DUAL SPREADS GENERATED BY COLLINEATIONS

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Dedicated to Walter Wunderlich on his 80th birthday

The present paper establishes in particular a relationship between certain dual spreads which are not spreads and collineations with an invariant line but without invariant points of a desarguesian projective plane.

1. Introduction

Suppose that we are given two different planes in a 3-dimensional projective space and a collineation of the first onto the second plane leaving invariant their common line without fixing any point. Joining points corresponding under this collineation yields a *dual spread generated by a collineation*. This construction is well known from classical geometry over the real numbers and has also been discussed e.g. in finite projective spaces. In either case such a dual spread is even a spread. But this result *does not carry over* to the general case, as will be illustrated by several examples which are based upon the following result: There exists a collineation which generates a dual spread that is not a spread if, and only if, for a plane of the given projective space there is a collineation which has an invariant line but lacks to have invariant points. Finally, it is shown that any dual spread generated by a collineation determines a translation plane which is also a dual translation plane. Necessary and sufficient conditions for this plane to be desarguesian or pappian are stated.

2. Dual Spreads

Throughout this section let $(\mathcal{P},\ell)$ be a 3-dimensional projective space. We assume that the reader is familiar with the definitions of *spread*, *dual spread*, *regular spread* and *partial spread*; cf. e.g. [2,86-87], [3,163], [4,801]. AB denotes the line joining different points A and B. The term *field* is used for a *not necessarily commutative field*.

2.1. Main Results

THEOREM 1. *Let $\mathcal{B}_0$, $\mathcal{B}_1$ be two different planes of $\mathcal{P}$, $s := \mathcal{B}_0 \cap \mathcal{B}_1$ and suppose that $\kappa : \mathcal{B}_0 \to \mathcal{B}_1$ is a collineation such that $s^\kappa = s$ and $\kappa|s$ has no invariant points. Then a dual spread is determined by*

$$\{XX^\kappa \mid X \in \mathcal{B}_0\}. \tag{1}$$

We refer to the dual spread (1) as being *generated by the collineation* κ.

THEOREM 2. *There exists a collineation which generates a dual spread that is not a spread if, and only if, for a plane of $\mathcal{P}$ there is a collineation which has an invariant line but lacks to have invariant points.*

THEOREM 3. *If $\mathcal{P}$ is pappian and if $(\kappa|s)^m$ is a projectivity for some $m \in \{1,2,3,\dots\}$, then the dual spread* (1) *is a spread.*

COROLLARY 1. *There exist dual spreads generated by collineations which are not spreads.*

2.2. Proofs

Proof of Theorem 1. If $X \in \mathcal{B}_0 \setminus s$, then XX^κ is skew to s. Let $Y \in \mathcal{B}_0 \setminus s$ be a point other than X. Suppose that XX^κ and YY^κ have a point in common. Hence X, X^κ, Y, Y^κ are incident with a plane $\mathcal{F}$, say, and $\mathcal{F}\cap s$ is a κ-invariant point, a contradiction. Thus (1) is a partial spread.

Putting $E := (\mathcal{E}\cap\mathcal{B}_0)\cap(\mathcal{E}\cap\mathcal{B}_1)^{\kappa^{-1}}$ for any plane $\mathcal{E} \not\ni s$ shows that the line EE^{κ} of (1) is contained in $\mathcal{E}$. □

Proof of Theorem 2. Choose s, $\mathcal{B}_0$, $\mathcal{B}_1$ subject to conditions in Theorem 1 and fix any point $Z \in \mathcal{P}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)$. Each $P \in \mathcal{P}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)$ defines a collineation $\omega(P) : \mathcal{B}_0 \to \mathcal{B}_1$, $X \mapsto (XP)\cap\mathcal{B}_1$. Denote by σ a collineation of $\mathcal{B}_0$ with invariant line s but no invariant points and put $\kappa := \sigma\omega(Z)$. If $P \in \mathcal{P}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)$, then

$$\pi(P) := \sigma\omega(Z)\omega(P)^{-1} = \kappa\omega(P)^{-1} \qquad (2)$$

is a collineation of $\mathcal{B}_0$ and $\pi(P)|s = \kappa|s = \sigma|s$. Furthermore $F = F^{\pi(P)}$ is equivalent to $FF^{\kappa} \ni P$. Thus $\pi(Z) = \sigma$ implies that no element of the dual spread (1) is incident with Z.

Conversely, any dual spread (1) which is not a spread gives rise to at least one collineation (2) with an invariant line but no invariant points. □

Proof of Theorem 3. We shall make use of the following result: In an n-dimensional desarguesian projective space ($2\leq n<\infty$) let σ be a collineation with an invariant hyperplane $\mathcal{H}$. Denote by K an underlying field. So $\sigma|(\mathcal{P}\setminus\mathcal{H})$, regarded as an affinity, is described, up to a translation, by a map of $\Gamma L(n,K)$ with companion automorphism $\beta \in \mathrm{Aut}(K)$. If β is of finite order, then at least one point is fixed under σ by the first part of a proof in [17,377].

Returning to the settings of Theorem 3 and Theorem 1, each $P \in \mathcal{B}_0\cup\mathcal{B}_1$ is on a line of the dual spread (1). If $P \in \mathcal{P}\setminus(\mathcal{B}_0\cup\mathcal{B}_1)$, then $\kappa|s = \pi(P)|s$ by (2), whence $\pi(P)^m$ is projective. The remarks given above and the commutativity of an underlying field K, say, establish that $\pi(P)|(\mathcal{B}_0\setminus s)$ corresponds to $\beta \in \mathrm{Aut}(K)$ of finite order. Thus $\pi(P)$ has an invariant point off s or, equivalently, a line of (1) is incident with P. □

Proof of Corrollary 1. There is a 3×3 matrix (with entries in a certain non-commutative field) which has a right eigenvalue but no left eigenvalues [6,155], [7,206]. This implies the existence of a *projective collineation* σ with an invariant line but without invariant points. Cf. also Example 5 in [17]. By Theorem 3 in a pappian projective plane such a σ never is projective. On the other hand, Example 1 in [17] establishes that in some pappian projective plane there exists a *non-projective collineation* σ fitting for our purposes. Applying Theorem 2 completes the proof. □

2.3. Comments

Clearly Theorem 1 has a dual counterpart which involves a collineation of two different stars of planes and yields a spread by intersecting corresponding planes.

If $\mathcal{P}$ is pappian and if κ is a projective collineation, then, by [1,186], [12,53], (1) is a *regular spread* or, equivalently, an *elliptic linear congruence of lines*. Cf. [19,69-75] for references on earlier papers. Conversely, assume that we are given a regular spread of $\mathcal{P}$. By [3,163], [11,136] or [20,319], $\mathcal{P}$ is pappian. In [1,189-190] it is shown that any *dual elliptic linear congruence of lines* in a 3-dimensional pappian projective space permits a representation (1) with κ being projective. The proof given there only makes use of the fact that such a congruence is a regular spread. Thus this result remains true for any regular spread.

If (1) contains at least one regulus, then, as above, $\mathcal{P}$ is pappian. Furthermore κ is projective; cf. [1,176], [1,181] and the construction of *aregular spreads* in [9], [12,64]. Thus now (1) is a regular spread.

With $(\mathcal{P},\ell)$ = PG$(3,q)$, q being finite, any dual spread has q^2+1 elements, whence it is a spread; cf. Theorem 3. By [5] in a 3-dimensional projective space of infinite order the concepts of spread and dual spread need not coincide. Corollary 1 provides some more examples.

In [10] a definition of linear congruences of lines is given for any 3-dimensional projective space. Theorem 1 improves a result on linear congruences of type (iii) in that paper: Any such congruence is a dual spread.

3. The Corresponding Translation Planes

At first we repeat the construction given in Theorem 1 in terms of a 4-dimensional left vector space $\mathfrak{V}$ over a field K, whence $\mathfrak{V}^* = \mathrm{Hom}_K(\mathfrak{V},K)$ is a right vector space over K. The centre of K will be denoted by $Z(K)$. With $\mathfrak{U} \subset \mathfrak{V}$, write $\mathfrak{U}^\perp := \{\mathfrak{x}^* \in \mathfrak{V}^* \mid \langle\mathfrak{u},\mathfrak{x}^*\rangle = 0, \text{ for all } \mathfrak{u}\in\mathfrak{U}\}$.

Denote by $\mathfrak{B}_0$, $\mathfrak{B}_1$ two different hyperplanes of $\mathfrak{V}$. Let $\varphi : \mathfrak{B}_0 \to \mathfrak{B}_1$ be a bijective semilinear map with $(\mathfrak{B}_0\cap\mathfrak{B}_1)^\varphi = \mathfrak{B}_0\cap\mathfrak{B}_1$ and the property that $\mathfrak{x}$, $\mathfrak{x}^\varphi$ are linearly independent for all $\mathfrak{x} \in (\mathfrak{B}_0\cap\mathfrak{B}_1)\setminus\{\mathfrak{o}\}$. As we are only interested in the collineation induced by φ, we may request that the companion automorphism of φ is $\alpha \in \mathrm{Aut}(K)$, say, such that

$$\alpha = \mathrm{id}_K \text{ or } \alpha \text{ is outer automorphism.} \tag{3}$$

Choose a non-zero vector $\mathfrak{p}_0 \in \mathfrak{B}_0\cap\mathfrak{B}_1$ and set $\mathfrak{p}_1 := \mathfrak{p}_0{}^\varphi$. Hence $\mathfrak{p}_1{}^\varphi = a\mathfrak{p}_0 + b\mathfrak{p}_1$ with a, $b \in K$, $a \neq 0$. Next take any vector $\mathfrak{p}_2 \in \mathfrak{B}_0\setminus\mathfrak{B}_1$. Putting $\mathfrak{p}_3 := \mathfrak{p}_2{}^\varphi \in \mathfrak{B}_1$ yields a basis $\{\mathfrak{p}_0,\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$ of $\mathfrak{V}$ whose dual basis is written as $\{\mathfrak{p}_0{}^*,\mathfrak{p}_1{}^*,\mathfrak{p}_2{}^*,\mathfrak{p}_3{}^*\}$. We deduce from $\mathfrak{x}$, $\mathfrak{x}^\varphi$ linearly independent for each $\mathfrak{x} \in (\mathfrak{B}_0\cap\mathfrak{B}_1)\setminus\{\mathfrak{o}\}$ that any matrix

$$\begin{pmatrix} u_0 & u_1 \\ u_1{}^\alpha a & u_0{}^\alpha + u_1{}^\alpha b \end{pmatrix}, \quad (u_0,u_1) \in K\times K, \tag{4}$$

has left row rank 2 provided that $(0,0) \neq (u_0,u_1)$. This is equivalent, by elementary transformations and $u_0 := 1$, $u_1 := -a^{-1}x$, to

$$x^\alpha x + x^\alpha b - a^\alpha \neq 0 \text{ for all } x \in K. \tag{5}$$

Thus $\mathcal{P}$, $\mathcal{B}_0$, $\mathcal{B}_1$, κ, as have been introduced in section 2, now are determined via $\mathfrak{V}$, $\mathfrak{B}_0$, $\mathfrak{B}_1$, φ, respectively. On the other hand it is easily seen that the existence of a, $b \in K$ and $\alpha \in \mathrm{Aut}(K)$ such that (5) holds implies the existence of a semilinear map φ with the required properties. If α is a non-trivial inner automorphism, then a, b, α can be replaced by a', b', $\alpha'=\mathrm{id}_K$ in order to satisfy (3).

Set $\mathfrak{S}^* := (\mathfrak{B}_0 \cap \mathfrak{B}_1)^\perp$; given $u_0, u_1 \in K$ then write

$$\mathfrak{S}^*(u_0,u_1) := \{u_0\mathfrak{p}_0+u_1\mathfrak{p}_1-\mathfrak{p}_2, (u_0\mathfrak{p}_0+u_1\mathfrak{p}_1-\mathfrak{p}_2)^\varphi\}^\perp$$

and denote by $\lambda(u_0,u_1) : \mathfrak{S}^*(0,0) \to \mathfrak{S}^*$ the linear map whose matrix with respect to ordered bases $(\mathfrak{p}_0^*, \mathfrak{p}_1^*)$ and $(\mathfrak{p}_2^*, \mathfrak{p}_3^*)$ equals (4). Hence

$$\mathfrak{S}^*(u_0,u_1) = \{\mathfrak{s}^* \oplus \mathfrak{s}^{*\lambda(u_0,u_1)} \mid \mathfrak{s}^* \in \mathfrak{S}^*(0,0)\}.$$

Apart from notational differences this is the description of a spread given in [2,90-93], [3,154-158] and e.g. [16,7-10]. As $\{\mathfrak{S}^*(u_0,u_1) \mid u_0,u_1 \in K\} \cup \{\mathfrak{S}^*\}$ is a spread of $\mathfrak{B}^*$, we obtain a *translation plane* $\mathcal{T}$; cf. e.g. [16,2]. Let D be a 2-dimensional right vector space over K with basis elements 1 and d; assign to $\mathfrak{p}_0^*u_0+\mathfrak{p}_1^*u_1$, $\mathfrak{p}_2^*u_0+\mathfrak{p}_3^*u_1$ the element $u_0+du_1 \in D$. Note that $\lambda\left(m_0,(m_1a^{-1})^{\alpha^{-1}}\right)$ takes $\mathfrak{p}_0^*$ to $\mathfrak{p}_2^*m_0+\mathfrak{p}_3^*m_1$. The image of $\mathfrak{p}_0^*x_0+\mathfrak{p}_1^*x_1$ under this map yields the multiplication rule

$$\begin{aligned}(m_0+dm_1)\circ(x_0+dx_1) := \; & m_0x_0+(m_1a^{-1})^{\alpha^{-1}}x_1 + \\ & + d(m_1x_0+m_0^{\alpha}x_1+m_1a^{-1}bx_1)\end{aligned} \tag{6}$$

making $(D,+,\circ)$ a *left quasifield* coordinatizing $\mathcal{T}$. It is immediate from (6) that D satisfies the right distributive law. So D is a *division ring* (semifield, distributive quasifield) and $\mathcal{T}$ is also a *dual translation plane*. The subfield $S := \{x+d0 \mid x \in K\}$ of D is isomorphic to K. We shall identify K and S via $x \equiv x+d0$. The special role of $d \in D$ is illustrated by

$$d\circ d = (a^{-1})^{\alpha^{-1}}+d(a^{-1}b),\ d\circ x = dx,\ x\circ d = dx^{\alpha} \tag{7}$$

for all $x \in K$. Multiplication rule (6) is a generalization of formula (7.17, IV) in [15,215]. Cf. also formula (19) in [8,241]. The field K is contained in both $\mathrm{N}_1(D)$ and $\mathrm{N}_r(D)$, the left and right nucleus of

D, respectively. By (7), D is a 2-dimensional left vector space over K, whence either $\mathrm{N}_r(D) = K = \mathrm{N}_l(D)$ or $\mathrm{N}_r(D) = D = \mathrm{N}_l(D)$.

THEOREM 4. *The division ring D is a field if, and only if, one of the following conditions holds true:*

$$b = 0 \wedge a = a^{\alpha} \wedge x^{\alpha\alpha} = axa^{-1} \text{ for all } x \in K; \tag{8}$$

$$b \neq 0 \wedge a, b \in \mathrm{Z}(K) \wedge \alpha = \mathrm{id}_K. \tag{9}$$

Proof. The associator (cf. e.g. [14,140]) of x_0+dx_1, y_0+dy_1, $z_0+dz_1 \in D$ equals

$$\begin{aligned}&(x_1a^{-1})^{\alpha^{-1}}\Big((ay_0a^{-1})^{\alpha^{-1}}-y_0^{\alpha}+(by_1a^{-1})^{\alpha^{-1}}-y_1a^{-1}b\Big)z_1+\\&+ d\Big(x_1\Big(y_0a^{-1}b-a^{-1}by_0^{\alpha}+a^{-1}y_1^{\alpha}-(y_1a^{-1})^{\alpha^{-1}}\Big)z_1\Big).\end{aligned} \tag{10}$$

Thus D is a field if, and only if,

$$y = a^{-1}y^{\alpha\alpha}a,\ y = (a^{-1})^{\alpha}y^{\alpha\alpha}a \tag{11}$$

and

$$ya^{-1}b = a^{-1}by,\ a^{-1}by = a^{-1}(ya^{-1}b)^{\alpha}a \tag{12}$$

for all $y \in K$. If $b = 0$, then (12) holds trivially and conditions (8) and (11) are equivalent. Now let $b \neq 0$. We infer from the first equation of (12) that α is an inner automorphism. But this forces $\alpha = \mathrm{id}_K$ and $a^{-1}b \in \mathrm{Z}(K)$ by (3). Finally $a, b \in \mathrm{Z}(K)$ follows from the second equation of (12). Conversely, (11) and (12) are implied by (9). □

We remark that, by (10), D never is a proper alternative field. If D is a *commutative division ring*, then K is commutative too, and $\alpha = \mathrm{id}_K$ by (7). Hence D is a *commutative field*. Conversely, commutativity of K and $\alpha = \mathrm{id}_K$ make D being a commutative field. These remarks together with Theorem 4 give neccessary and sufficient conditions for the translation plane $\mathcal{T}$ to be *pappian* or *desarguesian*, respectively.

If K is finite, then (5) and (8) cannot be fulfilled simultaniously. On the other hand, let $K = \mathbb{C}$ be the field of complex numbers, $a = -1$, $b = 0$ and α the conjugation in $\mathbb{C}$. Then (5) and (8) hold and D is the skew field of real quaternions.

Set $\mathfrak{S} := (\mathfrak{B}_0 \cap \mathfrak{B}_1)$; given $u_0, u_1 \in K$ then write

$$\mathfrak{S}(u_0,u_1) := \operatorname{span}\{u_0\mathfrak{p}_0+u_1\mathfrak{p}_1+\mathfrak{p}_2, (u_0\mathfrak{p}_0+u_1\mathfrak{p}_1+\mathfrak{p}_2)^{\varphi}\}.$$

Now regard (4) as the matrix of a linear map $\nu(u_0,u_1) : \mathfrak{S}(0,0) \to \mathfrak{S}$ with respect to ordered bases $(\mathfrak{p}_2,\mathfrak{p}_3)$ and $(\mathfrak{p}_0,\mathfrak{p}_1)$. Thus

$$\mathfrak{S}(u_0,u_1) = \{\mathfrak{s}\oplus\mathfrak{s}^{\nu(u_0,u_1)} \mid \mathfrak{s}\in\mathfrak{S}(0,0)\}.$$

Let D' be a 2-dimensional left vector space over K with basis elements 1 and d'; assign $u_0\mathfrak{p}_2+u_1\mathfrak{p}_3$, $u_0\mathfrak{p}_0+u_1\mathfrak{p}_1$ to the element $u_0+u_1d' \in D'$. If we pick any vector $m_0\mathfrak{p}_0+m_1\mathfrak{p}_1 \in \mathfrak{S}$, then $\nu(m_0,m_1)$ takes $\mathfrak{p}_0$ to this chosen vector. This permits to define a multiplication on D' by the action of $\nu(m_0,m_1)$ on $x_0\mathfrak{p}_2+x_1\mathfrak{p}_3$. One obtains

$$\begin{aligned}(x_0+x_1d')*(m_0+m_1d') &:= \\ = x_0m_0+x_1m_1{}^{\alpha}a&+(x_0m_1+x_1m_0{}^{\alpha}+x_1m_1{}^{\alpha}b)d'.\end{aligned} \tag{13}$$

Cf. formulae (1) in [13,390] (reverse multiplication), (7.17,II) in [15,215], (17) in [8,241] and (3) in [14,191] with K being finite or commutative, respectively. It is easily seen that

$$\{\mathfrak{S}(u_0,u_1) \mid u_0,u_1\in K\}\cup\{\mathfrak{S}\} \tag{14}$$

is a spread of $\mathfrak{B}$ if, and only if, $(D',+,*)$ is a *right quasifield*; see Theorem 3, Theorem 9.7 in [14,191] for sufficient conditions. An alternative proof of Theorem 3 is possible by virtue of that Theorem 9.7. Moreover, if D is a field, then all matrices (4) form a subfield F of the ring of 2×2 matrices over K, whence $D \cong F \cong D'$.

With (14) being a partition of $\mathfrak{B}$, we get a translation plane $\mathcal{T}'$ and a division ring D' whose left and middle nuclei contain $\{x+0d' \mid x\in K\}$, a subfield of D' isomorphic to K. Generalizing the terminology in [15,205], $\mathcal{T}'$ is the *transpose translation plane* of $\mathcal{T}$; cf. [4,531], [18,366]. If we would have changed from the left vector space $\mathfrak{B}$ over K to the associated *right vector space* over the *opposite field* of K, then transposition of the matrices (4) would have become necessary.

By combination of various results, we finally state

COROLLARY 2. *Let σ be a collineation of a projective plane with underlying field K. Suppose that σ has an invariant line s. Then σ has an invariant point if either σ^2 is a perspective collineation with axis s, or $\sigma|s$ is induced by $\psi \in \mathrm{GL}(2,K)$ with $\psi^2 = a\cdot\mathrm{id}+b\cdot\psi$, where a, b are non-zero elements in the centre of K.*

Proof. Suppose that σ has no invariant point on s and regard σ as a collineation of a plane within a 3-dimensional projective space $\mathcal{P}$. According to the construction in Theorem 2 we get a dual spread. Writing down a vector space representation, as has been done at the beginning of this section, yields that (8) or (9) holds. Thus D and D' are isomorphic fields which in turn shows that (14) is a partition of $\mathfrak{B}$ or, in other words, σ has an invariant point off s. □

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