

On bijections that preserve complementarity of subspaces

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6th May 2004

Abstract

The set \mathcal{G} of all m -dimensional subspaces of a $2m$ -dimensional vector space V is endowed with two relations, complementarity and adjacency. We consider bijections from \mathcal{G} onto \mathcal{G}' , where \mathcal{G}' arises from a $2m'$ -dimensional vector space V' . If such a bijection φ and its inverse leave one of the relations from above invariant, then also the other. In case $m \geq 2$ this yields that φ is induced by a semilinear bijection from V or from the dual space of V onto V' .

As far as possible, we include also the infinite-dimensional case into our considerations.

2000 Mathematics Subject Classification: 51A10, 51A45, 05C60.

Keywords: Distant graph, Grassmann graph, complementary subspaces.

1 Introduction

The purpose of the present article is to characterize the semilinear bijections between vector spaces in terms of their action on certain sets of subspaces. If V is vector space (of finite or infinite dimension) over a (not necessarily commutative) field K then the set of all subspaces X of V that are isomorphic to the quotient space V/X is denoted by \mathcal{G} . We rule out all vector spaces of finite odd dimension, since then \mathcal{G} is empty. It will be convenient to turn \mathcal{G} into an undirected graph with vertex set \mathcal{G} ; two vertices form an edge, whenever they are complements of each other. This gives the *distant graph* on \mathcal{G} (see 2.1). We adopt this name, as this graph is isomorphic to the distant graph of the projective line over the ring $\text{End}_K(U)$, where U is any element of \mathcal{G} . Recall that two points of a projective line over a ring are called *distant* if, and only if, they arise from a basis of the underlying module. We shall also make use of the well known *Grassmann graph* on \mathcal{G} ; cf. 2.3. It has the same set of vertices as the distant graph, but $X, Y \in \mathcal{G}$ comprise an edge if, and only if, they are *adjacent*, i.e., both X and Y have codimension 1 in $X + Y$.

Suppose now that

$$\varphi : \mathcal{G} \rightarrow \mathcal{G}'$$

is an isomorphism of the distant graphs arising from vector spaces V and V' , respectively. So φ preserves complementarity in both directions. It follows that φ is also an isomorphism

of Grassmann graphs; this will be shown in Theorem 4.2 (a), which in turn is based upon a characterization of adjacency in terms of the distant graph (Theorem 3.2). Up to this point the dimension of V plays no role. But now we have to distinguish two cases:

If the dimension of V is finite then the isomorphisms $\mathcal{G} \rightarrow \mathcal{G}'$ of the Grassmann graphs are well known: Apart from two trivial cases ($\dim V = 0, 2$) they arise from semilinear bijections $V \rightarrow V'$ or from semilinear bijections $V^* \rightarrow V'$, where V^* denotes the dual space of V . This result is due to W.-L. CHOW [8]. Hence for $4 \leq \dim V < \infty$ the given isomorphism φ is induced by a semilinear bijection. Also, in the finite dimensional case, every isomorphism $\mathcal{G} \rightarrow \mathcal{G}'$ of Grassmann graphs is an isomorphism of distant graphs (Theorem 4.2 (b)). This leads then to a complete description of the isomorphisms of distant graphs (Theorem 4.4).

If the dimension of V is infinite then so is the dimension of V' , but at this moment it remains open whether or not the isomorphism φ from above is induced by a semilinear bijection. There are two reasons for this: Firstly, we do not know an algebraic description of all isomorphisms of the corresponding Grassmann graphs, secondly, an automorphism of the Grassmann graph on \mathcal{G} need not be an automorphism of the distant graph on \mathcal{G} (Example 4.3).

2 Preliminaries

2.1 Let V be a left vector space over a (not necessarily commutative) field K and denote by \mathcal{G} the set of those subspaces X of V that are isomorphic to the quotient space V/X . Clearly, this condition is equivalent to saying that X is isomorphic to one (and hence all) of its complements with respect to V . We assume that $\mathcal{G} \neq \emptyset$, but there is no other restriction on the dimension of V .

So, if $\dim V$ is finite, then it is an even number $2m$, say, and the elements of \mathcal{G} are just the m -dimensional subspaces of V , whence all elements of \mathcal{G} form an orbit under the action of the general linear group of V .

If $\dim V$ is infinite then \mathcal{G} is non-empty and, as before, it is an orbit under the action of the general linear group: For, if \mathfrak{m} denotes the cardinality of a basis of V then $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}$ shows that $V \times V$ is isomorphic to V . But in $V \times V$ the subspaces $V \times 0$ and $0 \times V$ are complementary and isomorphic, whence $\mathcal{G} \neq \emptyset$. Now let $X, Y \in \mathcal{G}$. So there are subspaces $X_1, Y_1 \in \mathcal{G}$ such that $V = X \oplus X_1 = Y \oplus Y_1$. Then X, X_1, Y , and Y_1 have bases with cardinality \mathfrak{m} , whence they are mutually isomorphic. But the direct sum of any two K -linear bijections $X \rightarrow Y$ and $X_1 \rightarrow Y_1$ is a K -linear bijection $V \rightarrow V$ taking X to Y .

2.2 We say that $X, Y \in \mathcal{G}$ are *distant* (in symbols: $X \triangle Y$) whenever they are complementary, i.e., $X \oplus Y = V$. The *distant graph* on \mathcal{G} is the graph whose vertex set is \mathcal{G} and whose edges are the unordered pairs of distant elements. More generally, a distant graph can be associated with every projective line over a ring [5, p. 108]. The distant graph from above is – up to isomorphism – the distant graph of the projective line over a ring $\text{End}_K(U)$, where $U \in \mathcal{G}$. For a proof we refer to [3, Theorem 2.1]. The particular case that this endomorphism ring is a finite-dimensional K -algebra is treated in [24, 2.3], where distant points are said to be “in clear position”, and in [14, 4.5. Example (4)]. The distant graph is always connected

and its diameter is given in the following table:

$\dim V$	0	2	$4, 6, \dots$	∞
Diameter	0	1	2	3

(1)

This is immediate for $\dim V < \infty$ and follows from [5, Theorem 5.3] when $\dim V = \infty$. The distant graph has no loops except for $\dim V = 0$.

2.3 Two elements $X, Y \in \mathcal{G}$ are called *adjacent* (in symbols: $X \sim Y$) if

$$\dim((X + Y)/X) = \dim((X + Y)/Y) = 1. \quad (2)$$

This terminology goes back to W.-L. CHOW [8] in the finite-dimensional case. Clearly, adjacency is an antireflexive and symmetric relation. The *Grassmann graph* on \mathcal{G} is the graph whose vertex set is \mathcal{G} and whose edges are the 2-sets of adjacent vertices. (References are given in 2.5.) So the graph theoretic adjacency¹ coincides with the relation \sim defined according to (2). By a simple induction, the following formula for the distance of $X, Y \in \mathcal{G}$ in the Grassmann graph can be obtained:

$$\begin{aligned} \text{dist}(X, Y) = d &\Leftrightarrow \dim((X + Y)/X) = \dim((X + Y)/Y) = d \\ &\Leftrightarrow \dim(X/(X \cap Y)) = \dim(Y/(X \cap Y)) = d \end{aligned} \quad (3)$$

If $\dim V = 2m$ is finite then

$$\dim((X + Y)/X) = \dim((X + Y)/Y) < \infty$$

is true for all $X, Y \in \mathcal{G}$, whence the Grassmann graph is connected. Obviously, its diameter is equal to m . We refer also to [14, 4.4].

If $\dim V$ is infinite then there are always subspaces $X, Y \in \mathcal{G}$ such that

$$\dim((X + Y)/X) \neq \dim((X + Y)/Y),$$

even if both dimensions are finite. Let, for example, Y be a hyperplane of $X \in \mathcal{G}$. So there exists a 1-dimensional subspace A such that $X = Y \oplus A$ and there is an $X_1 \in \mathcal{G}$ with $V = X \oplus X_1$. Hence $V = Y \oplus (A \oplus X_1)$. Since $\dim Y = \dim(A \oplus X_1)$, we obtain that $Y \in \mathcal{G}$. Now, obviously,

$$\dim((X + Y)/X) = 0 \neq 1 = \dim((X + Y)/Y).$$

This explains why both the second and the third condition in equation (3) involve two equations. Also, it follows that the Grassmann graph is not connected: In fact, the connected component of $X \in \mathcal{G}$ is formed by all subspaces $Y \in \mathcal{G}$ satisfying (3) for some integer $d \geq 0$. Thus, for example, every complement of X or every element $Y \in \mathcal{G}$ with² $Y < X$ or $Y > X$ is not in this connected component. Moreover, for each $d \geq 0$ there are a d -dimensional subspace $A \leq X$ and a subspace $B \leq X$ such that $A \oplus B = X$. Also, there are a complement Y of X , a d -dimensional subspace $C \leq Y$, and a subspace D such that $Y = C \oplus D$. Then

¹In order to avoid ambiguity we shall refrain from speaking of ‘‘adjacent vertices’’ of the distant graph.

²We use the sign \leq for the inclusion of subspaces and reserve $<$ for strict inclusion.

$C \oplus B \in \mathcal{G}$, since $A \oplus D$ is an isomorphic complement, and the distance of $C \oplus B$ and X is d . So in this case all connected components of the Grassmann graph have infinite diameter.

If we add a loop at each vertex of the Grassmann graph then we obtain a *Plücker space* in the sense of W. BENZ [1, p. 199].

2.4 Let M, N be subspaces of V such that there is a $Y \in \mathcal{G}$ with

$$M \leq Y \leq N \text{ and } \dim(Y/M) = \dim(N/Y) = 1.$$

Then

$$\mathcal{G}[M, N] := \{X \in \mathcal{G} \mid M < X < N\} \quad (4)$$

is called a *pencil* in \mathcal{G} . If $\dim V = 2m$ is finite then $\dim M = m - 1$ and $\dim N = m + 1$, whence $M, N \notin \mathcal{G}$. However, when V is infinite-dimensional, then it follows that $M, N \in \mathcal{G}$. But the strict inclusion signs in (4) guarantee that neither M nor N belongs to $\mathcal{G}[M, N]$. If \mathcal{L} denotes the set of all pencils in \mathcal{G} then $(\mathcal{G}, \mathcal{L})$ is easily seen to be a partial linear space with “point set” \mathcal{G} and “line set” \mathcal{L} . Two elements of \mathcal{G} are adjacent if, and only if, they are distinct “collinear points” of the partial linear space $(\mathcal{G}, \mathcal{L})$. Every “line” is – up to isomorphism – a projective line over K , whence it has $\#K + 1$ “points”. There are three essentially different cases:

$0 \leq \dim V = 2m \leq 2$: Then $(\mathcal{G}, \mathcal{L})$ is an m -dimensional projective space over K .

$2 < \dim V = 2m < \infty$: Then $(\mathcal{G}, \mathcal{L})$ is an example of a *Grassmann space*. It is a connected proper partial linear space; see [2].

$\dim V = \infty$: It seems that in this case the partial linear space $(\mathcal{G}, \mathcal{L})$ has not been discussed in the literature so far. An essential difference to the previous case is that $(\mathcal{G}, \mathcal{L})$ is not connected: Two “points” $X, Y \in \mathcal{G}$ can be joined by a “polygonal path” if, and only if, they are in the same connected component of the Grassmann graph.

2.5 There is a widespread literature dealing with characterizations of (finite) Grassmann graphs and Grassmann spaces which are based upon the set $\mathcal{G}_{m,n}$ of all m -dimensional subspaces of an n -dimensional vector space, $2 \leq m \leq n - 2 < \infty$. We shall not be concerned with such results, but we refer to [2], [7, p. 268–272], [10], [12], [17], [18], [22], [23], [25] and, in addition, to the many other papers which are cited there.

The problem to determine and characterize all isomorphisms (or automorphisms) of Grassmann graphs and Grassmann spaces has been studied by many authors. See [1, Kapitel 5], [6], [8], [9, p. 81], [11] [13], [15], [16], [20], [21], and [26, p. 155].

3 A characterization of adjacency

3.1 In order to show the announced result on isomorphisms of distant graphs we describe the adjacency relation in terms of the distant graph. We shall use the terminology of projective geometry, i.e., the one-, two-, and three-dimensional subspaces of V will be called *points*, *lines*, and *planes*, respectively.

Theorem 3.2 *For all $P, Q \in \mathcal{G}$ the following statements are equivalent:*

(a) P and Q are adjacent.

(b) There is an element $R \in \mathcal{G}$ satisfying the following conditions:

$$R \neq P, Q, \quad (5)$$

$$\forall X \in \mathcal{G} : X \triangle R \Rightarrow X \triangle P \text{ or } X \triangle Q. \quad (6)$$

Proof: (a) \Rightarrow (b): By $P \sim Q$, (3) holds for $d = 1$. So the set $\mathcal{G}[P \cap Q, P + Q]$ is a pencil containing P, Q . This pencil contains an element, say R , which is different from P and Q , since every pencil contains $\#K + 1$ elements.

Let $X \triangle R$. Then $X \cap (P + Q) =: A$ is a point, since $\dim((P + Q)/R) = 1$. We deduce from $P \cap Q < R$ that the point A cannot lie in both P and Q . So, for example,

$$X \cap P = 0, \quad (7)$$

whence $P + Q = A \oplus P$. This gives

$$V = X \oplus R \leq X + (P + Q) = X + (A \oplus P) = (X + A) + P = X + P. \quad (8)$$

By (7) and (8), we have $X \triangle P$, as required.

(b) \Rightarrow (a): We proceed by showing several assertions. Some of them are symmetric with respect to P and Q , so it will be sufficient to treat the assertion for P .

(i) The first step is to establish that if $A, B \leq V$ then

$$0 \neq A \leq P \text{ and } 0 \neq B \leq Q \Rightarrow (A + B) \cap R \neq 0. \quad (9)$$

Assume to the contrary that $(A + B) \cap R = 0$. We infer that there exists a complement X of R containing $A + B$, whence

$$\begin{aligned} 0 \neq A \leq X \cap P &\Rightarrow X \not\triangle P, \\ 0 \neq B \leq X \cap Q &\Rightarrow X \not\triangle Q, \end{aligned}$$

which contradicts (6).

(ii) Next we claim that if $A, B \leq V$ then

$$P \leq A \neq V \text{ and } Q \leq B \neq V \Rightarrow (A \cap B) + R \neq V. \quad (10)$$

Assume to the contrary that $(A \cap B) + R = V$. We infer that there exists a complement X of R contained in $A \cap B$, whence

$$\begin{aligned} X + P \leq A \neq V &\Rightarrow X \not\triangle P, \\ X + Q \leq B \neq V &\Rightarrow X \not\triangle Q, \end{aligned}$$

which contradicts (6). Clearly, if $\dim V$ is finite then (10) follows from (9) by the principle of duality.

(iii) We show that

$$P \cap Q \leq R. \quad (11)$$

This is true for $P \cap Q = 0$. Otherwise choose any point A in $P \cap Q$. By (9), applied to $A = B$, we get $A \cap R \neq 0$. Hence $A \leq R$, and since $A \leq P \cap Q$ can be chosen arbitrarily, $P \cap Q \leq R$.

(iv) We claim that

$$P + Q \geq R. \quad (12)$$

This is true for $P + Q = V$. Otherwise choose any hyperplane A through $P + Q$. By (10), applied to $A = B$, we get $A + R \neq V$. Hence $A \geq R$, and $P + Q \geq R$ since $P + Q$ is equal to the intersection of all such hyperplanes.

(v) Our next assertion is that

$$P \not\leq Q \text{ and } Q \not\leq P. \quad (13)$$

Assume to the contrary that $P \leq Q$ so that $P < R < Q$ follows from (11), (12), and (5). So there is a point $B \leq Q$ with $B \not\leq R$, and there exists a complement X of R containing B . By the law of modularity,

$$(P + X) \cap R = P + (X \cap R) = P + 0 = P \neq R,$$

whence $P + X \neq V$. Consequently, $X \not\leq P$. Furthermore, $0 \neq B \leq X \cap Q$ shows that $X \not\leq Q$. Altogether, this contradicts (6).

(vi) We continue by showing that

$$P \not\leq R \text{ and } Q \not\leq R. \quad (14)$$

Assume to the contrary that $P \leq R$, whence (5) yields $P < R$. We know from (13) that $P \not\leq Q$, whence there is a hyperplane H containing Q with $P \not\leq H$. Thus $V = P + H \leq R + H \leq V$. So there is a complement X of R which lies in H . We have $X + Q \leq H$ and, consequently, $X \not\leq Q$. As in the proof of equation (13), the strict inclusion $P < R$ implies that no complement of R can be a complement of P . This gives $X \not\leq P$ which is absurd by (6).

(vii) Now it is our task to verify that

$$\dim(P/(P \cap R)) = \dim(Q/(Q \cap R)) = 1. \quad (15)$$

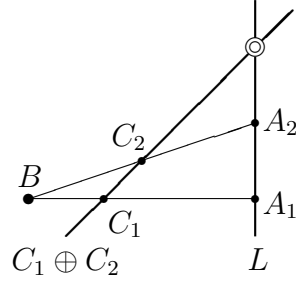
By (14), $P \not\leq R$ so that $P \cap R \neq P$ and $\dim P \geq 1$. So, for $\dim P = 1$, we obtain $\dim(P/(P \cap R)) = 1$. If $\dim P \geq 2$ then it suffices to show that the inequality

$$L \cap (P \cap R) \neq 0$$

holds for all lines $L \leq P$. We deduce from (14) that there is a point $B \leq Q$ with $B \not\leq R$. By (11), $B \not\leq P$ and so for every line $L \leq P$ the subspace $L \oplus B$ is a plane. Let $A_1, A_2 \leq L \leq P$ be distinct points. Each point A_i together with the point $B \leq Q$ meets the requirements of (9). This shows that

$$C_i := (A_i \oplus B) \cap R$$

is a point other than B for $i = 1, 2$. As A_1 and A_2 are different, so are C_1 and C_2 . Hence the subspace $C_1 \oplus C_2 \leq R$ is a line. But L and $C_1 \oplus C_2$ are coplanar, whence they have a common point which lies in $L \cap (P \cap R)$. (See the figure below which illustrates the general case when $A_1 \neq C_1$ and $A_2 \neq C_2$.)



(viii) The penultimate step is to show that

$$\dim((P + Q)/R) = 1. \quad (16)$$

By (15) there are points A, B with

$$\begin{aligned} P &= (P \cap R) \oplus A, \\ Q &= (Q \cap R) \oplus B. \end{aligned}$$

We cannot have $A = B$, since then (11) would give $A = B \leq P \cap Q \leq R$ which is impossible because obviously

$$A, B \not\leq R.$$

So (9) yields that $C := (A \oplus B) \cap R$ is a point. As the point C lies in R , it is different from A and B . Next, it follows that

$$\begin{aligned} P + Q &= (P \cap R) + (Q \cap R) + A + B \\ &= (P \cap R) + (Q \cap R) + C + B && \text{(by the exchange lemma)} \\ &\leq R + C + B && \text{(by } P \cap R \leq R \text{ and } Q \cap R \leq R) \\ &= R \oplus B && \text{(by } C \leq R \text{ and } B \not\leq R) \\ &\leq P + Q && \text{(by (12) and } B \leq Q). \end{aligned}$$

Therefore $P + Q = R \oplus B$.

(ix) Now, finally, we are in a position to show that P and Q are adjacent. This is equivalent, by definition, to

$$\dim((P + Q)/P) = \dim((P + Q)/Q) = 1. \quad (17)$$

We infer from the proof of (16) that there is a point $B \leq Q$ such that $P + Q = R \oplus B$. Denote by D a complement of $P + Q$. (The following is trivial if $P + Q = V$, since then $D = 0$.) So we get

$$V = (P + Q) \oplus D = (R \oplus B) \oplus D,$$

whence $X := D \oplus B$ is a complement of R . But $B \leq X \cap Q$ yields $X \not\leq Q$. Now (6) forces that $X \triangle P$. So, the three formulas

$$\begin{aligned} V = X \oplus P &= D \oplus (P \oplus B), \\ V &= D \oplus (P + Q), \\ P \oplus B &\leq P + Q, \end{aligned}$$

together yield that $P \oplus B = P + Q$, whence $\dim((P + Q)/P) = 1$, as required.

This completes the proof. \square

Let us remark that the adjacency relation can be expressed in terms of the distant graph in a trivial way in the following cases: For $\dim V = 0$ we have $X \sim Y \Leftrightarrow X \not\triangleleft Y$, since both sides are identically false. For $\dim V = 2$ we have $X \sim Y \Leftrightarrow X \triangleleft Y$. For $\dim V = 4$ we have $X \sim Y \Leftrightarrow X \not\triangleleft Y \neq X$.

4 Isomorphisms of distant graphs

4.1 Suppose now that V and V' are left vector spaces over K and K' , respectively. As before, we assume that neither \mathcal{G} nor \mathcal{G}' is empty. Let $f : V \rightarrow V'$ be a semilinear bijection. Such an f induces a bijection $\mathcal{G} \rightarrow \mathcal{G}'$ by $X \mapsto X^f$.

Let V^* denote the dual vector space of V ; we consider V^* as a right vector space over K and we assume now that $\dim V < \infty$. If $f : V^* \rightarrow V'$ is a semilinear bijection (with respect to an antiisomorphism $K \rightarrow K'$) then $X \mapsto (X^\perp)^f$ is a bijection $\mathcal{G} \rightarrow \mathcal{G}'$, since the annihilator map \perp yields a bijection of \mathcal{G} onto \mathcal{G}^* .

In both cases this bijection $\mathcal{G} \rightarrow \mathcal{G}'$ is an isomorphism of distant graphs and an isomorphism of Grassmann graphs. Furthermore, it is a collineation of the partial linear space $(\mathcal{G}, \mathcal{L})$ onto $(\mathcal{G}', \mathcal{L}')$.

Theorem 4.2 *Suppose that V and V' are left vector spaces over K and K' , respectively. Then the following assertions hold:*

- (a) *If $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$ is an isomorphism of distant graphs then it is also an isomorphism of Grassmann graphs.*
- (b) *Suppose, moreover, that $\dim V = 2m$ is finite. If $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$ is an isomorphism of Grassmann graphs then it is also an isomorphism of distant graphs.*

Proof: (a) This is an immediate consequence of the characterization of adjacency in terms of the distant graph given in Theorem 3.2.

(b) Let $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$ be an isomorphism of Grassmann graphs. By $\dim V = 2m$ and 2.3, the Grassmann graph on \mathcal{G} is connected and its diameter is m . By virtue of φ , the Grassmann graph on \mathcal{G}' is also connected and it has the same diameter m . We read off from 2.3 that $\dim V' = 2m$. For all $X, Y \in \mathcal{G}$ we have $X \triangleleft Y$ if, and only if, X and Y are at distance m in the Grassmann graph on \mathcal{G} . Hence X^φ and Y^φ , too, are at distance m in the Grassmann graph on \mathcal{G}' which in turn is equivalent to $X^\varphi \triangleleft Y^\varphi$. \square

We refer to [19] for the logical background of our reasoning in part (a) of the previous proof. For $\dim V = \infty$ the assertion in (b) need not be true:

Example 4.3 Let $\dim V = \infty$ and suppose that $P \triangleleft Q$. Choose points $A \leq P$ and $B \leq Q$. There are subspaces A_1 and B_1 such that

$$P = A \oplus A_1, \quad Q = B \oplus B_1.$$

Then $R := B \oplus A_1 \in \mathcal{G}$, since it is isomorphic and complementary to $A \oplus B_1$. There exists a K -linear bijection $f : V \rightarrow V$ taking P to R . We define the following map:

$$\varphi : \mathcal{G} \rightarrow \mathcal{G} : \begin{cases} X \mapsto X^f & \text{if } \dim(P/(X \cap P)) = \dim(X/(X \cap P)) < \infty \\ X \mapsto X & \text{otherwise} \end{cases}$$

This means that f is applied to all elements of the connected component of P in the Grassmann graph, whereas all other elements of \mathcal{G} remain fixed. As f preserves adjacency and non-adjacency, the connected component of P , which coincides with the connected component of R , is mapped bijectively onto itself. So φ is an automorphism of the Grassmann graph. However, φ is not an automorphism of the distant graph, since $P^\varphi = R \not\triangleleft Q = Q^\varphi$.

Theorem 4.4 *Let V and V' be left vector spaces over K and K' , respectively, where $\dim V = 2m$ is finite. A bijection $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$ is an isomorphism of distant graphs if, and only if, one of the following assertions holds:*

- (a) $\dim V = \dim V' = 0$.
- (b) $\dim V = \dim V' = 2$ and $\#K = \#K'$.
- (c) $4 \leq \dim V = \dim V' = 2m < \infty$ and there is either a semilinear bijection $f : V \rightarrow V'$ such that $X^\varphi = X^f$ or a semilinear bijection $f : V^* \rightarrow V'$ such that $X^\varphi = (X^\perp)^f$.

Proof: Let φ be an isomorphism of distant graphs. By Theorem 4.2, φ is an isomorphism of Grassmann graphs. Furthermore, we see from the proof of Theorem 4.2 (b) that $\dim V = \dim V' = 2m$.

If $\dim V = 2$ then the distant graph on \mathcal{G} is a complete graph with $\#K + 1$ vertices. Hence the same properties are shared by the isomorphic distant graph on \mathcal{G}' . Therefore $\#K = \#K'$.

If $\dim V \geq 4$ then the assertion follows from a theorem due to W.L. CHOW on the isomorphisms of Grassmann graphs. See [8] or [9, p. 81].

The converse is trivially true, if one of the assertions (a) or (b) is satisfied. If (c) holds then φ is an isomorphism of distant graphs according to 4.1. \square

So only the case $\dim V = \dim V' = \infty$ remains open. In view of Theorem 4.2 (a) and Example 4.3, a promising strategy could be as follows: First, describe all isomorphisms of Grassmann graphs and then single out the isomorphisms of distant graphs.

Another problem is as follows: Suppose that $\dim V \geq 4$ and that $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$ is a bijection such that

$$X \triangle Y \Rightarrow X^\varphi \triangle Y^\varphi$$

for all $X, Y \in \mathcal{G}$. Is such a φ an isomorphism of distant graphs? By [4, Theorem 5.1], the answer is affirmative if $\dim V = \dim V' = 4$.

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