The Dual of a Chain Geometry

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We introduce and discuss the dual of a chain geometry. Each chain geometry is canonically isomorphic to its dual. This allows us to show that there are isomorphisms of chain geometries that arise from antiisomorphisms of the underlying rings.
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1 Introduction

For each left module over a ring R there is the dual module. It may be considered as a right R-module or as a left module over the opposite ring R°. A chain geometry $\Sigma(K, R)$ is based upon a proper subfield K of a ring R and the left R-module R^2. Observe that we do not assume that K is in the centre of R. The *dual chain geometry* $\widehat{\Sigma}(K, R)$ of $\Sigma(K, R)$ is defined via the dual module of R^2. Up to notation $\widehat{\Sigma}(K, R)$ is the same as the chain geometry $\Sigma(K^\circ, R^\circ)$. There is a "canonical isomorphism" from each chain geometry onto its dual. However, in general it seems difficult to describe it explicitly for all points in terms of coordinates unless the underlying ring R has some additional properties.

We establish that each residue of a chain geometry can be identified with a residue of its dual in a natural way. However, the (algebraically defined) relation of compatibility given on the set of blocks of each residue is not always preserved under the canonical isomorphism, whence one obtains also a notion of *dual compatibility*.

From [3, Theorem 4.1], the point set of each residue together with one compatibility class of blocks forms a partial affine space which is embeddable in the affine space on the left vector space R over K. This result remains true if "compatibility" and "left vector space" are replaced with "dual compatibility" and "right vector space", respectively. In addition, we give an example of a chain geometry with the following property: The point set of a residue

together with certain blocks of different compatibility classes forms not only a partial affine space but a non-desarguesian affine plane.

Finally, we show that two chain geometries $\Sigma(K, R)$ and $\Sigma(K', R')$ are isomorphic if there is an antiisomorphism $R \to R'$ that takes K onto a subfield of R' which is conjugate to K'. Again, an explicit description in terms of coordinates of such an isomorphism of chain geometries does not seem at hand for arbitrary rings, but we are able to give a formula which allows to calculate the images of all points in the connected component of the point $R(1, 0)$. This generalizes, in part, a result on isomorphisms of chain geometries in [1].

2   PRELIMINARIES

Throughout this paper we shall only consider associative rings with a unit element 1, which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of invertible elements of a ring R will be denoted by R^*. We refer to [6, Chapter II] for the basic properties of free modules.

Consider the free left R-module R^2 and the group $\mathrm{GL}_2(R)$ of invertible 2×2-matrices with entries in R. A pair $(a, b) \in R^2$ is called *admissible*, if there exists a matrix in $\mathrm{GL}_2(R)$ with (a, b) being its first row. The *projective line over R* is the orbit of the free cyclic submodule $R(1, 0)$ under the action of $\mathrm{GL}_2(R)$. In other words, $\mathbb{P}(R)$ is the set of all $p \leq R^2$ such that $p = R(a, b)$ for an admissible pair $(a, b) \in R^2$; compare [10, p. 785]. From [4, Proposition 2.1], in certain cases $R(x, y) \in \mathbb{P}(R)$ does not imply the admissibility of $(x, y) \in R^2$. However, we adopt the convention that points are represented by admissible pairs only. Two admissible pairs represent the same point exactly if they are left-proportional by a unit in R.

Points $p = R(a, b)$ and $q = R(c, d)$ are called *distant* if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$. The vertices of the *distant graph* on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, the edges of this graph are the unordered pairs of distant points. The set $\mathbb{P}(R)$ can be decomposed into *connected components* (maximal connected subsets of the distant graph), for each connected component there is a *distance function* ($\mathrm{dist}(p, q)$ is the minimal number of edges needed to go from vertex p to vertex q). All connected components share a common *diameter* (the supremum of all distances between its points). See [5, Theorem 3.2].

Let $K \subset R$ be a (not necessarily commutative) proper subfield. The projective line over K can be embedded in $\mathbb{P}(R)$ via $K(k, l) \mapsto R(k, l)$. The image of $\mathbb{P}(K)$ under this embedding is a subset $\mathcal{C} \subset \mathbb{P}(R)$ called the *standard chain*. The orbit of $\mathcal{C}$ under the action of $\mathrm{GL}_2(R)$ is denoted by $\mathfrak{C}(K, R)$ and each of its elements is called a *chain*. Altogether the *chain geometry* $\Sigma(K, R)$ is the incidence structure with point set $\mathbb{P}(R)$ and chain set $\mathfrak{C}(K, R)$ [3]. Observe that a chain geometry according to this definition has been called a *generalized chain geometry* in [4] and [5] in order to distinguish from an "ordinary" chain geometry where K is in the centre of R. However, in the present paper such a distinction will not be essential.

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Reversing the multiplication in the ring R yields the *opposite ring* R° and the projective line $\mathbb{P}(R^\circ)$. Further, if K is a proper subfield of R, then the opposite field K° appears as a proper subfield of R° and we obtain the chain geometry $\Sigma(K^\circ, R^\circ)$. The left R°-module $(R^\circ)^2$ can be considered as a *right R-module* in a natural way. It will then be denoted by $\widehat{R}^2$ and its elements will be written as *columns* rather than rows. The right R-module $\widehat{R}^2$ will be identified with the *dual module* of R^2 as usual, i.e., the image of $(a, b) \in R^2$ under $(v, w)^\mathrm{T} \in \widehat{R}^2$ is given by their matrix product. For a subset $U \subset R^2$ we write

$$U^\perp := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \widehat{R}^2 \mid \forall (a, b) \in U : (a, b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0 \right\}. \tag{1}$$

Furthermore, we have

$$(U \cdot M)^\perp = M^{-1} \cdot U^\perp \tag{2}$$

for all $M \in \mathrm{GL}_2(R)$ and all $U \subset R^2$.

By changing from $(R^\circ)^2$ to $\widehat{R}^2$ we obtain the *dual projective line* $\widehat{\mathbb{P}}(R)$ of $\mathbb{P}(R)$ as alternative algebraic description of the projective line $\mathbb{P}(R^\circ)$. So an element of $\widehat{\mathbb{P}}(R)$ has the form $M \cdot (1, 0)^\mathrm{T} R$, with $M \in \mathrm{GL}_2(R)$. Similarly, one obtains $\widehat{\Sigma}(K, R)$, the *dual chain geometry* of $\Sigma(K, R)$. Its set of chains is written as $\widehat{\mathfrak{C}}(K, R)$. Since the module R^2 is free, it can be identified with its bidual module. Up to this identification, the dual of $\widehat{\Sigma}(K, R)$ is again $\Sigma(K, R)$.

Theorem 3.1 *Let $\Sigma(K, R)$ be a chain geometry. Then the mapping*

$$\iota : \mathbb{P}(R) \to \widehat{\mathbb{P}}(R) : p \mapsto p^\perp \tag{3}$$

is an isomorphism of $\Sigma(K, R)$ onto its dual.

Proof: Obviously,

$$(R(1, 0))^\iota = \begin{pmatrix} 0 \\ 1 \end{pmatrix} R, \tag{4}$$

and $R(1, 0)$ is the only ι-preimage of $(0, 1)^\mathrm{T} R$. Each point $p \in \mathbb{P}(R)$ can be written in the form $p = R(1, 0) \cdot M$ with $M \in \mathrm{GL}_2(R)$. So (2) implies that $p^\iota = M^{-1} \cdot (0, 1)^\mathrm{T} R \in \widehat{\mathbb{P}}(R)$ and that ι is bijective. Let

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t \in R. \tag{5}$$

Then $E(t) \in \mathrm{GL}_2(R)$ with

$$E(t)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} = E(0) \cdot E(-t) \cdot E(0). \tag{6}$$

Hence (2) and (4) imply

$$(R(t_1, 1))^\iota = (-1, t_1)^\mathrm{T} R \tag{7}$$

for all $t_1 \in R$. So ι maps the standard chain $\mathcal{C} = \{R(k, 1) \mid k \in K\} \cup \{R(1, 0)\} \in \mathfrak{C}(K, R)$ onto $\{(-1, k)^\mathrm{T} R \mid k \in K\} \cup \{(0, 1)^\mathrm{T} R\}$, which is the standard chain in $\widehat{\Sigma}(K, R)$. From the definition of chains and (2), the mapping ι yields a bijection of $\mathfrak{C}(K, R)$ onto $\widehat{\mathfrak{C}}(K, R)$. $\square$

We refer to ι as the *canonical isomorphism* $\Sigma(K, R) \to \widehat{\Sigma}(K, R)$.

Remark 3.2 Each point $p \in \mathbb{P}(R)$ is spanned by the first row of a matrix $M \in \mathrm{GL}_2(R)$. From (2) and (4) it follows that p^ι is spanned by the second column of M^{-1}. Thus, whenever one has an algorithm to invert matrices of $\mathrm{GL}_2(R)$ then it is also possible to calculate explicitly the ι-image of a point given in that form. For example, when R is commutative then $R(a, b)^\iota = (-b, a)^\mathrm{T} R$ for all admissible pairs $(a, b) \in R^2$.

Remark 3.3 We recall that the *elementary subgroup* $\mathrm{E}_2(R)$ of $\mathrm{GL}_2(R)$ is generated by the set of all matrices (5); cf. [8, p. 5]. Each pair

$$(a, b) := (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1) \tag{8}$$

with $t_1, t_2, \ldots, t_n \in R$ and $n \geq 0$ is admissible. A point of $\mathbb{P}(R)$ is in the connected component of $R(1, 0)$ if, and only if, it has a representative (a, b) of this form; see [5, Theorem 3.2].

Suppose now that $(a, b) \in R^2$ is given according to (8). From (2), (6), and $E(0)^2 = -I$, where I denotes the identity in $\mathrm{GL}_2(R)$, the point $R(a, b)^\iota$ is represented by

$$\begin{pmatrix} v \\ w \end{pmatrix} := (-I)^{n-1} E(0) \cdot E(-t_1) \cdot E(-t_2) \cdots E(-t_n) \cdot E(0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9}$$

Clearly, the irrelevant factor $(-I)^{n-1}$ may be omitted. In particular, this includes formulae (4) and (7) by letting $n = 0$ and $n = 1$, respectively. On the other hand, for $n = 2, 3$ we get from (8) and (9)

$$(R(t_2 t_1 - 1, t_2))^\iota = (-t_2, t_1 t_2 - 1)^\mathrm{T} R, \tag{10}$$

$$(R(t_3 t_2 t_1 - t_3 - t_1, t_3 t_2 - 1))^\iota = (-t_2 t_3 + 1, t_1 t_2 t_3 - t_1 - t_3)^\mathrm{T} R \tag{11}$$

for all $t_1, t_2, t_3 \in R$.

If the connected component of $R(1, 0)$ has finite diameter m then each of its points has a representative of the form (8) with $0 \leq n \leq m$. See [5, formula (10)]. Also, from $E(0)^2 = -I$ and $(1, 0) \cdot E(t) = (1, 0) \cdot E(1) \cdot E(t + 1)$ for all $t \in R$, it is enough to consider products where $n = \max\{2, m\}$.

The explicit formula (10) describes the ι-images of *all* points of $\mathbb{P}(R)$ if R is a ring of *stable rank* 2, since here $\mathbb{P}(R)$ is connected and $m \leq 2$. See [10, Proposition 1.4.2] and [5, Example 5.2 (b)]. We add in passing that the stable rank of R equals the stable rank of its opposite ring R° [14, 2.2]. See Example 5.5 below for an application of formula (11).

4   Compatibility and Dual Compatibility

We consider a chain geometry $\Sigma(K, R)$. For a fixed point $p \in \mathbb{P}(R)$ the set $\mathbb{P}(R)_p$ consists of all points distant from p, and $\mathfrak{C}(K, R)_p$ consists of all sets $\mathcal{D} \setminus \{p\}$, where $\mathcal{D}$ is a chain through p. An element of $\mathfrak{C}(K, R)_p$ will be called a *block*. Altogether the *residue of $\Sigma(K, R)$ at p* is the incidence structure

$$\Sigma(K, R)_p = (\mathbb{P}(R)_p, \mathfrak{C}(K, R)_p). \tag{12}$$

Cf. [3, Section 4].

Let $\infty := R(1,0)$. The chains $\mathcal{D}_1, \mathcal{D}_2$ through ∞ are called *compatible at* ∞ if they belong to the same orbit under the action of the group

$$\Delta := \{ \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \mid a \in R^*, c \in R \} \subset \mathrm{GL}_2(R) \tag{13}$$

on $\mathfrak{C}(K,R)$. Then also the blocks $\mathcal{D}_1 \setminus \{\infty\}$ and $\mathcal{D}_2 \setminus \{\infty\}$ of $\Sigma(K,R)_\infty$ will be called compatible. By definition, the compatibility of chains (at a common point) is a $\mathrm{GL}_2(R)$-invariant notion (see [3, Section 3]), whence one has a compatibility relation on the set of blocks of each residue $\Sigma(K,R)_p$.

It suffices to consider the case where $p = \infty$. A point $R(a,b)$ is distant from ∞ exactly if b is a unit in R. The bijection

$$\mathbb{P}(R)_\infty \to R : R(x,1) \mapsto x \tag{14}$$

will be used to identify $\mathbb{P}(R)_\infty$ with R. By [3, Theorem 4.1], a subset of $\mathfrak{C}(K,R)_\infty$ is a *compatibility class* exactly if it has the form

$$\{ (u^{-1}Ku)a + c \mid a \in R^*, \ c \in R \} \text{ with } u \in R^*. \tag{15}$$

Recall that a *partial affine space* is an incidence structure resulting from an affine space by removing certain parallel classes of lines (but no points). If K^* is not normal in R^* then the residue $\Sigma(K,R)_\infty$ cannot be embedded in any affine space, since the points $0, 1 \in R$ are joined by more than one block, namely by all subfields $u^{-1}Ku$, where $u \in R^*$. However, the point set $\mathbb{P}(R)_\infty$ together with one compatibility class (15) forms a partial affine space which extends to the affine space $\mathbb{A}(u^{-1}Ku, R)$ on the *left vector space* R over $u^{-1}Ku$; see [3, Theorem 4.2].

The construction described above can be carried over to the dual chain geometry $\widehat{\Sigma}(K,R)$. We restrict ourselves to the residue of $\widehat{\Sigma}(K,R)$ at $\infty^\iota = (0,1)^\mathrm{T} R$, where ι is the canonical isomorphism. The counterpart of (14) is the bijection

$$\widehat{\mathbb{P}}(R)_{\infty^\iota} \to R : \begin{pmatrix} -1 \\ x \end{pmatrix} R \mapsto x. \tag{16}$$

Two chains of $\widehat{\Sigma}(K,R)$ through $(1,0)^\mathrm{T} R$ are compatible at $(1,0)^\mathrm{T} R$, if they belong to the same orbit with respect to the group $\Delta^\mathrm{T} := \{ D^\mathrm{T} \mid D \in \Delta \}$ acting on $\widehat{\mathbb{P}}(R)$ from the left; cf. (13). So the compatibility in $\widehat{\Sigma}(K,R)_{\infty^\iota}$ is governed by the group

$$\widehat{\Delta} := M \cdot \Delta^\mathrm{T} \cdot M^{-1} = \{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \mid c \in R, d \in R^* \} \subset \mathrm{GL}_2(R), \tag{17}$$

where $M \in \mathrm{GL}_2(R)$ is any matrix taking $(1,0)^\mathrm{T} R$ to $\infty^\iota = (0,1)^\mathrm{T} R$. The partial affine spaces defined by the compatibility classes in $\widehat{\Sigma}(K,R)_{\infty^\iota}$ are embedded in affine spaces $\widehat{\mathbb{A}}(uKu^{-1}, R)$, where R is considered as *right vector space* over uKu^{-1}.

Let $\mathcal{D}_1, \mathcal{D}_2$ be chains of $\Sigma(K,R)$ with common point p. We say that $\mathcal{D}_1, \mathcal{D}_2$ are *dually compatible at p* if, and only if, $\mathcal{D}_1^\iota, \mathcal{D}_2^\iota \in \widehat{\mathfrak{C}}(K,R)$ are compatible at p^ι. Analogously, we define dual compatibility of blocks of a residue.

Theorem 4.1 *Suppose that $\mathbb{P}(R)_\infty$ and $\widehat{\mathbb{P}}(R)_{\infty^\iota}$ are identified with the ring R according to (14) and (16), respectively. Then the following holds:*

(a) *Each point of $\mathbb{P}(R)_\infty$ and its ι-image are the same.*

(b) *The residue of $\Sigma(K, R)$ at ∞ coincides with the residue of $\widehat{\Sigma}(K, R)$ at ∞^ι.*

(c) *The equivalence relations of "compatibility" and "dual compatibility" on the set of blocks are the same exactly if the multiplicative group K^* is normal in the multiplicative group R^*.*

Proof: (a) This is obviously true.

(b) In both residues the blocks are exactly the sets $dKa + c$ with $a, d \in R^*$ and $c \in R$.

(c) Suppose that K^* is not normal in R^*. Then there is a $u \in R^*$ with $uK \neq Ku$. The compatibility class of the block K contains exactly the blocks $Ka + c$ with $a \in R^*$ and $c \in R$. The only block of this class running through 0 and u is Ku. We read off from $0, u \in uK \neq Ku$ that the block uK is not compatible to K. However, the dual compatibility class of K contains uK. Therefore the relations are different. On the other hand, if K^* is normal in R^* then all blocks are compatible and dually compatible; see [3, Theorem 4.2]. This completes the proof. $\square$

Remark 4.2 From Theorem 4.1(c) and formula (2) we obtain the following: The canonical isomorphism $\iota : \Sigma(K, R) \to \widehat{\Sigma}(K, R)$ preserves compatibility (at all points) if, and only if, K^* is normal in R^*. In particular, this shows that the notion of compatibility needs not be invariant under isomorphisms of chain geometries.

Let S_∞ be the set of all $\mathfrak{L} \subset \mathfrak{C}(K, R)_\infty$ such that $(\mathbb{P}(R)_\infty, \mathfrak{L})$ is a partial affine space. We have seen before that each (dual) compatibility class of blocks belongs to S_∞. From [3, Lemma 2.1] and [3, Proposition 2.2], two distinct points $R(x, 1), R(y, 1)$ of the residue are joined by at least one block exactly if they are distant. This is equivalent to $y - x \in R^*$. From (15), the set of blocks through two distant points of $\mathbb{P}(R)_\infty$ has exactly one element in common with each (dual) compatibility class. This means that each (dual) compatibility class is a maximal element of S_∞ with respect to inclusion. One could conjecture that the maximal elements of S_∞ were exactly the (dual) compatibility classes. However, there may also be other maximal elements of S_∞:

Example 4.3 Let $R = \mathbb{H}$ be the field of real quaternions with the usual $\mathbb{R}$-basis $\{1, i, j, k\}$. Further, let $K = \mathbb{C} = \mathbb{R} + \mathbb{R}i$ be a subfield of complex numbers. The blocks of $\mathfrak{C}(\mathbb{C}, \mathbb{H})_\infty$ compatible to $\mathbb{C}$ are exactly the lines of the complex affine plane $\mathbb{A}(\mathbb{C}, \mathbb{H})$. Put

$$\mathcal{B} := \{a(\mathbb{R} + \mathbb{R}j) + c \mid a \in \mathbb{C}^*, \; c \in \mathbb{H}\}. \tag{18}$$

Each element of $\mathcal{B}$ is a block, since $\mathbb{R} + \mathbb{R}j = (1 + k)^{-1}\mathbb{C}(1 + k)$, but not a line of $\mathbb{A}(\mathbb{C}, \mathbb{H})$. Obviously, the elements of $\mathcal{B}$ are Baer subplanes of $\mathbb{A}(\mathbb{C}, \mathbb{H})$. We apply the well known procedure of *derivation*: All lines of $\mathbb{A}(\mathbb{C}, \mathbb{H})$ that are parallel to a line of the Baer subplane $\mathbb{R} + \mathbb{R}j$ are removed and instead the Baer subplanes belonging to $\mathcal{B}$ are introduced as "new lines". This gives a (non-desarguesian) affine plane with point set $\mathbb{H}$. Cf. [11, Theorem 3.14].

By construction, the set of lines of the derived plane is a maximal element of S_∞, but it is neither a compatibility class nor a dual compatibility class.

A reader who is familiar with *elliptic geometry* will easily verify the following: The lines of the projective 3-space $\mathbb{P}(\mathbb{R}, \mathbb{H})$ (which carries the structure of an *elliptic space* coming from the Euclidean norm on quaternions) are exactly the blocks of $\mathfrak{C}(\mathbb{C}, \mathbb{H})_\infty$ through $0 \in \mathbb{H}$. Two blocks B_1, B_2 through 0 are compatible exactly if there is a *right Clifford translation* ($\mathbb{R}x \mapsto \mathbb{R}xa$, $a \in \mathbb{H}^*$) of $\mathbb{P}(\mathbb{R}, \mathbb{H})$ taking B_1 to B_2. This characterizes the lines B_1 and B_2 as *left Clifford parallel*. Similarly, dual compatibility corresponds to *right* Clifford parallelism. The set of blocks through 0 that are compatible to $\mathbb{C}$ appears as a regular spread (elliptic linear congruence of lines) of the elliptic space (fig. 1). All lines of this spread are left parallel. See, among others, [9, Chapter VII] and [12, p. 76]. It is well known that here the process of derivation means that one regulus $\mathcal{R}$ of this spread is replaced with its opposite regulus $\mathcal{R}^\circ$ (fig. 2); see, e.g., [7, p. 101–102]. The lines of $\mathcal{R}^\circ$ are mutually right Clifford parallel, since $\mathcal{B}$ is a subset of a dual compatibility class.
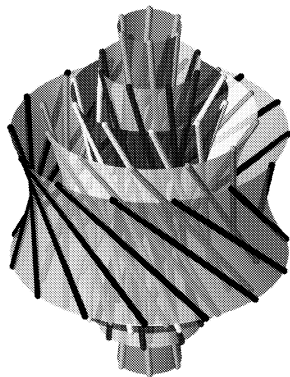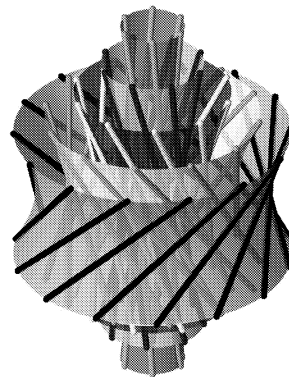

Fig. 1.   Fig. 2.

5   ISOMORPHISMS

In this section we consider two chain geometries $\Sigma(K, R)$ and $\Sigma(K', R')$.

Remark 5.1 For each $u' \in R'^*$ we have $\Sigma(K', R') = \Sigma(u'^{-1}K'u', R')$ by virtue of the transformation of $\mathbb{P}(R')$ given by the matrix $\mathrm{diag}(u', u')$. So, if $\varphi : R \to R'$ is an isomorphism of rings such that $K^\varphi = u'^{-1}K'u'$ holds for a suitable $u' \in R'^*$, then

$$\overline{\varphi} : \mathbb{P}(R) \to \mathbb{P}(R') : R(a, b) \mapsto R'(a^\varphi, b^\varphi) \tag{19}$$

is an isomorphism $\Sigma(K, R) \to \Sigma(K', R')$ mapping $\infty = R(1, 0)$ to $\infty' := R'(1', 0')$.

The restriction of $\overline{\varphi}$ to $\mathbb{P}(R)_\infty$ is an isomorphism from the residue $\Sigma(K, R)_\infty$ onto $\Sigma(K', R')_{\infty'}$. According to the identification (14), applied to $\mathbb{P}(R)_\infty$ and $\mathbb{P}(R')_{\infty'}$, this restriction coincides with φ. Using (15), one sees that φ preserves compatibility of blocks. The same holds for the restriction of $\overline{\varphi}$ to any other residue, because the actions of $\mathrm{GL}_2(R)$ and $\mathrm{GL}_2(R')$ on $\mathbb{P}(R)$ and $\mathbb{P}(R')$, respectively, are isomorphic via $\overline{\varphi}$. Altogether, $\overline{\varphi}$ preserves compatibility of chains.

We now study the case of antiisomorphisms.

Theorem 5.2 *Let $\varphi : R \to R'$ be an antiisomorphism of rings such that $K^\varphi = u'^{-1}K'u'$ for some $u' \in R'^*$. Then the product of the canonical isomorphism $\iota : \mathbb{P}(R) \to \widehat{\mathbb{P}}(R)$ and the mapping*

$$\widehat{\varphi} : \widehat{\mathbb{P}}(R) \to \mathbb{P}(R') : \binom{v}{w}R \mapsto R'(v^\varphi, w^\varphi). \tag{20}$$

is an isomorphism of $\Sigma(K, R)$ onto $\Sigma(K', R')$.

Proof: The antiisomorphism $\varphi : R \to R'$ is an isomorphism $R^\circ \to R'$. So, from Remark 5.1, the mapping $\widehat{\varphi}$ is an isomorphism of $\widehat{\Sigma}(K, R)$ onto $\Sigma(K', R')$, whence $\iota\widehat{\varphi}$ has the required properties. $\square$

Remark 5.3 We conclude from Remark 4.2 and Remark 5.1 that the isomorphism $\iota\widehat{\varphi}$ preserves compatibility if, and only if, K^* is normal in R^*.

Theorem 5.2 does not give an explicit description of the isomorphism $\iota\widehat{\varphi}$ from $\Sigma(K, R)$ onto $\Sigma(K', R')$, since we did not describe the canonical isomorphism ι explicitly either. As in Remark 3.3, we know more for certain points:

Remark 5.4 Let $\varphi : R \to R'$ be given as in Theorem 5.2. For $M \in \mathrm{GL}_2(R)$ let M^φ be the matrix in $\mathrm{GL}_2(R')$ obtained by applying φ to the entries of M. We observe that $M \mapsto (M^{\mathrm{T}})^\varphi$ is an antiisomorphism of groups. Also, for each point $q \in \widehat{\mathbb{P}}(R)$ and each matrix $M \in \mathrm{GL}_2(R)$ we have

$$(M \cdot q)^{\widehat{\varphi}} = q^{\widehat{\varphi}} \cdot (M^{\mathrm{T}})^\varphi. \tag{21}$$

The product $\iota\widehat{\varphi}$ is an isomorphism of $\Sigma(K, R)$ onto $\Sigma(K', R')$. However, by (4) and (20), it takes $R(1,0)$ to $R'(0', 1')$ rather than to $R'(1', 0')$. So let $\eta : R'(a', b') \mapsto R'(b', -a')$ be the transformation of $\mathbb{P}(R')$ induced by $E(0')^{-1}$. We focus our attention to the isomorphism

$$\sigma := \iota\widehat{\varphi}\eta \tag{22}$$

of $\Sigma(K, R)$ onto $\Sigma(K', R')$. By construction,

$$(R(1, 0))^\sigma = R'(1', 0'). \tag{23}$$

We aim at an explicit computation of the σ-images of all points in the connected component of $R(1, 0)$: Let $p = R(a, b)$ with (a, b) as in (8), whence $p^\iota = (v, w)^{\mathrm{T}}R$ with $(v, w)^{\mathrm{T}}$ as in (9). Using (21) and $E(-t)^{\mathrm{T}} = (-I) \cdot E(t)$ we obtain from (9) and (22) that

$$p^\sigma = R'(1', 0') \cdot E(t_n^\varphi) \cdot E(t_{n-1}^\varphi) \cdots E(t_1^\varphi). \tag{24}$$

In particular, we have

$$
\begin{align}
(R(t_1, 1))^\sigma &= R'(t_1^\varphi, 1'), \tag{25}\\
(R(t_2 t_1 - 1, t_2))^\sigma &= R'(t_2^\varphi t_1^\varphi - 1', t_2^\varphi), \tag{26}\\
R((t_3 t_2 t_1 - t_3 - t_1, t_3 t_2 - 1))^\sigma &= R'(t_3^\varphi t_2^\varphi t_1^\varphi - t_3^\varphi - t_1^\varphi, t_3^\varphi t_2^\varphi - 1') \tag{27}
\end{align}
$$

for all $t_1, t_2, t_3 \in R$, as counterparts of formulae (7), (10), and (11).

From [1, Theorem 2.4], for rings of stable rank 2 an isomorphism of $\Sigma(K, R)$ onto $\Sigma(K', R')$ can be *defined* according to (26), even when $\varphi : R \to R'$ is a *Jordan isomorphism* satisfying certain conditions on the image of K. See also [2], [10, 9.1], and [13] for related results.

Example 5.5 Let $R = \mathrm{End}_K(V)$ be the endomorphism ring of an infinite dimensional vector space V over a commutative field K. For each $a \in R$ the transpose mapping a^{T} is an endomorphism of the dual vector space. We put $R' := \{a^{\mathrm{T}} \mid a \in R\}$ so that $\varphi : R \to R' : a \mapsto a^{\mathrm{T}}$ is an antiisomorphism of rings. The field $K =: K'$ can be embedded in R via $k \mapsto k \cdot \mathrm{id}_V$ and in R' via $k \mapsto k \cdot (\mathrm{id}_V)^{\mathrm{T}}$ ($k \in K$), whence $K^{\varphi} = K'$. Then an isomorphism σ of the corresponding chain geometries is given by (20) and (22). From [5, Theorem 5.3], the projective line $\mathbb{P}(R)$ is connected and its diameter equals 3. So formula (27) can be used to calculate the σ-images of *all* points. Further, the canonical isomorphism $\Sigma(K, R) \to \widehat{\Sigma}(K, R)$ is given by (11).

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