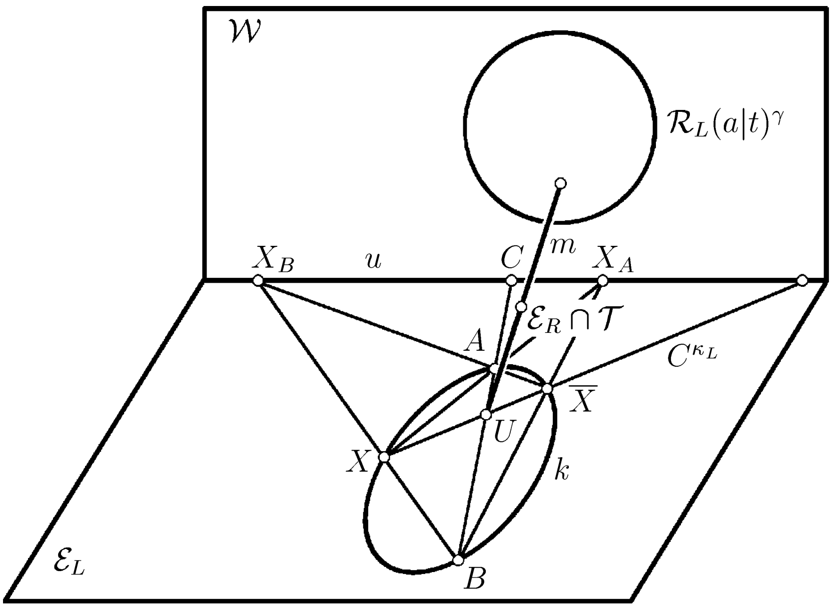
$$\mathcal{R}_L(c|a) \approx a, \ \mathcal{R}_L(a|b) \approx b, \ \mathcal{R}_L(b|c) \approx c. \tag{18}$$

If $\mathcal{R}$ denotes any of these three reguli, then the restriction $\gamma\rho|\mathcal{R}$ is not injective, but the fibre of $x \in \mathcal{R}$ is given by $\{x, x^\pi\}$. There is an infinite number of such unordered pairs, whence the image $\mathcal{R}^{\gamma\rho}$ is an infinite subset on one side of the triangle $\{a^{\gamma\rho}, b^{\gamma\rho}, c^{\gamma\rho}\}$; e.g., $\mathcal{R}_L(c|a)^{\gamma\rho} \subset (b^{\gamma\rho} \vee c^{\gamma\rho})$. By (17) and Proposition 7, $\operatorname{im}\varphi_L$ contains the κ_L-self-polar triangle $\{a^{\varphi\gamma\rho}, b^{\varphi\gamma\rho}, c^{\varphi\gamma\rho}\}$ as well as infinitely many points on each side of this triangle. We read off from [6, p. 4] that $\operatorname{im}\varphi_L$ is either a projective subplane or a near-pencil (degenerate subplane) of $\mathcal{E}_L$. Obviously, the second possibility cannot occur, whence there exists a quadrangle in $\operatorname{im}\varphi_L$, i.e., φ_L is full. □

The next result is immediate from Proposition 7, whenever we have a surjective gnomonic projection $\rho|\mathcal{S}_L(a)^\gamma$ for at least one line $a \in \mathcal{L}$.

PROPOSITION 9 *The lineation $\varphi_L : \mathcal{E}_L \to \mathcal{E}_L$ injective.*

Proof. Choose intersecting lines $a, t \in \mathcal{L}$ with $a \not\approx t$. Set $\mathcal{T} := \operatorname{span}\mathcal{S}_L(a)^\gamma$. The regulus $\mathcal{R}_L(a|t)$ does not contain a^π so that the plane $\mathcal{W} := \operatorname{span}\mathcal{R}_L(a|t)^\gamma \subset \mathcal{T}$ does not contain the



point $\mathcal{E}_R \cap \mathcal{T}$; cf. the figure. Therefore the image of the conic $\mathcal{R}_L(a|t)^\gamma$ under the gnomonic projection $\rho|\mathcal{S}_L(a)^\gamma$, is also a conic, say

$$k := \mathcal{R}_L(a|t)^{\gamma\rho}. \tag{19}$$

Put $u := \mathcal{W} \cap \mathcal{E}_L$ and write m for the polar line of u with respect to the polarity of $\mathcal{S}_L(a)^\gamma$. Then $m \cap \mathcal{W}$ is the pole of u with respect to $\mathcal{R}_L(a|t)^\gamma$. As$\mathcal{E}_R \cap \mathcal{T}$ is on m, the point $U := m \cap \mathcal{E}_L = (m \cap \mathcal{W})^\rho$ is the pole of u with respect to k and κ_L. The elliptic polarity κ_L and the polarity of the conic $\mathcal{R}_L(a|t)^\gamma$ induce the same elliptic involutory projectivity on u, since both are arising from the polarity κ of the Klein quadric. The gnomonic projection fixes u pointwise, whence this involutory projectivity on u is also induced by the polarity of the conic[4] k. Hence U is an interior point of k. By Proposition 7 and (19),

$$\varphi_L|k \text{ is injective.} \tag{20}$$

Since k is infinite, we may find points $A, B \in k$, collinear with U, such that A^{φ_L}, B^{φ_L} and U^{φ_L} are mutually distinct. By Proposition 8,

$$U^{\varphi_L} \in A^{\varphi_L} \vee B^{\varphi_L}. \tag{21}$$

Let $X \in k \setminus \{A, B\}$. As U is interior point of k, the line $X \vee U$ meets k at one more point $\overline{X} \neq X$. Then $\{A, B, X, \overline{X}\} \subset k$ is a quadrangle, whence its diagonal points form a self-polar triangle with respect to k, say $\{U, X_A, X_B\}$. Hence $u = X_A \vee X_B$, and $\{U, X_A, X_B\}$ is also a κ_L-self-polar triangle. We infer from (20) that $A^{\varphi_L}, B^{\varphi_L}, X^{\varphi_L}, \overline{X}^{\varphi_L}$ are mutually distinct. Proposition 8 tells us that $\{U^{\varphi_L}, X_A^{\varphi_L}, X_B^{\varphi_L}\}$ is a κ_L-self-polar triangle. We claim that

$$X^{\varphi_L} \notin A^{\varphi_L} \vee B^{\varphi_L}; \tag{22}$$

otherwise we would have

$$X_A^{\varphi_L} \in A^{\varphi_L} \vee X^{\varphi_L} = A^{\varphi_L} \vee B^{\varphi_L} \text{ and } X_B^{\varphi_L} \in B^{\varphi_L} \vee X^{\varphi_L} = A^{\varphi_L} \vee B^{\varphi_L},$$

whence $X_A^{\varphi_L}$, $X_B^{\varphi_L}$ and U^{φ_L} would be collinear, an absurdity.
Since $X \in k \setminus \{A, B\}$ has been chosen arbitrarily, we can deduce from (21) and (22) that

$$P^{\varphi_L} \neq U^{\varphi_L} \text{ for all } P \in k. \tag{23}$$

It is obvious now that $A^{\varphi_L}, B^{\varphi_L}, X^{\varphi_L}, \overline{X}^{\varphi_L}$ is a quadrangle with U^{φ_L} being one of its diagonal points. Thus we have arrived at the well-known construction of the fourth harmonic point for $A^{\varphi_L}, B^{\varphi_L}, U^{\varphi_L}$. Setting

$$u' := U^{\varphi_L \kappa_L} = X_A^{\varphi_L} \vee X_B^{\varphi_L}$$

yields that

$$u' \neq A^{\varphi_L} \vee B^{\varphi_L}. \tag{24}$$

With $C := u \cap (A \vee B)$, we obtain $C^{\varphi_L} = u' \cap (A^{\varphi_L} \vee B^{\varphi_L})$ and

$$C^{\varphi_L} \neq X_A^{\varphi_L}. \tag{25}$$

As X varies in $k \setminus \{A, B\}$, the point X_A is running in $u \setminus (\{C\} \cup (C^{\kappa_L} \cap u))$, that is, we are reaching all points of u but two. By (25), $C^{\varphi_L} \in u'$ has only a finite number of pre-images on the line u. Now [1, Satz 3.2] establishes that φ_L is injective. $\square$

[4] k is a circle of the elliptic plane $\mathcal{E}_L$.

PROPOSITION 10 *If $\varphi : \mathcal{L} \to \mathcal{L}$ is direct, then any two points of $\mathcal{E}_L$ that are not κ_L-conjugate remain non-conjugate under φ_L.*

Proof. The injective and full lineation φ_L is preserving non-collinearity of points [6, p. 4]. If $U, V \in \mathcal{E}_L$ are not κ_L-conjugate, then $V \notin U^{\kappa_L}$, whence $V^{\varphi_L} \notin \operatorname{span}(U^{\kappa_L \varphi_L}) = U^{\varphi_L \kappa_L}$. Thus $U^{\varphi_L}, V^{\varphi_L}$ are not κ_L-conjugate. □

PROPOSITION 11 *If $\varphi : \mathcal{L} \to \mathcal{L}$ is direct, then*

$$a \not\sim b \Longrightarrow a^{\varphi} \not\sim b^{\varphi} \text{ for all } a, b \in \mathcal{L}. \tag{26}$$

Proof. We infer from the left hand side of (26) and from (5) that (up to interchanging the terms "left" and "right") $a^{\gamma\rho}, b^{\gamma\rho}$ are not κ_L-conjugate. Under φ_L this property remains unchanged by Proposition 10, whence the assertion follows from (5) and the definition of φ_L in Proposition 6. □

We recall a concept introduced in [8]: Two collineations $\zeta : \mathcal{E}_L \to \mathcal{E}_L$ and $\eta : \mathcal{E}_R \to \mathcal{E}_R$ are called *admissible* if the following conditions hold true:

Ad1. ζ and η are commuting with κ_L and κ_R, respectively.
Ad2. $(X \vee Y) \cap \Gamma \neq \emptyset \Longrightarrow (X^{\zeta} \vee Y^{\eta}) \cap \Gamma \neq \emptyset$ for all $X \in \mathcal{E}_L$, $Y \in \mathcal{E}_R$.

When writing [8], the author considered the next Proposition to be self-evident. It seems, however, that it deserves a formal proof.

PROPOSITION 12 *If collineations $\zeta : \mathcal{E}_L \to \mathcal{E}_L$ and $\eta : \mathcal{E}_R \to \mathcal{E}_R$ are admissible, then their inverse mappings are also admissible.*

Proof. Condition Ad1 is evidently true for ζ^{-1}, η^{-1}. Given $X \in \mathcal{E}_L$ and $Y \in \mathcal{E}_R$ such that $(X \vee Y) \cap \Gamma = \emptyset$, then, by the surjectivity of $\gamma\lambda$, there is a line $a \in \mathcal{L}$ with $Y = a^{\gamma\lambda}$. By Ad2, there exists a line $a' \in \mathcal{L}$ with $a'^{\gamma} \in (a^{\gamma\rho\zeta} \vee Y^{\eta}) \cap \Gamma$. It is straightforward to verify that ζ extends to a collineation

$$\delta : Y \vee \mathcal{E}_L \to Y^{\eta} \vee \mathcal{E}_L$$

with $Y \mapsto Y^{\eta}$ and $a^{\gamma} \mapsto a'^{\gamma}$. Under δ the elliptic quadric $\mathcal{S}_L(a)^{\gamma}$ goes over to an elliptic quadric within $Y^{\eta} \vee \mathcal{E}_L$. This quadric coincides with the quadric $(Y^{\eta} \vee \mathcal{E}_L) \cap \Gamma$, since a'^{γ} is a common point, $\mathcal{E}_L$ is the common polar plane of Y^{η} and $\kappa_L = \zeta^{-1}\kappa_L\zeta$ is the induced polarity in $\mathcal{E}_L$ for both quadrics[5]; cf., e.g., [4, vol. I, p. 191]. Thus $(X^{\zeta} \vee Y^{\eta}) \cap \Gamma = ((X \vee Y) \cap \Gamma)^{\delta} = \emptyset$. □

Proof of Theorem 1. If φ is direct, then the result is immediate from (26). Otherwise choose a point $Q \in \mathcal{P}$. The harmonic homology with centre Q and axis Q^{π} is an elliptic reflection and yields an opposite Plücker transformation χ. By (26), $\chi\varphi$ is a direct Plücker transformation so that φ too is a Plücker transformation. □

Proof of Theorem 2. If φ is direct, then $\mathcal{E}_L^{\varphi_L}$ is a subplane of $\mathcal{E}_L$ isomorphic to $\mathcal{E}_L$. Hence the underlying field of $\mathcal{E}_L^{\varphi_L}$ is a subfield of F isomorphic to F [18, p. 266]. This implies, by

[5] This is a projective generalization of the well-known fact that a sphere in Euclidean space is uniquely determined by one point and its mid-point.

our assumption on F, that φ_L is surjective. We infer from Propositions 8 and 9 that φ_L is a collineation of $\mathcal{E}_L$ commuting with κ_L. Similarly, φ_R is a collineation of $\mathcal{E}_R$ commuting with κ_R. By their definition, φ_L and φ_R are admissible.

Given a line $a' \in \mathcal{L}$ we may apply φ_L^{-1} and φ_R^{-1} to the points $a'^{\gamma\rho}$ and $a'^{\gamma\lambda}$, respectively. Thus we obtain points $X \in \mathcal{E}_L$ and $Y \in \mathcal{E}_R$, say. By Proposition 12, there exists a line $a \in \mathcal{L}$ such that a^γ is on the line $X \vee Y$. Therefore $\{a, a^\pi\}^\varphi = \{a', a'^\pi\}$ so that φ is surjective. The assertion follows now from Theorem 1.

If φ is opposite, then let χ be an elliptic reflection. Hence both $\chi\varphi$ and φ are Plücker transformations. $\square$

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