On automorphisms of flag spaces*

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Abstract

We show that the automorphisms of the flag space associated with a 3-dimensional projective space can be characterized as bijections preserving a certain binary relation on the set of flags in both directions. From this we derive that there are no other automorphisms of the flag space than those coming from collineations and dualities of the underlying projective space. Further, for a commutative ground field, we discuss the corresponding flag variety and characterize its group of automorphic collineations.
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1 Introduction

The aim of the present article is to determine all automorphisms of the *flag space* associated with a 3-dimensional projective space $(\mathcal{P}, \mathcal{L})$; cf. Section 2. Such an automorphism is a bijection on the set $\mathcal{F}$ of flags that preserves pencils of flags in both directions. However, we adopt a slightly different point of view: Two flags are called *related* ($\sim$) if they differ in at most one of their three components. Now we ask for all bijections $\alpha : \mathcal{F} \to \mathcal{F}$ such that

$$\Phi \sim \Psi \Leftrightarrow \Phi^\alpha \sim \Psi^\alpha \text{ for all } \Phi,\, \Psi \in \mathcal{F}. \tag{1}$$

Clearly, each collineation and each duality of $(\mathcal{P}, \mathcal{L})$, via its action on the set of flags, is a solution of (1). It will be established in Section 3, that there are no other solutions. Since the pencils of flags are exactly the maximal sets of mutually related flags, this solves at the same time the problem to find all automorphisms of the flag space.

Our result may also be seen as a characterization of the group of collineations and dualities of a 3-dimensional projective space under a *mild hypothesis* [1], [2]. See also [21] for the logical background of such characterizations.

In Section 4 we turn to the classical point model of $\mathcal{F}$, i.e. a *flag variety.* (It is necessary here to assume that the ground field is commutative.) We sketch a coordinate-free approach using tools from multilinear algebra. So, finally, we get an intrinsic characterization of the group of collineations fixing such a flag variety.

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The flag variety associated with the n-dimensional projective space over the complex numbers $\mathbb{C}$ has been discussed by W. BURAU in [8], [9], [11], and [12]. One of the crucial tools in those papers is that this flag variety yields an irreducible representation of the projective group $\mathrm{PGL}(n+1,\mathbb{C})$. If $\mathbb{C}$ is replaced with an arbitrary commutative ground field K then one still gets a representation of $\mathrm{PGL}(n+1,K)$ as a group of projective collineations fixing the associated flag variety. However, this representation is not necessarily irreducible. So, in general the connection to irreducible representations of linear groups is lost. For example, if the ground field K has characteristic 3, then there is an invariant point in the ambient space of the variety representing the flags of a projective plane over K [22]. Let us just mention here that also for $n=3$ the representation of $\mathrm{PGL}(4,K)$ turns out to be reducible in case of characteristic 3, but this will be discussed elsewhere.

2 The flag space

Let $(\mathcal{P},\mathcal{L})$ be a 3-dimensional projective space with point set $\mathcal{P}$ and line set $\mathcal{L}$. The subspaces of $(\mathcal{P},\mathcal{L})$ are considered as sets of points. We shall not distinguish between a point $Q\in\mathcal{P}$ and the subspace $\{Q\}\subset\mathcal{P}$. The sign $\vee$ is used to denote the join of projective subspaces.

Recall that a *flag* is a triple (P,g,ε) consisting of a point P, a line g, and a plane ε such that $P\in g\subset\varepsilon$. We put $\mathcal{F}$ for the set of all flags of $(\mathcal{P},\mathcal{L})$. Two flags $\Phi,\Psi\in\mathcal{F}$ are called *related* ($\Phi\sim\Psi$) if they differ in at most one of their components. We say that Φ and Ψ are *adjacent* if they are related and distinct.

It is easy to show that $(\mathcal{F},\sim)$ is a *Plücker space* in the sense of W. BENZ [1, p. 199]: Clearly, the relation $\sim$ is reflexive and symmetric. In order to show that $(\mathcal{F},\sim)$ is connected we consider two arbitrary flags (P,g,ε) and (P',g',ε'). Then there is a line h skew to g and g', a point $Q\in h$ that is neither in ε nor in ε' and a plane $\varphi\supset h$ that contains neither P nor P'. We infer that the four points $P_0:=P$, $P_1:=g\cap\varphi$, $P_2:=h\cap\varepsilon$, and $P_3:=Q$ form a tetrahedron (figure 1). Put g_{ij} for the edge joining P_i with P_j and ε_i for the face opposite to P_i. Then $(P,g,\varepsilon)=(P_0,g_{01},\varepsilon_3)\sim(P_1,g_{01},\varepsilon_3)\sim(P_1,g_{12},\varepsilon_3)\sim(P_1,g_{12},\varepsilon_0)\sim(P_2,g_{12},\varepsilon_0)\sim(P_2,g_{23},\varepsilon_0)\sim(P_3,g_{23},\varepsilon_0)=(Q,h,\varphi)$. Similarly, (Q,h,φ) and (P',g',ε') give rise to a tetrahedron, whence the assertion holds.

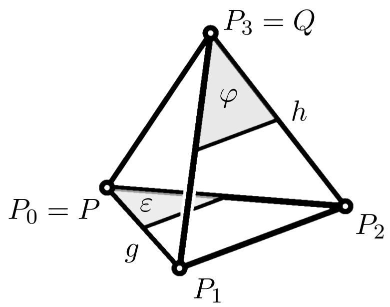

Fig. 1.

Let $P\in\mathcal{P}$ be a point. Then $\mathcal{F}[P]\subset\mathcal{F}$ stands for all flags with point component P and arbitrary other components. Given a line $g\in\mathcal{L}$ or a plane

$\varepsilon \subset \mathcal{P}$ then $\mathcal{F}[g]$ and $\mathcal{F}[\varepsilon]$ are defined similarly. In addition we put $\mathcal{F}[P, g] := \mathcal{F}[P] \cap \mathcal{F}[g]$, $\mathcal{F}[P, \varepsilon] := \mathcal{F}[P] \cap \mathcal{F}[\varepsilon]$, and $\mathcal{F}[g, \varepsilon] := \mathcal{F}[g] \cap \mathcal{F}[\varepsilon]$.

The set $\mathcal{F}$ has three families of distinguished subsets, namely the set $\mathcal{B}_i$ of *pencils of type* i, where $i \in \{0, 1, 2\}$: A pencil of type $0, 1, 2$ is a non-empty set of the form

$$\mathcal{F}[g, \varepsilon],\ \mathcal{F}[P, \varepsilon],\ F[P, g], \tag{2}$$

respectively, where $P \in \mathcal{P}$, $g \in \mathcal{L}$, and $\varepsilon \subset \mathcal{P}$ is a plane (figure 2, 3, 4). Observe that $\mathcal{F}[g, \varepsilon] \neq \varnothing$ is equivalent to $g \subset \varepsilon$ etc.

For all flags of a pencil of type i the components of (projective) dimension $\neq i$ are the same, whence a pencil of type i and a pencil of type $j \neq i$ cannot coincide. We put $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ for the set of all *pencils*. Each pencil contains as many flags as there are points on a line.

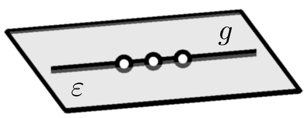

Fig. 2.

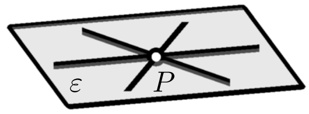

Fig. 3.

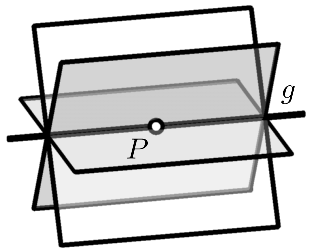

Fig. 4.

If $(P, g, \varepsilon) \in \mathcal{F}$ then the pencils given in (2) are the only pencils through it. So each flag is on exactly one pencil of type 0, 1, and 2. Two distinct pencils of the same type are disjoint. Two adjacent flags Φ, Ψ are joined by exactly one pencil. It will be denoted by $\Phi\Psi$. The following result describes pencils in the terms of the Plücker space $(\mathcal{F}, \sim)$:

Proposition 1. *The pencils of flags are exactly the maximal sets of mutually related flags.*

Proof. By definition, the elements of a fixed pencil $\mathcal{M} \subset \mathcal{F}$ are mutually related. Let $\Phi = (P, g, \varepsilon)$ be a flag in $\mathcal{M}$. A flag Ψ is related to Φ exactly if Ψ is contained in one of the pencils given in (2); in particular one of these pencils is $\mathcal{M}$.

However, if $\Psi \sim \Phi$ is chosen in $\mathcal{F} \setminus \mathcal{M}$, then Φ is the only element of $\mathcal{M}$ related to Ψ, because every other flag of $\mathcal{M}$ differs from Ψ in more than one component. So the pencil $\mathcal{M}$ is a maximal set of mutually related flags.

On the other hand, let $\mathcal{M} \subset \mathcal{F}$ be a maximal set of mutually related flags. Such an $\mathcal{M}$ contains at least two adjacent flags, say Φ_1 and Φ_2. We assume that Φ_1 and Φ_2 differ exactly in their i-dimensional component. Hence $\Psi \in \mathcal{M} \setminus \{\Phi_1, \Phi_2\}$ implies that the components of dimension $\neq i$ of Ψ, Φ_1, and Φ_2 are the same. In other words, Ψ is a flag of the pencil $\Phi_1\Phi_2$. So $\mathcal{M} \subset \Phi_1\Phi_2$ and, by the maximality of $\mathcal{M}$, we have $\mathcal{M} = \Phi_1\Phi_2$. □

The pair $(\mathcal{F}, \mathcal{B})$ is a partial linear space [7, p. 70] with "point set" $\mathcal{F}$ and "line set" $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$. A. Bichara and C. Somma have given an axiomatic description of this *flag space* associated with the projective space

$(\mathcal{P}, \mathcal{L})$; see [3], [4], and [5]. In terms of this partial linear space related flags are "collinear points" and adjacent flags are "distinct collinear points".

3 Plücker Transformations

A *Plücker transformation* of $(\mathcal{F}, \sim)$ is a bijection $\alpha : \mathcal{F} \to \mathcal{F}$ preserving the relation $\sim$ in both directions [1, p. 199].

From Proposition 1, a bijection $\alpha : \mathcal{F} \to \mathcal{F}$ is a Plücker transformation exactly if it is an automorphism of the flag space $(\mathcal{F}, \mathcal{B})$. Note that here (in contrast to [4, p. 61]) we do not require that the type of a pencil is preserved under α. It is straightforward to show that each collineation κ of $(\mathcal{P}, \mathcal{L})$ gives rise to a Plücker transformation $\mathcal{F} \to \mathcal{F} : (P, g, \varepsilon) \mapsto (P^\kappa, g^\kappa, \varepsilon^\kappa)$. Similarly, each duality δ of $(\mathcal{P}, \mathcal{L})$ yields a Plücker transformation $\mathcal{F} \to \mathcal{F} : (P, g, \varepsilon) \mapsto (\varepsilon^\delta, g^\delta, P^\delta)$. It is our aim to show that there are no other Plücker transformations of $(\mathcal{F}, \sim)$.

Proposition 2. *Let $\alpha : \mathcal{F} \to \mathcal{F}$ be a Plücker transformation of $(\mathcal{F}, \sim)$. Then there exists a bijection $\beta : \mathcal{L} \to \mathcal{L}$ such that*

$$\mathcal{F}[g]^\alpha = \mathcal{F}[g^\beta] \text{ for all lines } g \in \mathcal{L}. \tag{3}$$

Proof. (a) Choose a line $g \in \mathcal{L}$ and write $\mathcal{B}_i[g]$ for the set of all pencils of type $i \in \{0, 1, 2\}$ that are contained in $\mathcal{F}[g]$. Clearly, $\mathcal{B}_1[g] = \varnothing$, whereas $\mathcal{B}_0[g]$ consists of all pencils $\mathcal{F}[g, \varepsilon]$, $\varepsilon \supset g$ an arbitrary plane, and $\mathcal{B}_2[g]$ consists of all pencils $\mathcal{F}[P, g]$, $P \in g$ an arbitrary point.

Each flag $\Phi \in \mathcal{F}[g]$ is on a unique pencil of $\mathcal{B}_0[g]$ and on a unique pencil of $\mathcal{B}_2[g]$. Further, each pencil of $\mathcal{B}_0[g]$, say $\mathcal{F}[g, \varepsilon]$, and each pencil of $\mathcal{B}_2[g]$, say $\mathcal{F}[P, g]$, meet at exactly one flag, namely (P, g, ε). Finally, every pencil has at least three elements. This means that the incidence structure $(\mathcal{F}[g], \mathcal{B}_0[g], \mathcal{B}_2[g])$ is a 2-net [19, p. 79–80]. Cf. also [3, p. 99].

(b) Let, as before, $g \in \mathcal{L}$. We claim that under α no pencil of $\mathcal{B}_0[g]$ goes over to a pencil of type 1: Assume to the contrary that $\mathcal{F}[g, \varepsilon]$, $\varepsilon \supset g$ a plane, is such a pencil. There exist distinct points $P, Q \in g$ and a plane $\varphi \supset g$ other than ε. We put

$$\Phi' := (P, g, \varepsilon)^\alpha,\ \Psi' := (Q, g, \varepsilon)^\alpha,\ \Phi'' := (P, g, \varphi)^\alpha, \Psi'' := (Q, g, \varphi)^\alpha.$$

As $\Phi'\Psi'$ is a pencil of type 1, we get $\Phi' = (P', g', \varepsilon')$, $\Psi' = (P', h', \varepsilon')$ with distinct lines $g', h' \in \mathcal{L}$. Since $\Phi'\Psi'$ is the only pencil of type 1 through Φ' and Ψ', the pencils $\Phi'\Phi''$ and $\Psi'\Psi''$ cannot be of type 1, whence the line components of Φ'' and Ψ'' are g' and h', respectively. However, $g' \cap h' = P'$ and $g' \vee h' = \varepsilon'$ implies $\Phi''\Psi'' = \mathcal{F}[P', \varepsilon'] = \Phi'\Psi'$ so that $\Phi' \sim \Psi''$ which contradicts $(P, g, \varepsilon) \not\sim (Q, g, \varphi)$.

Similarly, no pencil of $\mathcal{B}_2[g]$ goes over to a pencil of type 1.

(c) Let (P, g, ε) and (Q, g, φ) be distinct flags. We put $(P, g, \varepsilon)^\alpha =: (P', g', \varepsilon')$. From (a), the pencils $\mathcal{F}[g, \varepsilon] \in \mathcal{B}_0[g]$ and $\mathcal{F}[Q, g] \in \mathcal{B}_2[g]$ meet at $(Q, g, \varepsilon) \in \mathcal{F}[g]$. Now (b) implies that $\mathcal{F}[g, \varepsilon]^\alpha$ and $\mathcal{F}[Q, g]^\alpha$ both are not of type 1. Hence g' is also line component of $(Q, g, \varepsilon)^\alpha \in \mathcal{F}[g, \varepsilon]^\alpha \cap \mathcal{F}[Q, g]^\alpha$ and $(Q, g, \varphi)^\alpha \in \mathcal{F}[Q, g]^\alpha$. Consequently, $\mathcal{F}[g]^\alpha \subset \mathcal{F}[g']$.

In the same manner we obtain $\mathcal{F}[g']^{\alpha^{-1}} \subset \mathcal{F}[g]$, whence $\mathcal{F}[g]^\alpha = \mathcal{F}[g']$. Therefore, by (3) we have a well-defined mapping $\beta : \mathcal{L} \to \mathcal{L}$. Similarly, α^{-1} defines a mapping $\mathcal{L} \to \mathcal{L}$ which is easily seen to be the inverse of β; so β is bijective. $\square$

We note that in the terminology of [3], [4], and [5] the four pencils $\Phi'\Psi'$, $\Psi'\Psi''$, $\Psi''\Phi''$, and $\Phi''\Phi'$ that have been introduced in (b) form a *closed* 4*-path*. Such a path cannot contain a pencil of type 1. This is one of the axioms used in the cited papers.

From part (b) above, it is clear that pencils of type 1 go over to pencils of the same type under α^{-1} as well as under α.

Proposition 3. *The bijection $\beta : \mathcal{L} \to \mathcal{L}$ defined in (3) and its inverse mapping β^{-1} take intersecting lines to intersecting lines.*

Proof. Suppose that $g, h \in \mathcal{L}$ intersect, i.e., $g \cap h =: P$ is a point and $g \vee h =: \varepsilon$ is a plane.

We infer that $\Phi := (P, g, \varepsilon) \in \mathcal{F}[g]$ and $\Psi := (P, h, \varepsilon) \in \mathcal{F}[h]$ span the pencil $\mathcal{F}[P, \varepsilon]$ of type 1. Hence its image under α is again a pencil of type 1. So $\Phi^\alpha = (P', g^\beta, \varepsilon')$ implies $\Psi^\alpha = (P', h^\beta, \varepsilon')$. Therefore g^β and h^β intersect.

The proof for β^{-1} runs in an analogous way. $\square$

We are now in a position to show the announced result.

Theorem 1. *Let $\alpha : \mathcal{F} \to \mathcal{F}$ be a Plücker transformation of $(\mathcal{F}, \sim)$. Then there exists either a unique collineation κ of $(\mathcal{P}, \mathcal{L})$ with*

$$(P, g, \varepsilon)^\alpha = (P^\kappa, g^\kappa, \varepsilon^\kappa) \tag{4}$$

or a unique duality δ of $(\mathcal{P}, \mathcal{L})$ with

$$(P, g, \varepsilon)^\alpha = (\varepsilon^\delta, g^\delta, P^\delta) \tag{5}$$

for all $(P, g, \varepsilon) \in \mathcal{F}$.

Proof. From Propositions 2 and 3, the given Plücker transformation α determines a bijection $\beta : \mathcal{L} \to \mathcal{L}$ such that β and β^{-1} map intersecting lines to intersecting lines. By a result of W.L. Chow (see [13, Theorem 1] or [15, p. 80–82]), there exists either a collineation κ of $(\mathcal{P}, \mathcal{L})$ with $g^\beta = g^\kappa$ or a duality δ of $(\mathcal{P}, \mathcal{L})$ with $g^\beta = g^\delta$ for all $g \in \mathcal{L}$.

In order to verify (4) or (5) choose any flag $\Phi = (P, g, \varepsilon) \in \mathcal{F}$. There is a flag $\Psi = (P, h, \varepsilon)$ adjacent to Φ. Now there are two possibilities:

If β is induced by a collineation κ then $\Phi^\alpha \in \mathcal{F}[g^\kappa]$ and $\Psi^\alpha \in \mathcal{F}[h^\kappa]$ are adjacent too. But the only flag in $\mathcal{F}[g^\kappa]$ that is adjacent to some flag of $\mathcal{F}[h^\kappa]$ is $(g^\kappa \cap h^\kappa, g^\kappa, g^\kappa \vee h^\kappa) = (P^\kappa, g^\kappa, \varepsilon^\kappa)$, whence (4) holds.

If β is induced by a duality δ then $\Phi^\alpha = (g^\delta \cap h^\delta, g^\delta, g^\delta \vee h^\delta) = (\varepsilon^\delta, g^\delta, P^\delta)$ follows similarly.

Finally from (4) or (5), the collineation κ or the duality δ is uniquely determined, since each point $P \in \mathcal{P}$ is a component of at least one flag. $\square$

4 The flag variety

In this section let $(\mathcal{P}, \mathcal{L})$ be a 3-dimensional pappian projective space. We denote by V the underlying vector space with (commutative) ground field K. So $\dim V = 4$. Furthermore, the points, lines, and planes of $(\mathcal{P}, \mathcal{L})$ are the 1-, 2-, and 3-dimensional subspaces of V, respectively.

Put $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}})$ for the projective space on the (6-dimensional) exterior square $V \wedge V$ of V. Recall the *Klein mapping*

$$\gamma : \mathcal{L} \to \widehat{\mathcal{P}} : \; Kq + Kr \mapsto K(q \wedge r), \tag{6}$$

where $q, r \in V$ are linearly independent. It is injective and its image $\mathcal{Q} := \mathcal{L}^\gamma$ is the *Klein quadric* representing the lines of the projective space $(\mathcal{P}, \mathcal{L})$. See, for example, [10, p. 301–302], [16, p. 224], or [17, p. 28–31].

Further, let V^* be the dual space of V. The 1-dimensional subspaces of V^* correspond bijectively to the 3-dimensional subspaces of V via $Ke^* \mapsto \ker e^*$ ($e^* \in V^* \backslash \{0\}$). We shall identify the planes of $(\mathcal{P}, \mathcal{L})$ with the 1-dimensional subspaces of V^* or, in other words, the points of the projective space $(\mathcal{P}^*, \mathcal{L}^*)$ on V^*.

Next, we consider the projective space $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$ on the (96-dimensional) tensor product $V \otimes (V \wedge V) \otimes V^* =: \widetilde{V}$. The *Segre mapping*

$$\sigma : \mathcal{P} \times \widehat{\mathcal{P}} \times \mathcal{P}^* \to \widetilde{\mathcal{P}} : \; (Kp, Kt, Ke^*) \mapsto K(p \otimes t \otimes e^*), \tag{7}$$

where $p \in V$, $t \in V \wedge V$, and $e^* \in V^*$ are non-zero, is injective and its image is a *Segre variety* $\mathcal{S}$ of type $(3, 5, 3)$ [10, p. 111], [18, Chapter 25.5]. If we restrict σ to the product $\mathcal{P} \times \mathcal{Q} \times \mathcal{P}^*$ then we get a point model for all triples consisting of a point, a line, and a plane of $(\mathcal{P}, \mathcal{L})$. In particular, from (6) and (7) we obtain an injective mapping

$$\varphi : \mathcal{F} \to \widetilde{\mathcal{P}} : (P, g, \varepsilon) \mapsto (P, g^\gamma, \varepsilon)^\sigma \tag{8}$$

whose image $\mathcal{G} := \mathcal{F}^\varphi$ is a variety representing the flags of $(\mathcal{P}, \mathcal{L})$. The following property of $\mathcal{G}$ is essential.

Proposition 4. *The φ-images of the pencils of flags are exactly the lines contained in the flag variety $\mathcal{G}$.*

Proof. Let $(P, T, \varepsilon) \in \mathcal{P} \times \widehat{\mathcal{P}} \times \mathcal{P}^*$. Then

$$(\mathcal{P} \times \{T\} \times \{\varepsilon\})^\sigma, (\{P\} \times \widehat{\mathcal{P}} \times \{\varepsilon\})^\sigma, (\{P\} \times \{T\} \times \mathcal{P}^*)^\sigma$$

are the only maximal subspaces contained in the Segre variety $\mathcal{S}$ that pass through the point $(P, T, \varepsilon)^\sigma$ [10, p. 127–128]. Furthermore, by (7), the mapping $Q \mapsto (Q, T, \varepsilon)^\sigma$ is a collineation $\mathcal{P} \to (\mathcal{P} \times \{T\} \times \{\varepsilon\})^\sigma$. Similarly, we have collineations $\widehat{\mathcal{P}} \to (\{P\} \times \widehat{\mathcal{P}} \times \{\varepsilon\})^\sigma$, and $\mathcal{P}^* \to (\{P\} \times \{T\} \times \mathcal{P}^*)^\sigma$ (cf. [17, Theorem 25.5.2]).

Suppose we are given a line ℓ contained in $\mathcal{G}$. Choose an arbitrary point of that line, say $(P, T, \varepsilon)^\sigma$. Since ℓ is also a line of the Segre variety $\mathcal{S}$, its σ-preimage can be found with the inverse of one of the three collineations described above. By taking into account that the lines on the Klein quadric are exactly the γ-images of the pencils of lines in $\mathcal{L}$ [10, p. 301], we see that ℓ is the φ-image of a pencil of flags.

It is immediately clear from (6) and (7) that under φ each pencil of flags is mapped onto a line contained in $\mathcal{G}$. $\square$

The flag variety $\mathcal{G}$ is the intersection of the Segre variety $\mathcal{S}$ with a subspace of $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$ [8, Satz 2]. More precisely, we have the following:

Proposition 5. *The variety $\mathcal{G}$ representing the flags of $(\mathcal{P}, \mathcal{L})$ is the intersection of the Segre variety $\mathcal{S}$ given by (7) with a 63-dimensional projective subspace $(\overline{\mathcal{P}}, \overline{\mathcal{L}})$ of the 95-dimensional projective space $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$.*

Proof. (a) We shall argue in terms of the (16-dimensional) exterior algebra $\bigwedge V$, its dual $(\bigwedge V)^*$ which will be identified with $\bigwedge V^*$, and the inner product $\llcorner : \bigwedge V \times \bigwedge V^* \to \bigwedge V$. See, among others, [6], [14], or [20].

(b) Our first aim is to describe incidence of a point and a line: The mapping

$$V \times (V \wedge V) \times V^* \to \textstyle\bigwedge^3 V \otimes V^* : (p, t, e^*) \mapsto (p \wedge t) \otimes e^*$$

is trilinear. By the universal property of the tensor product $V \otimes (V \wedge V) \otimes V^* = \widetilde{V}$, there is a unique linear mapping

$$i_{01} : \widetilde{V} \to \textstyle\bigwedge^3 V \otimes V^*$$

with $p \otimes t \otimes e^* \mapsto (p \wedge t) \otimes e^*$ for all $(p, t, e^*) \in V \times (V \wedge V) \times V^*$. As i_{01} is surjective, the dimension of $I_{01} := \ker i_{01}$ equals $96 - 16 = 80$.

Choose any triple $(p, t, e^*) \in V \times (V \wedge V) \times V^*$ with $p, t, e^* \neq 0$. Then

$$p \otimes t \otimes e^* \in I_{01} \Leftrightarrow p \wedge t = 0. \tag{9}$$

The subspace $T := \{x \in V \mid x \wedge t = 0\}$ is at most 2-dimensional and $\dim T = 2$ characterizes t as being decomposable [20, 47.5]. Further, the product of the bilinear mapping $V \times V \to \bigwedge^4 V : (v, w) \mapsto v \wedge w \wedge t$ with an (arbitrarily chosen) isomorphism $\bigwedge^4 V \to K$ is a non-zero alternating bilinear form with radical T. The rank of this form is necessarily even. So, it follows that $\dim T \in \{0, 2\}$. We infer from $p \neq 0$ that the right hand side of (9) is equivalent to the existence of $q, r \in V$ such that $t = q \wedge r$ and such that the point Kp is on the line $Kq + Kr$ represented by the point Kt of the Klein quadric.

(c) Next, we turn to the incidence of a line and a plane: The mapping

$$V \times (V \wedge V) \times V^* \to V \otimes V : (p, t, e^*) \mapsto p \otimes (t \llcorner e^*)$$

is trilinear. Hence, as before, there is a unique linear mapping

$$i_{12} : \widetilde{V} \to V \otimes V$$

with $p \otimes t \otimes e^* \mapsto p \otimes (t \llcorner e^*)$ for all $(p, t, e^*) \in V \times (V \wedge V) \times V^*$. The image of i_{12} is the 16-dimensional tensor product $V \otimes V$, whence $I_{12} := \ker i_{12}$ is 80-dimensional.

Choose any triple $(p, t, e^*) \in V \times (V \wedge V) \times V^*$ with $p, t, e^* \neq 0$. Then

$$p \otimes t \otimes e^* \in I_{12} \Leftrightarrow t \llcorner e^* = 0. \tag{10}$$

The bilinear form $V^* \times V^* \to K : (v^*, w^*) \mapsto \langle t, v^* \wedge w^* \rangle$ is non-zero and alternating. (Here $\langle\, , \rangle$ denotes the canonical pairing.) From [20, 47.4, 47.5] the rank of this bilinear form is 2 exactly if t is decomposable. By the definition of the inner product,

$$\langle t \llcorner e^*, w^* \rangle = \langle t, e^* \wedge w^* \rangle \text{ for all } w^* \in V^*. \tag{11}$$

Suppose that $t \llcorner e^* = 0$. This implies that $e^* \neq 0$ is in the radical of the bilinear form from above so that there are $q, r \in V$ with $t = q \wedge r$. Now (11) gives

$$\langle q \wedge r, e^* \wedge w^* \rangle = \det \begin{pmatrix} \langle q, e^* \rangle & \langle r, e^* \rangle \\ \langle q, w^* \rangle & \langle r, w^* \rangle \end{pmatrix} = 0 \text{ for all } w^* \in V^*. \tag{12}$$

Consequently, $\langle q, e^* \rangle = \langle r, e^* \rangle = 0$. By reversing these arguments it follows that the right hand side of (10) is equivalent to the fact that Kt is a point of the Klein quadric which describes a line of the plane Ke^*.

(d) It remains to show that $I_{01} \cap I_{12}$ is a 64-dimensional subspace of $\widetilde{V}$. We establish instead that $I_{01} + I_{12} = \widetilde{V}$ which is equivalent by the dimension formula.

Let b_0, b_1, b_2, b_3 be a basis of V and put b_l^* for the vectors of the dual basis. Then the 96 product vectors

$$b_i \otimes (b_j \wedge b_k) \otimes b_l^* \tag{13}$$

where $i, j, k, l \in \{0, 1, 2, 3\}$ and $j < k$ form a basis of $\widetilde{V}$. This implies that $\widetilde{\mathcal{P}}$ is spanned by the 96 points of $\mathcal{S}$ that represent all triples formed by a vertex, an edge, and a face of the tetrahedron Kb_0, Kb_1, Kb_2, Kb_3. Hence also $(\mathcal{P} \times \mathcal{Q} \times \mathcal{P}^*)^\sigma$ generates $\widetilde{\mathcal{P}}$. So it is enough to show that the points of the Segre variety $\mathcal{S}$ that belong to I_{01} or I_{12} generate a subspace that contains all points of $(\mathcal{P} \times \mathcal{Q} \times \mathcal{P}^*)^\sigma$.

So let $(P, g, \varepsilon) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P}^*$ with $P \notin g$ and $g \not\subset \varepsilon$. We distinguish two cases:

If $P \notin \varepsilon$ then put $g' := P \vee (g \cap \varepsilon)$ and $g'' := (P \vee g) \cap \varepsilon$ (figure 5). As g, g', and g'' are three distinct elements of a pencil of lines we obtain that $(P, g^\gamma, \varepsilon)^\sigma$, $(P, g'^\gamma, \varepsilon)^\sigma \subset I_{01}$, and $(P, g''^\gamma, \varepsilon)^\sigma \subset I_{12}$ are three distinct collinear points of $\mathcal{S}$, whence $(P, g^\gamma, \varepsilon)^\sigma \subset I_{01} + I_{12}$.

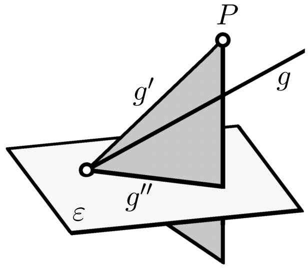

Fig. 5.

If $P \in \varepsilon$ then there are points $P', P'' \notin \varepsilon \cup g$ such that P, P', and P'' are three distinct collinear points. We obtain from the previous case that also $(P, g^\gamma, \varepsilon)^\sigma$, $(P', g^\gamma, \varepsilon)^\sigma \subset I_{01} + I_{12}$, and $(P'', g^\gamma, \varepsilon)^\sigma \subset I_{01} + I_{12}$ are three distinct collinear points of $\mathcal{S}$, whence $(P, g^\gamma, \varepsilon)^\sigma \subset I_{01} + I_{12}$, as required. $\square$

Clearly, the incidence conditions of (b) and (c) can now easily be expressed in terms of coordinates; cf. also [8, Satz 1].

Remark. The previous result does not answer the question whether or not the flag variety $\mathcal{G}$ actually spans the 63-dimensional projective space $(\overline{\mathcal{P}}, \overline{\mathcal{L}})$. In fact, the answer is affirmative. For $K = \mathbb{C}$ this follows from a dimension formula in [9, p. 142]. However, at present we can only establish this result for an arbitrary ground field in terms of (96×64)-matrices by explicit computer based calculations (using Maple V). We just sketch our approach and we use vector space dimensions throughout: Let b_0, b_1, b_2, b_3 be a basis of V and define a basis of $\widetilde{V}$ as in (13). Then each flag can be represented by its 96 homogeneous coordinates with respect to this basis.

In a first step it is easy to show that for each point $Q \in \mathcal{P}$ the φ-image of $\mathcal{F}[Q]$ spans an 8-dimensional subspace of $\widetilde{V}$; cf. [22, 3.2]. Next we consider the four points $P_i := Kb_i$ of the coordinate tetrahedron and the four unit points in the faces of this tetrahedron, i.e. the points $U_i := K(b_0 + b_1 + b_2 + b_3 - b_i)$ where $i \in \{0, 1, 2, 3\}$. Observe that the four points U_i are coplanar exactly if the ground field K has characteristic 3.

The subspace $W_P \subset \widetilde{V}$ spanned by the φ-images of the flags belonging to the union of all subsets $\mathcal{F}[P_i]$ has dimension 32. The same holds (irrespective of $\operatorname{char} K$) for the subspace W_U spanned by all flags belonging to the union of all subsets $\mathcal{F}[U_i]$. But now there are two cases:

If $\operatorname{char} K \neq 3$ then $\dim(W_P + W_U) = 64$. Otherwise $\dim(W_P + W_U) = 63$, but the φ-image of the flag given by the point with coordinates $(1, 1, 1, -1)$, the line with Plücker coordinates $(1, 1, -1, 0, 0, 0)$, and the plane with dual coordinates $(0, 0, 1, 1)$ is not in $W_P + W_U$, whence the assertion follows.

Remark. In textbooks on multilinear algebra *Kronecker products* and *exterior powers* are usually defined only for *linear* mappings. However, this can easily be extended to semilinear mappings that share the *same* accompanying automorphism. As an example we treat the exterior square of a semilinear mapping:

Let $f : X \to Y$ be a semilinear mapping of vector spaces over K with accompanying automorphism $\zeta \in \operatorname{Aut}(K)$. We define Y_ζ as the vector space with the same additive group as Y, but with the modified multiplication $k * y := k^\zeta y$ for all $k \in K$ and all $y \in Y$; cf. [6, p. 221]. Then the linear mappings of X into Y_ζ are exactly the ζ-semilinear mappings $X \to Y$. Moreover, $(Y \wedge Y)_\zeta = Y_\zeta \wedge Y_\zeta$. The usual exterior square of the linear mapping $f : X \to Y_\zeta$ is a linear mapping $\hat{f} : X \wedge X \to (Y \wedge Y)_\zeta$ and at the same time a ζ-semilinear mapping $X \wedge X \to Y \wedge Y$.

Let us say that two points of the flag variety $\mathcal{G}$ are *related* if they are on a line which is contained in $\mathcal{G}$. Here is our final result.

Theorem 2. *Let $\eta : \mathcal{G} \to \mathcal{G}$ be a bijection of the variety representing the flags of a 3-dimensional pappian projective space such that under η and η^{-1} related* *points go over to related points. Then η extends to a unique collineation of the subspace spanned by $\mathcal{G}$.*

Proof. (a) From Proposition 4, the given bijection η is the φ-transform of a Plücker transformation α of $(\mathcal{F}, \sim)$. By Theorem 1, we obtain that there is either a collineation κ or a duality δ whose action on $\mathcal{F}$ coincides with α.

(b) Let κ be such a collineation. Then κ is induced by a semilinear bijection $f : V \to V$. The exterior square of f, say $\hat{f}$, describes the action (via the Klein mapping γ) of κ on the Klein quadric, and the inverse of the transpose of f, say $f^* : V^* \to V^*$, describes the action of κ on the set of planes. Observe that all three mappings belong to the same automorphism of K. Then their Kronecker product $\tilde{f} := f \otimes \hat{f} \otimes f^*$ is a semilinear bijection too and hence induces a collineation μ of the projective space $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$.

(c) Let δ be such a duality. Then δ is induced by a semilinear bijection $f : V \to V^*$. The polarity of the Klein quadric determines a linear bijection $d : V \wedge V \to V^* \wedge V^*$. (Here we identify $V^* \wedge V^*$ with the dual of $V \wedge V$). The product of the exterior square of f, say $\hat{f}$, with d^{-1} describes the action (via the Klein mapping γ) of δ on the Klein quadric, and the inverse of the transpose of f, say $f^* : V^* \to V$, describes the action of δ on the set of planes. Now the remaining proof runs as before by virtue of the commutativity of the tensor product, i.e. the canonical isomorphism $V^* \otimes (V \wedge V) \otimes V \cong V \otimes (V \wedge V) \otimes V^*$.

(d) The collineation μ leaves invariant the subspace generated by the flag variety $\mathcal{G}$ and, by construction, extends the given bijection η. On the other hand, let ρ be a collineation with the required properties. Then μ and ρ coincide for all lines contained in $\mathcal{G}$. Since $(\mathcal{F}, \sim)$ is connected, any two points of $\mathcal{G}$ can be joined by a polygonal path contained in $\mathcal{G}$ consisting of m lines, say. Then it is an easy induction on $m \geq 1$ that μ and ρ coincide on the subspace spanned by the lines of the polygon. Thus, finally, the two collineations are the same on the subspace spanned by $\mathcal{G}$. $\square$

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