On embedded products of Grassmannians*

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Abstract

Let Γ' and Γ be two Grassmannians. The standard embedding $\phi : \Gamma' \times \Gamma \to \overline{\mathbb{P}}$ is obtained by combining the Plücker and Segre embeddings. Given a further embedding $\eta : \Gamma' \times \Gamma \to \mathbb{P}'$, we find a sufficient condition for the existence of $\alpha \in \mathrm{Aut}(\Gamma)$ and of a collineation $\psi : \overline{\mathbb{P}} \to \mathbb{P}'$ such that $\eta = (\mathrm{id}_{\Gamma'} \times \alpha)\phi\psi$.
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1 Introduction

1.1 Background

Several authors have proved that the classical embeddings of geometries such as Grassmannians and product spaces are essentially unique. For example, Havlicek [4] showed that every embedding of a Grassmann space can be represented as the product of the standard embedding, which is obtained by means of Plücker coordinates, and a linear morphism between projective spaces (essentially a projection between complementary subspaces).

Such a strong universal property does not hold in general for product spaces. However, if $\gamma : \mathbb{P}_1 \times \mathbb{P}_2 \to \overline{\mathbb{P}}$ is the Segre embedding, and $\chi : \mathbb{P}_1 \times \mathbb{P}_2 \to \mathbb{P}'$ is any embedding, then there exist $\alpha \in \mathrm{Aut}(\mathbb{P}_2)$ and a linear morphism $\psi : \overline{\mathbb{P}} \to \mathbb{P}'$ such that $\chi = (\mathrm{id}_{\mathbb{P}_1} \times \alpha)\gamma\psi$ (cf. Zanella [8]).

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As a consequence of the results in [4, 8], the image of any embedding of a Grassmann space or a product space is projectively equivalent to a projection of the related variety. So, the incidence geometrical characterizations given by Tallini and several other authors are also intrinsic characterizations of those varieties (cf. the surveys [2, 6]).

We are attempting to give a result analogous to [8] for an embedding η of the product of two Grassmannians Γ' and Γ. As a first step, in this paper we characterize the product of two Grassmannians up to collineations.

1.2 Preliminaries

A *semilinear space* is a pair $\Sigma = (\mathcal{P}, \mathcal{G})$, where $\mathcal{P}$ is a set, whose elements are called *points*, and $\mathcal{G} \subseteq 2^{\mathcal{P}}$. The elements of $\mathcal{G}$ are *lines*. The axioms defining a semilinear space are the following: (i) $|g| \geq 2$ for every line g; (ii) $\bigcup_{g \in \mathcal{G}} g = \mathcal{P}$; (iii) $g, h \in \mathcal{G}$, $g \neq h \Rightarrow |g \cap h| \leq 1$. Two points $X, Y \in \mathcal{P}$ are *collinear*, $X \sim Y$, if a line g exists such that $X, Y \in g$ (for $X \neq Y$ we will also write $XY := g$). An *isomorphism* between the semilinear spaces $(\mathcal{P}, \mathcal{G})$ and $(\mathcal{P}', \mathcal{G}')$ is a bijection $\alpha : \mathcal{P} \to \mathcal{P}'$ such that both α and α^{-1} map lines onto lines.

The *join* of $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{P}$ is:

$$\mathcal{M}_1 \vee \mathcal{M}_2 := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \Big(\bigcup_{\substack{X_i \in \mathcal{M}_i \\ X_1 \sim X_2,\, X_1 \neq X_2}} X_1 X_2 \Big).$$

If X is a point, we will often write X instead of $\{X\}$.

Let S and T be sets. A *generalized mapping*, briefly *g-map*, $f : S \to T$ is a mapping of a subset $\mathbf{D}(f)$ of S into T. $\mathbf{D}(f)$ is the *domain* of f and $\mathbf{A}(f) := S \setminus \mathbf{D}(f)$ is the *exceptional set*. If $X \in \mathbf{A}(f)$, then $Xf = \emptyset$. If $\mathbf{D}(f) = S$, then f is called a *global* g-map.

Let $\mathbb{P}' = (\mathcal{P}', \mathcal{G}')$ be a projective space. A *linear morphism* $\chi : \Sigma \to \mathbb{P}'$ is a g-map of $\mathcal{P}$ into $\mathcal{P}'$ satisfying the following axioms (L1) and (L2) [3, 4]:

(L1) $(X \vee Y)\chi = X\chi \vee Y\chi$ for $X, Y \in \mathcal{P}$, $X \sim Y$;

(L2) $X, Y \in \mathbf{D}(\chi)$, $X\chi = Y\chi$, $X \neq Y$, $X \sim Y \Rightarrow \exists A \in XY$ such that $A\chi = \emptyset$.

The linear morphism χ is called *embedding* if it is global and injective. It should be noted the last definition is somewhat particular, since for instance the inclusion of an affine space into its projective extension is not an embedding.

The *(projective) rank* of χ, $\mathrm{rk}\chi$, is the projective dimension of $[\mathcal{P}\chi]$, where the square brackets $[\,]$ denote projective closure.

2 Bilinear g-maps

Let Σ', Σ'' be semilinear spaces, $\mathbb{P}$ a projective space, and $f : \Sigma' \times \Sigma'' \to \mathbb{P}$ a g-map. If for every point P of Σ' the g-map ${}_Pf : \Sigma'' \to \mathbb{P}$ defined by $X_Pf := (P, X)f$ is a linear morphism, then we say that f is *right linear*. The definition of a left linear g-map is similar. If the g-map f is both left linear and right linear, then it is called *bilinear*. The bilinear mappings are exactly the linear morphisms of the product spaces. If f is a bilinear g-map and for every point P of Σ' the g-map ${}_Pf$ is an embedding, then we say that f is a *right embedding*. So, a right embedding is a special type of global linear morphism.

Let Σ be a semilinear space embedded in an n-dimensional projective space $\mathbb{P}$ (that is, the inclusion is an embedding). This semilinear space satisfies the *chain condition* (with respect to $\mathbb{P}$) if there are a plane $\mathcal{E}$ and $n-2$ lines (say ℓ_3, ℓ_4, ..., ℓ_n) of Σ, such that for every $i = 3, \ldots, n$, $\dim [\mathcal{E} \cup \ell_3 \cup \ell_4 \cup \ldots \cup \ell_i] = i$.

Let Σ be a semilinear space embedded in a projective space $\mathbb{P}$. If every embedding $f : \Sigma \to \mathbb{P}'$ can be uniquely extended to a linear morphism $\overline{f} : \mathbb{P} \to \mathbb{P}'$, then we say that Σ is *universally embedded* in $\mathbb{P}$. Obviously, if Σ is universally embedded in $\mathbb{P}$, then Σ spans $\mathbb{P}$. For instance: Every Grassmann variety is universally embedded in its ambient space [4]; a set of three pairwise skew lines of a projective space $\mathbb{P}$ of dimension 3 is not universally embedded in $\mathbb{P}$.

Proposition 2.1 *Let Σ' and Σ'' be two semilinear spaces, and $F : \Sigma' \times \Sigma'' \to \mathbb{P}'$ a right embedding. Assume that Σ'' is universally embedded in a projective space $\mathbb{P}$ of dimension n. Let ℓ be a line of Σ' and ℓ_1, ℓ_2 lines of Σ'' such that $\ell_1 \cap \ell_2$ is a point P^*. If*

$$\dim [(\ell \times \Sigma'')F] \geq 2n + 1, \tag{1}$$

then for $i = 1, 2$, $(\ell \times \ell_i)F$ is a hyperbolic quadric of the threedimensional subspace $U_i = [(\ell \times \ell_i)F]$ of $\mathbb{P}'$. Furthermore, $U_1 \cap U_2 = (\ell \times P^*)F$.

Proof. Let $i \in \{1,2\}$. In order to prove that $(\ell \times \ell_i)F$ is a hyperbolic quadric, it is enough to check that given two distinct points $A, B \in \ell$, the lines $(A \times \ell_i)F$ and $(B \times \ell_i)F$ are skew. Since the embedding ${}_AF : \Sigma'' \to \mathbb{P}'$, mapping X into $(A, X)F$, can be linearly extended to $\mathbb{P}$, we have $\dim\,[(A \times \Sigma'')F] \leq n$. Then, in view of (1), we obtain:

$$\text{For every } A, B \in \ell,\ A \neq B:\ [(A \times \Sigma'')F] \cap [(B \times \Sigma'')F] = \emptyset. \tag{2}$$

So, $(\ell \times \ell_i)F$ is a hyperbolic quadric. As a further consequence of (2), since $(A, P^*)F \neq (B, P^*)F$, and F is left linear, $(\ell \times P^*)F$ is a line. Such a line is contained in $U_1 \cap U_2$. If $\dim(U_1 \cap U_2) > 1$, then $U_1 \cap U_2$ contains a plane $\tilde{\mathcal{E}}$ that is tangent to both quadrics $(\ell \times \ell_i)F$, $i = 1, 2$. Then $\tilde{\mathcal{E}}$ contains two lines of type $(Q_i \times \ell_i)F$, with $Q_i \in \ell$, $i = 1, 2$.

If $Q_1 = Q_2$, then $(Q_1 \times \ell_1)F \neq (Q_2 \times \ell_2)F$ since we have a right embedding, whence $\tilde{\mathcal{E}} \subset [(Q_1 \times \Sigma'')F]$; but $\tilde{\mathcal{E}}$ also contains a point of type $(Q^*, P^*)F \in [(Q^* \times \Sigma'')F]$ with $Q^* \in \ell \setminus Q_1$. This implies $[(Q_1 \times \Sigma'')F] \cap [Q^* \times \Sigma'')F] \neq \emptyset$, contradicting (2).

If $Q_1 \neq Q_2$ we can obtain a similar contradiction because $(Q_1 \times \ell_1)F$ and $(Q_2 \times \ell_2)F$ have a common point, and $(Q_1 \times \Sigma'')F \cap (Q_2 \times \Sigma'')F \neq \emptyset$.□

Let Σ' and Σ be two semilinear spaces, and $f : \Sigma' \times \Sigma \to \mathbb{P}'$ a right embedding. If Σ is universally embedded in a projective space $\mathbb{P}$, then for every point P of Σ' the g-map ${}_Pf : \Sigma \to \mathbb{P}'$ has a unique linear extension $\overline{{}_Pf} : \mathbb{P} \to \mathbb{P}'$. So, by setting $(P, Q)\overline{f} := Q\overline{{}_Pf}$, we obtain a right linear g-map $\overline{f} : \Sigma' \times \mathbb{P} \to \mathbb{P}'$ which extends f.

Proposition 2.2 *Let Σ' and Σ be two semilinear spaces. Assume that (i) Σ is universally embedded in an n-dimensional projective space $\mathbb{P}$ and satisfies the chain condition; (ii) $f : \Sigma' \times \Sigma \to \mathbb{P}'$ is a right embedding; (iii) for every line ℓ of Σ', $\dim\,[(\ell \times \Sigma)f] \geq 2n + 1$.*

Then the right linear extension $\overline{f} : \Sigma' \times \mathbb{P} \to \mathbb{P}'$ is bilinear.

Proof. *(a)* Taking into account the chain condition for Σ we define

$$S_1 := \emptyset, \quad S_2 := \mathcal{E}, \quad S_i := \mathcal{E} \cup \ell_3 \cup \ell_4 \cup \ldots \cup \ell_i,\ i = 3, 4, \ldots, n,$$

$$T_i := [S_i],\ i = 1, 2, \ldots, n.$$

Set $(\mathcal{P}, \mathcal{G}) := \Sigma$ and let $\mathcal{G}_i$ be the line set of T_i $(i = 1, 2, \ldots, n)$. We will show that the semilinear spaces

$$\Sigma_i := (\mathcal{P} \cup T_i, \mathcal{G} \cup \mathcal{G}_i)$$

are universally embedded in $\mathbb{P}$ for all $i = 1, 2, \ldots, n$. So let $F_i : \Sigma_i \to \tilde{\mathbb{P}}$ be an embedding in some projective space $\tilde{\mathbb{P}}$. By (i), the embedding $F_{i|\Sigma}$ can be extended uniquely to a linear morphism $F'_i : \mathbb{P} \to \tilde{\mathbb{P}}$. Furthermore, $F'_{i|T_i}$ and $F_{i|T_i}$ are two linear morphisms which agree on S_i. In particular, $F_{i|\mathcal{E}} = F'_{i|\mathcal{E}}$ is a collineation. From [4], Satz 1.3, $F_{i|\ell_j} = F'_{i|\ell_j}$ for $j = 3, 4, \ldots, i$ and an easy induction on j, we obtain that $F'_{i|T_i} = F_{i|T_i}$; so, Σ_i is universally embedded in $\mathbb{P}$, as required.

(b) For a point P of $\mathbb{P}$, let $\overline{f}_P : \Sigma' \to \mathbb{P}'$ be the g-map defined by $X\overline{f}_P := (X, P)\overline{f}$. It is enough to prove the following statement by induction on $i = 2, 3, \ldots, n$:

(P$_i$) *If $P \in T_i \setminus (T_{i-1} \cup \mathcal{P})$, then $\overline{f}_P$ is a linear morphism.*

First, (P$_2$) is trivial. Next, assume that (P$_i$) holds for some $1 < i < n$. Take a point $P \in T_{i+1} \setminus (T_i \cup \mathcal{P})$, and define $g := (\ell_{i+1} \vee P) \cap T_i$.

Let ℓ be any line of Σ'; we shall prove that $\overline{f}_{P|\ell}$ is injective and that $\ell\overline{f}_P$ is a line of $\mathbb{P}'$; this will conclude the proof. For any point A of Σ' take into account the linear morphism $\overline{_Af} : \mathbb{P} \to \mathbb{P}'$. There is a line a of Σ' with $A \in a$ and a point $A' \in a \setminus A$. Then the subspace spanned by $(a \times \Sigma)f$ is also spanned by the image of $\overline{_Af}$ together with the image of $\overline{_{A'}f}$, so that (iii) implies $\mathrm{rk}(\overline{_Af}) = n$. Hence $\overline{_Af}$ is an embedding and

$$F := \overline{f}_{|\Sigma' \times (T_i \cup \mathcal{P})}$$

is a right embedding which allows to apply prop. 2.1 with $\Sigma'' := \Sigma_i$, $\ell_1 := g$, $\ell_2 := \ell_{i+1}$, $P^* := g \cap \ell_{i+1}$. Let r' and r'' be two lines through P such that $r' \vee r'' = \ell_1 \vee \ell_2$, $\ell_1 \cap \ell_2 \notin r' \cup r''$. Furthermore, let $B'_j := r' \cap \ell_j$ and $B''_j := r'' \cap \ell_j$ $(j = 1, 2)$. From prop. 2.1 the five lines

$$(\ell \times P^*)f,\ (\ell \times B'_1)\overline{f},\ (\ell \times B'_2)f,\ (\ell \times B''_1)\overline{f},\ (\ell \times B''_2)f \qquad (3)$$

are mutually skew and their span is 5-dimensional. As L varies in ℓ, the four mappings $(L, P^*)f \mapsto (L, B'_j)\overline{f}$, $(L, P^*)f \mapsto (L, B''_j)\overline{f}$ $(j = 1, 2)$ are projectivities. Hence $(L, B'_1)\overline{f} \mapsto (L, B'_2)f$ is a projectivity too and the family of lines $(L \times r')\overline{f}$ with $L \in \ell$ is a regulus with transversal lines $(\ell \times B'_1)\overline{f}$ and $(\ell \times B'_2)f$. Fix one $L \in \ell$: By the linearity of $\overline{_Lf}$, the line $(L \times r')\overline{f}$ carries the point $(L, P)\overline{f}$. Since the lines of a regulus are mutually skew, the mapping

$(\ell \times P)\overline{f} \ \rightarrow \ \{(L \times r')\overline{f} | L \in \ell\}, \quad (L, P)\overline{f} \ \mapsto \ (L, r')\overline{f}$ is bijective. Hence $\overline{f}_{P|\ell}$ is injective. Similar arguments hold true for r''. Now (3) implies that

$$((\ell \times B_1')\overline{f} \vee (\ell \times B_2')f) \cap ((\ell \times B_1'')\overline{f} \vee (\ell \times B_2'')f)$$

is a line $\tilde{\ell}$ which contains $\ell\overline{f}_P$. Hence $\tilde{\ell}$ is a common transversal line of the two reguli $\{(L \times r')\overline{f} | L \in \ell\}$ and $\{(L \times r'')\overline{f} | L \in \ell\}$ so that $\tilde{\ell} = \ell\overline{f}_P$.□

Let F be a commutative field. We consider the integers $0 < h' < N'$, $0 < h < N$, $n' = \binom{N'+1}{h'+1} - 1$, $n = \binom{N+1}{h+1} - 1$, the Grassmann spaces $\Gamma' = \Gamma^{h'}(\mathbb{P}_{N',F})$, $\Gamma = \Gamma^h(\mathbb{P}_{N,F})$, with their Plücker embeddings $\wp' : \Gamma' \to \mathbb{P}_{n',F}$, $\wp : \Gamma \to \mathbb{P}_{n,F}$. Let $\gamma : \mathbb{P}_{n',F} \times \mathbb{P}_{n,F} \to \overline{\mathbb{P}}$ be the Segre embedding ($\dim \overline{\mathbb{P}} = (n'+1)(n+1) - 1$). The *standard embedding* of $\Gamma' \times \Gamma$ is the composition $\phi := (\wp' \times \wp)\gamma$.

Theorem 2.3 *Let* $\eta : \Gamma' \times \Gamma \to \mathbb{P}'$ *be an embedding such that* $\mathrm{rk}\eta \geq \mathrm{rk}\phi$ *and* $[\mathrm{im}\eta] = \mathbb{P}'$. *Then there are* $\alpha \in \mathrm{Aut}(\Gamma)$ *and a collineation* $\psi : \overline{\mathbb{P}} \to \mathbb{P}'$ *such that* $\eta = (\mathrm{id}_{\Gamma'} \times \alpha)\phi\psi$. *Furthermore,* $\mathrm{im}\eta$ *is projectively equivalent to the image of the standard embedding of* $\Gamma' \times \Gamma$.

Proof. We will identify Γ' and Γ with their images under $\wp'$ and $\wp$, respectively. Let A be a point of Γ', and ${}_A\eta : \Gamma \to \mathbb{P}'$ the embedding of Γ defined by $X{}_A\eta := (A, X)\eta$. By the main result in [4], such an embedding can be extended to a linear morphism ${}_A\eta' : \mathbb{P}_{n,F} \to \mathbb{P}'$. So, by setting $(A, B)\overline{\eta} := B{}_A\eta'$, we have a right linear g-map $\overline{\eta} : \Gamma' \times \mathbb{P}_{n,F} \to \mathbb{P}'$. By prop. 2.2, $\overline{\eta}$ is bilinear. By [4] again, $\overline{\eta}$ has a left linear extension $\overline{\overline{\eta}} : \mathbb{P}_{n',F} \times \mathbb{P}_{n,F} \to \mathbb{P}'$. We can apply the symmetric of prop. 2.2; condition $\dim[(\Gamma' \times \ell)] \geq 2n' + 1$ is a consequence of the assumption $\mathrm{rk}\eta \geq \mathrm{rk}\phi$. By the main theorem in [8], there are $\tilde{\alpha} \in \mathrm{Aut}(\mathbb{P}_{n,F})$ and a collineation $\tilde{\psi} : \overline{\mathbb{P}} \to \mathbb{P}'$, such that $\overline{\overline{\eta}} = (\mathrm{id}_{\mathbb{P}_{n',F}} \times \tilde{\alpha})\gamma\tilde{\psi}$. Every collineation of a projective space transforms both Grassmann and Segre varieties in projectively equivalent ones (cf. e.g. [7] (1.1)). Then $\tilde{\alpha} = \tilde{\alpha}_1\tilde{\alpha}_2$, where $\tilde{\alpha}_1$ is a collineation of $\mathbb{P}_{n,F}$ such that $\Gamma\tilde{\alpha}_1 = \Gamma$, and $\tilde{\alpha}_2$ is a projectivity. Since $(\mathrm{id}_{\mathbb{P}_{n',F}} \times \tilde{\alpha}_2)\gamma = \gamma\pi$, π a projectivity of $\overline{\mathbb{P}}$, we obtain the first assertion with $\alpha = \tilde{\alpha}_{1|\Gamma}$ and $\psi = \pi\tilde{\psi}$.

Also the latter statement follows from the properties of the action of a collineation on the Grassmann and Segre varieties.□

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