Chow's Theorem for Linear Spaces

Hans Havlicek

Abstract

If $\varphi : \mathcal{L} \to \mathcal{L}'$ is a bijection from the set of lines of a linear space $(\mathcal{P}, \mathcal{L})$ onto the set of lines of a linear space $(\mathcal{P}', \mathcal{L}')$ ($\dim(\mathcal{P}, \mathcal{L}), \dim(\mathcal{P}', \mathcal{L}') \geq 3$), such that intersecting lines go over to intersecting lines in both directions, then φ is arising from a collineation of $(\mathcal{P}, \mathcal{L})$ onto $(\mathcal{P}', \mathcal{L}')$ or a collineation of $(\mathcal{P}, \mathcal{L})$ onto the dual linear space of $(\mathcal{P}', \mathcal{L}')$. However, the second possibility can only occur when $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ are 3–dimensional generalized projective spaces.

Keywords: Linear space, line geometry, Plücker space.

1 Introduction

The Grassmannian $\Gamma_{k,n}$ formed by the k–dimensional subspaces of an n–dimensional projective space ($1 \leq k \leq n-2$) carries the structure of a metric space: Two k–subspaces are at distance d if their meet is $(k-d)$–dimensional [11, p. 807]; subspaces with distance 1 are called *adjacent*. Any collineation and, however only if $n = 2k+1$, any duality of the underlying projective space yields a bijection on $\Gamma_{k,n}$ which is distance preserving in both directions. Conversely, any bijection of $\Gamma_{k,n}$ which is adjacency preserving in both directions arises in this way. The last result is due to W.–L. Chow; cf. [7] or [8, 80–82]. Thus Chow's theorem may be seen as an early result in a discipline which now is called *characterizations of geometrical transformations under mild hypotheses* [2], [3], [13]. Alternatively, one may consider $\Gamma_{k,n}$ as the set of vertices of a graph where two vertices are joined by an edge if, and only if, they represent adjacent subspaces. However, we shall not adopt this point of view.

Other results in the spirit of Chow's theorem can be found in [2], [4], [7], [8, 82–88], [9], and [10].

In an arbitrary linear space there are several concepts of "dimension". Cf. the remarks in [5, p. 73]. Hence one may ask if Chow's theorem still holds true for linear spaces. We shall not deal with this question in full generality, as we focus our attention on the case of 1–dimensional subspaces, i.e., the lines of a linear space. Loosely speaking, here the answer is in the affirmative. We refer to Theorem 1 for a precise statement.

2 Preliminaries

Let $(\mathcal{P}, \mathcal{L})$ be a linear space. We shall stick to the terminology but not to the notation used in [12]: Given a subset $\mathcal{S} \subset \mathcal{P}$ then $\langle \mathcal{S} \rangle$ denotes the *span* of $\mathcal{S}$, i.e., the intersection of all subspaces containing $\mathcal{S}$. The *join* of subsets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{P}$ is defined as $\langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle =: \mathcal{S}_1 \vee \mathcal{S}_2$. The *dimension* of $(\mathcal{P}, \mathcal{L})$ is given by[1]

$$\dim(\mathcal{P}, \mathcal{L}) := \min\{\#\mathcal{X} - 1 \mid \mathcal{X} \subset \mathcal{P},\ \langle \mathcal{X} \rangle = \mathcal{P}\}. \tag{1}$$

A *plane* of $(\mathcal{P}, \mathcal{L})$ is a two–dimensional subspace.

A linear space $(\mathcal{P}, \mathcal{L})$ is called an *exchange space*, if it satisfies the exchange axiom: Given any subset $\mathcal{S} \subset \mathcal{P}$ and points $A, B \in \mathcal{P}$ then

$$B \in (\mathcal{S} \vee \{A\}) \setminus \langle \mathcal{S} \rangle \Longrightarrow A \in \mathcal{S} \vee \{B\}. \tag{2}$$

We observe that (2) is true in any linear space, if A, B are collinear with some point $X \in \langle \mathcal{S} \rangle$.

A 2–dimensional linear space such that any two distinct lines have at least one point in common is called a *generalized projective plane*. A *generalized projective space* is a linear space $(\mathcal{P}, \mathcal{L})$ where any plane is a generalized projective plane [1, p. 9]; cf. also [6, p. 39]. The following property of generalized projective spaces is immediate: Given a subset $\mathcal{S} \neq \emptyset$ and a point $X \in \mathcal{P} \setminus \langle \mathcal{S} \rangle$ then

$$\mathcal{S} \vee \{X\} = \bigcup_{Y \in \langle \mathcal{S} \rangle} (\{X\} \vee \{Y\}). \tag{3}$$

Consequently, $(\mathcal{P}, \mathcal{L})$ is an exchange space.

We shall encounter 3–dimensional generalized projective spaces in Theorem 1. If $(\mathcal{P}, \mathcal{L})$ is such a space, then any plane $\mathcal{E} \subset \mathcal{P}$ and any line $y \in \mathcal{L}$ have a common point according to (3). Moreover, any two distinct planes meet at a unique line. Conversely, given a line $a \in \mathcal{L}$ then denote by a^* the *pencil of planes* with axis a, i.e., the set of all planes passing through a. As $(\mathcal{P}, \mathcal{L})$ is an exchange space, any two distinct points on a extend to a basis of $(\mathcal{P}, \mathcal{L})$, whence $\#a^* \geq 2$. Thus it is possible to define the *dual linear space* of $(\mathcal{P}, \mathcal{L})$ as follows: Let $\mathcal{P}^*$ be the set of all planes of $(\mathcal{P}, \mathcal{L})$ and let $\mathcal{L}^*$ be the set of all pencils of planes [5, p. 72].

3 Line Geometry

Let $(\mathcal{P}, \mathcal{L})$ be a linear space. Lines $a, b \in \mathcal{L}$ are called *related* ($a \sim b$) if $a \cap b \neq \emptyset$. Distinct related lines a, b are denoted by $a \approx b$ and will be named *adjacent*. The pair $(\mathcal{L}, \sim)$ is a *Plücker space*[2] in the sense of W. Benz [2, p. 199].

Let $\mathcal{N} \subset \mathcal{L}$ be a set of mutually related lines; we shall use the term *related set* for such an $\mathcal{N}$. It is easily seen, by virtue of Zorn's lemma, that there is a maximal related set $\mathcal{M} \subset \mathcal{L}$ containing $\mathcal{N}$. If $-1 \leq \dim(\mathcal{P}, \mathcal{L}) \leq 1$, then $\mathcal{L}$ is the only maximal related set. With $A \in \mathcal{P}$ write $\mathcal{L}(A)$ for the *star of lines* with vertex A, i.e., the set of all lines of $\mathcal{L}$ running through A. Any star of lines is a maximal related set for $\dim(\mathcal{P}, \mathcal{L}) \neq 2$; in the 2–dimensional case a star of lines obviously is a related set. It may be maximal (e.g. in affine planes) or not maximal (e.g. in projective planes). If $\dim(\mathcal{P}, \mathcal{L}) \geq 2$, then any trilateral is contained in a maximal related set which cannot be a star of lines.

[1] Cf., however, the definition in [1, p. 9].

[2] We refrain from assuming that $\mathcal{L}$ is non–empty.

Lemma 1 *Let* $\dim(\mathcal{P},\mathcal{L}) \geq 2$ *and let* $\mathcal{M} \subset \mathcal{L}$ *be a maximal related set different from a star of lines. Then the subspace* $\mathcal{E}$ *spanned by the lines of* $\mathcal{M}$ *is a plane satisfying*

$$\mathcal{E} = x \vee y \text{ for all } x, y \in \mathcal{M},\ x \neq y. \tag{4}$$

Proof. Choose any line $w \in \mathcal{M} \setminus \{x, y\}$. If $\{x, y, w\}$ is a trilateral, then $w \subset x \vee y \subset \mathcal{E}$. Otherwise, there exists a line $z \in \mathcal{M} \setminus \{x, y\}$ such that $\{x, y, z\}$ is a trilateral, since $\mathcal{M}$ cannot be contained in a star of lines. Hence $w \subset x \vee z = x \vee y \subset \mathcal{E}$ and

$$\mathcal{E} = \bigvee_{w \in \mathcal{M}} w \subset x \vee y \subset \mathcal{E} \tag{5}$$

is a plane. □

For $\dim(\mathcal{P},\mathcal{L}) \geq 3$ the maximal related sets in $\mathcal{L}$ fall into two classes: stars of lines and the *coplanar maximal related sets* described in Lemma 1. This result is crucial in proving Theorem 1, since stars allow to recover the points of $(\mathcal{P},\mathcal{L})$ in terms of line geometry. If $\dim(\mathcal{P},\mathcal{L}) = 2$, then this distinction of the maximal related sets is in general not available. Hence 2–dimensional linear spaces will not appear in the next section.

4 Chow's Theorem

Let $(\mathcal{P},\mathcal{L})$ and $(\mathcal{P}',\mathcal{L}')$ be linear spaces. Any collineation $\kappa : \mathcal{P} \to \mathcal{P}'$ gives rise to a bijection

$$\varphi : \mathcal{L} \to \mathcal{L}',\ \{A\} \vee \{B\} \mapsto \{A^\kappa\} \vee \{B^\kappa\}\ (A, B \in \mathcal{P},\ A \neq B) \tag{6}$$

taking related lines to related lines in both directions. We shall prove the following converse:

Theorem 1 *Let* $(\mathcal{P},\mathcal{L})$ *and* $(\mathcal{P}',\mathcal{L}')$ *be linear spaces such that* $\dim(\mathcal{P},\mathcal{L}) \geq 3$ *and* $\dim(\mathcal{P}',\mathcal{L}') \geq 3$*. Suppose that* $\varphi : \mathcal{L} \to \mathcal{L}'$ *is a bijection with the property*

$$a \sim b \iff a^\varphi \sim' b^\varphi \text{ for all } a, b \in \mathcal{L}. \tag{7}$$

Then the following assertions hold true:

1. *Under* φ, φ^{-1} *maximal related sets go over to maximal related sets.*

2. *If* φ *maps one star of lines onto a star of lines, then*[3]

   $$\kappa : \mathcal{P} \to \mathcal{P}',\ a \cap b \mapsto a^\varphi \cap b^\varphi\ (a, b \in \mathcal{L},\ a \approx b) \tag{8}$$

   is a collineation.

3. *If* φ *maps one star of lines onto a coplanar maximal related set, then* $(\mathcal{P}',\mathcal{L}')$ *is a 3–dimensional generalized projective space. The mapping*

   $$\delta : \mathcal{P} \to \mathcal{P}'^*,\ a \cap b \mapsto a^\varphi \vee b^\varphi\ (a, b \in \mathcal{L},\ a \approx b) \tag{9}$$

[3] Whenever it is convenient, we do not distinguish between a point X and the set $\{X\}$.

is a collineation onto the dual linear space of $(\mathcal{P}',\mathcal{L}')$. Therefore $(\mathcal{P},\mathcal{L})$ is a 3–dimensional generalized projective space too and δ is a correlation of $(\mathcal{P},\mathcal{L})$ onto $(\mathcal{P}',\mathcal{L}')$.

Proof. Ad 1. This is obviously true.

Ad 2. Suppose that there is a point $A\in\mathcal{P}$ with $(\mathcal{L}(A))^\varphi$ being a star of lines, say $\mathcal{L}'(A')$ where $A'\in\mathcal{P}'$.

Choose any point $B\in\mathcal{P}\setminus\{A\}$. Assume that $(\mathcal{L}(B))^\varphi$ is not a star of lines. By Lemma 1, $(\mathcal{L}(B))^\varphi$ is a coplanar maximal related set. Write $\mathcal{E}'$ for the plane spanned by the lines of $(\mathcal{L}(B))^\varphi$. We infer from $(\mathcal{L}(A)\cap\mathcal{L}(B))^\varphi=(\{A\}\vee\{B\})^\varphi$ that $A'\in\mathcal{E}'$. Since $\dim(\mathcal{P}',\mathcal{L}')\geq 3$, there exists a point $X'\in\mathcal{P}'\setminus\mathcal{E}'$. Choose $x\in\mathcal{L}(A)$ such that $x^\varphi=\{A'\}\vee\{X'\}$. Then $x\neq\{A\}\vee\{B\}$. There is a point $Y\in x\setminus\{A\}$. Put $y:=\{Y\}\vee\{B\}\notin\mathcal{L}(A)$. Therefore $y^\varphi\notin\mathcal{L}'(A')$ and $A'\notin y^\varphi$. Hence

$$x^\varphi\cap y^\varphi=x^\varphi\cap\mathcal{E}'\cap y^\varphi=\{A'\}\cap y^\varphi=\emptyset. \tag{10}$$

This implies $x^\varphi\not\sim' y^\varphi$ in contradiction to $x\sim y$.

Thus we have established that $(\mathcal{L}(B))^\varphi$ is a star of lines. Conversely, given a point $B'\in\mathcal{P}'\setminus\{A'\}$ it follows in the same manner that $(\mathcal{L}'(B'))^{\varphi^{-1}}$ is a star of lines. The previous discussion shows that (8) is a well–defined surjection. If points $Q,R\in\mathcal{P}$ are distinct, then

$$\#(\mathcal{L}(Q)\cap\mathcal{L}(R))=\#((\mathcal{L}(Q))^\varphi\cap(\mathcal{L}(R))^\varphi)=1, \tag{11}$$

whence $Q^\kappa\neq R^\kappa$. Three mutually distinct points $Q,R,S\in\mathcal{P}$ are collinear if, and only if, $\#(\mathcal{L}(Q)\cap\mathcal{L}(R)\cap\mathcal{L}(S))=1$. This in turn is equivalent to the collinearity of $Q^\kappa,R^\kappa,S^\kappa\in\mathcal{P}'$. Hence κ is a collineation.

Ad 3. (a) Suppose that there is a point in $A\in\mathcal{P}$ with $(\mathcal{L}(A))^\varphi$ being a coplanar maximal related set. By the second part of the present proof, φ maps all stars in $(\mathcal{P},\mathcal{L})$ onto coplanar maximal related sets in $(\mathcal{P}',\mathcal{L}')$. We infer from (4) that (9) is a well–defined mapping.

(b) Let $\mathcal{E}'$ be any plane of $(\mathcal{P}',\mathcal{L}')$. There exist adjacent lines $a',b'\in\mathcal{L}'$ with $a'\vee b'=\mathcal{E}'$. By putting

$$\{E\}:=a'^{\varphi^{-1}}\cap b'^{\varphi^{-1}}\text{ and }\mathcal{M}'(a',b'):=(\mathcal{L}(E))^\varphi, \tag{12}$$

we obtain a point in $\mathcal{P}$ and a coplanar maximal related set spanning $\mathcal{E}'$, respectively. We deduce from (9) that $E^\delta=\mathcal{E}'$, whence δ is surjective.

(c) Choose any line $y'\in\mathcal{L}'$. Then there is a line $x\in\mathcal{L}(E)$ with $x\sim y'^{\varphi^{-1}}$. Hence any $y'\in\mathcal{L}'$ is related to at least one line of $\mathcal{M}'(a',b')$.

(d) There exists a point $Y'\in\mathcal{P}'\setminus\mathcal{E}'$. By (c), each line on Y' meets $\mathcal{E}'$ at some point, whence $\dim(\mathcal{P}',\mathcal{L}')=3$.

(e) By (c), each line meets every plane. We infer from (d) that no plane is properly contained in $\mathcal{E}$, whence each line $c'\subset\mathcal{E}'$ can be written as $c'=\mathcal{E}'\cap(c'\vee Y')$. So any two distinct lines of $\mathcal{E}'$ have a common point and hence $(\mathcal{P}',\mathcal{L}')$ is a 3–dimensional generalized projective space.

(f) Let $P,Q\in\mathcal{P}$ be distinct. By (e), $(\mathcal{L}(P))^\varphi$ equals the set of all lines within the generalized projective plane P^δ. The bijectivity of φ forces that

$P^\delta \cap Q^\delta = (\{P\} \vee \{Q\})^\varphi$ is a single line. Thus $P^\delta \neq Q^\delta$. This means that δ is injective.

(g) Three mutually distinct points $Q, R, S \in \mathcal{P}$ are collinear if, and only if, $\#(\mathcal{L}(Q) \cap \mathcal{L}(R) \cap \mathcal{L}(S)) = 1$. This in turn is equivalent to the "collinearity" of $Q^\delta, R^\delta, S^\delta \in \mathcal{P}'^*$ with respect to the dual linear space of $(\mathcal{P}', \mathcal{L}')$. Hence δ is a collineation of $(\mathcal{P}, \mathcal{L})$ onto $(\mathcal{P}'^*, \mathcal{L}'^*)$. Finally, the dual linear space of $(\mathcal{P}', \mathcal{L}')$ is a 3–dimensional generalized projective space too, whence the assertion on $(\mathcal{P}, \mathcal{L})$ follows. □

References

[1] Batten, L.M., Beutelspacher, A.: *The Theory of Finite Linear Spaces*. Cambridge University Press, Cambridge, 1992.

[2] Benz, W.: *Geometrische Transformationen*. B.I. Wissenschaftsverlag, Mannheim Leipzig Wien Zürich, 1992.

[3] Benz, W.: *Real Geometries*. B.I. Wissenschaftsverlag, Mannheim Leipzig Wien Zürich, 1994.

[4] Brauner, H.: *Über die von Kollineationen projektiver Räume induzierten Geradenabbildungen*. Sitz. Ber. österr. Akad. Wiss., math.–naturw. Kl. Abt. II, **197** (1988), 327–332.

[5] Buekenhout, F.: *Foundations of Incidence Geometry*. in: Buekenhout, F. (ed.), Handbook of Incidence Geometry (pp. 63–106), Amsterdam, Elsevier, 1995.

[6] Buekenhout, F., Cameron, P.: *Projective and Affine Geometry over Division Rings*. in: Buekenhout, F. (ed.), Handbook of Incidence Geometry (pp. 27–62), Amsterdam, Elsevier, 1995.

[7] Chow, W.–L.: *On the Geometry of Algebraic Homogeneous Spaces*. Ann. of Math. **50** (1949), 32–67.

[8] Dieudonné, J.A.: *La Géométrie des Groupes Classiques*. 3rd ed., Berlin Heidelberg New York, Springer, 1971.

[9] Havlicek, H.: *On Isomorphisms of Grassmann Spaces*. Mitt. Math. Ges. Hamburg **14** (1995), 117–120.

[10] Havlicek, H.: *Isomorphisms of Affine Plücker Spaces*. Proceedings Congress of Geometry, Thessaloniki 1996, to appear.

[11] Herzer, A.: *Chain geometries*. in: Buekenhout, F. (ed.), Handbook of Incidence Geometry (pp. 781–842), Amsterdam, Elsevier, 1995.

[12] Karzel, H., Sörensen, K., Windelberg, D.: *Einführung in die Geometrie*. Göttingen, Vandenhoek & Ruprecht, 1973.

[13] Lester, J.A.: *Distance Preserving Transformations*. in: Buekenhout, F. (ed.), Handbook of Incidence Geometry (pp. 921–944), Amsterdam, Elsevier, 1995.

Hans Havlicek, Abteilung für Lineare Algebra und Geometrie, Technische Universität, Wiedner Hauptstraße 8–10/1133, A–1040 Wien, Austria
EMAIL: `havlicek@geometrie.tuwien.ac.at`