

Geometric structures on finite- and infinite-dimensional Grassmannians

Andrea Blunck Hans Havlicek

March 14, 2012

Abstract

In this paper, we study the Grassmannian of n -dimensional subspaces of a $2n$ -dimensional vector space and its infinite-dimensional analogues. Such a Grassmannian can be endowed with two binary relations (adjacent and distant), with pencils (lines of the Grassmann space) and with so-called Z -reguli. We analyse the interdependencies among these different structures.

Mathematics Subject Classification (2010): 51A45, 51B99, 51C05

Key words: Grassmann space, Grassmann graph, projective matrix space, distant graph, regulus, projective line over a ring, chain geometry

1 Introduction

Let V be a left vector space of arbitrary (not necessarily finite) dimension over an arbitrary (not necessarily commutative) field K . It will always be assumed that $\dim V > 2$. We study the set¹

$$\mathcal{G} := \{X \leq V \mid X \cong V/X\}$$

of subspaces X of V that are isomorphic to the quotient space V/X . Clearly, this condition is equivalent to saying that X is isomorphic to one (and hence all) of its complements. We assume that $\mathcal{G} \neq \emptyset$. So, if $\dim V$ is finite, then it is an even number $2n$, say, and \mathcal{G} is just the Grassmannian of n -dimensional subspaces of V . The set \mathcal{G} can be endowed with several structures such that \mathcal{G} becomes the vertex set of a graph or the point set of an incidence geometry. We investigate the interrelations among these structures and among their automorphism groups. Section 2

¹We use the sign \leq for the inclusion of subspaces and reserve $<$ for strict inclusion.

is devoted to the *adjacency relation* on \mathcal{G} and its associated *Grassmann graph*. The *Grassmann space* on \mathcal{G} is based on the notion of a *pencil* of subspaces. We extend results about these two structures, which are well known for $\dim V < \infty$, to infinite dimension. In Section 3 we recall the *distant relation* on \mathcal{G} , where “distant” is just another phrase for “complementary”, the *distant graph*, and the link with *chain geometries*. The *Z-reguli* (reguli over the centre Z of the ground field K) from Section 4 are distinguished subsets of \mathcal{G} . Our main result (Theorem 4.11) says that Z -reguli can be defined in terms of the distant graph. The key tool is a characterisation of Z -reguli in Theorem 4.10 and a description of adjacency in terms of the distant graph from [7]. Finally, in Section 5 we state a series of corollaries about automorphisms.

Throughout the paper we prefer the projective point of view, using the language of points and lines for one- and two-dimensional subspaces. Lower case letters are reserved for points, the join of subspaces is denoted by $+$. Note that dimensions are always understood in terms of vector spaces (rather than projective dimensions).

Although there is no principle of duality for infinite-dimensional vector spaces, for *some* statements in this article a *dual statement* can be obtained as follows: (i) Reverse all inclusion signs between subspaces. (ii) Change the order of subspaces defining a quotient space (e.g. X/Y turns into Y/X). (iii) For any integer $k \geq 0$ replace “subspace of V with dimension k ” by “subspace of V with codimension k ” and vice versa. (iv) Interchange signs for join and meet of subspaces.

For example, conditions (1) and (2) (see below) are dual to each other. We shall frequently claim that the dual of a certain result holds. In such case the reader will easily verify that the proof of the dual result can be accomplished by dualising the initial proof. Clearly, in case of finite dimension this is a consequence of the usual principle of duality, otherwise this is due to the specific content of the initial result.

2 Grassmann graph and Grassmann space

Two elements $X, Y \in \mathcal{G}$ are called *adjacent* (in symbols: $X \sim Y$) if

$$\dim((X + Y)/X) = \dim((X + Y)/Y) = 1, \quad (1)$$

or, equivalently, if

$$\dim(X/(X \cap Y)) = \dim(Y/(X \cap Y)) = 1. \quad (2)$$

This terminology goes back to W.-L. CHOW [10] in the finite-dimensional case. Clearly, adjacency is an antireflexive and symmetric relation. The *Grassmann*

graph on \mathcal{G} is the graph whose vertex set is \mathcal{G} and whose edges are the 2-sets of adjacent vertices. It is studied (also in the infinite-dimensional case) e.g. in [7], [25, 3.8]. In the finite-dimensional case, \mathcal{G} can also be viewed as the point set of a *projective geometry of matrices* (compare [28, 3.6]).

Let $M \leq V$ be a subspace such that there is an $X \in \mathcal{G}$ with $M \leq X$ and $\dim(X/M) = 1$. We define the set

$$\mathcal{G}[M] := \{E \leq V \mid M \leq E \text{ and } \dim(E/M) = 1\}$$

and call it the *star* with centre M . Dually, given an $N \leq V$ for which there exists an $X \in \mathcal{G}$ with $X \leq N$ and $\dim(N/X) = 1$, we set

$$\mathcal{G}\langle N \rangle := \{E \leq V \mid E \leq N \text{ and } \dim(N/E) = 1\}$$

and call it the *top* with carrier N .

Recall our global assumption $\dim V > 2$. It guarantees that a set of subspaces of V cannot be at the same time a top and a star: The subspaces of any star cover the entire space V , whereas the elements of any top, say $\mathcal{G}\langle N \rangle$, cover only the proper subspace $N < V$. Note also that the star with centre M coincides with the set $\mathcal{M} := \{E \in \mathcal{G} \mid M \leq E\}$ only for $\dim V < \infty$, because here $E \in \mathcal{M}$ implies $\dim(E/M) = 1$. On the other hand, in the infinite-dimensional case for any integer $n \geq 0$ there exists at least one $E_n \in \mathcal{M}$ with $\dim(E_n/M) = n$. It can be obtained as the join of M with n independent points in a complement of M .

A set of mutually adjacent elements from \mathcal{G} is nothing but a clique of the Grassmann graph. It will also be called an *adjacency clique*. Our first aim is to show that stars and tops are the maximal adjacency cliques, a fact which is well known in the finite-dimensional case (see, e.g., [25, Prop. 3.2]).

Lemma 2.1. *Let $\mathcal{G}[M]$ be a star. Then the following hold:*

1. $\mathcal{G}[M] \subseteq \mathcal{G}$.
2. Any two distinct elements $E, E' \in \mathcal{G}[M]$ are adjacent.
3. Two adjacent elements $E, E' \in \mathcal{G}$ belong to $\mathcal{G}[M]$ if, and only if, $E \cap E' = M$.

Proof. Ad 1.: By definition, $\dim(M/X) = 1$ for some $X \in \mathcal{G}$. Given any $E \in \mathcal{G}[M]$ we have $\dim E = \dim M + 1 = \dim X$ and $\dim(V/E) = \dim M - 1 = \dim(V/X)$. So $E \cong X \cong V/X \cong V/E$.

Ad 2.: Let E, E' be distinct elements of $\mathcal{G}[M]$. Since $E = E \cap E' = E'$ is impossible, we may assume w.l.o.g. that $E \cap E' < E$. From this and the definition of $\mathcal{G}[M]$, we obtain $M \leq E \cap E' < E$. Now $\dim(E/M) = 1$ yields $M = E \cap E'$, which implies $E \sim E'$ due to $\dim(E'/M) = 1$ and (2).

Ad 3.: The “if part” is trivial, the “only if part” follows from the proof of 2. \square

Dually, the following can be proved:

Lemma 2.2. *Let $\mathcal{G}\langle N \rangle$ be a top. Then the following hold:*

1. $\mathcal{G}\langle N \rangle \subseteq \mathcal{G}$.
2. Any two distinct $E, E' \in \mathcal{G}\langle N \rangle$ are adjacent.
3. Two adjacent elements $E, E' \in \mathcal{G}$ belong to $\mathcal{G}\langle N \rangle$ if, and only if, $E + E' = N$.

Lemma 2.3. *Let $A, B, C \in \mathcal{G}$ be mutually adjacent. Then there is a star or a top containing them.*

Proof. Assume that A, B, C do not belong to any star. This means by Lemma 2.1.3 that w.l.o.g. $A \cap B \neq A \cap C$. Let $a \leq A$ and $b \leq A \cap B$ be points with $a \notin B \cup C$ and $b \notin C$. Then the line $L = a + b \leq A$ does not lie in $A \cap C$, which is a hyperplane of A due to $\dim(A/(A \cap C)) = 1$. Consequently, L meets $A \cap C$ in a point $c \neq b$. So $a \leq L = b + c \leq B + C$. Altogether, $A \leq B + C$. This implies $B < A + B \leq B + C$, whence $A + B = B + C$ as B is a hyperplane in $B + C$. Analogously, $A + C = B + C$. So A, B, C belong to the top $\mathcal{G}\langle N \rangle$ with $N := A + B$. \square

Proposition 2.4. *The maximal adjacency cliques of the Grassmannian \mathcal{G} are precisely the stars and tops.*

Proof. (a) We show that any adjacency clique $\mathcal{A} \subseteq \mathcal{G}$ is a subset of a star or a subset of a top. For $|\mathcal{A}| < 2$ the assertion obviously holds. Otherwise there exist two distinct elements $A, B \in \mathcal{A}$. We read off from Lemma 2.1.3 and Lemma 2.2.3 that they belong to the star $\mathcal{G}[A \cap B] =: \mathcal{S}$ and to the top $\mathcal{G}\langle A + B \rangle =: \mathcal{T}$. If \mathcal{A} is contained in $\mathcal{S} \cap \mathcal{T}$ then we are done. Otherwise there exists a $C \in \mathcal{A} \setminus (\mathcal{S} \cap \mathcal{T})$. We infer from this and from Lemma 2.3, applied to A, B, C , that C belongs to the symmetric difference of \mathcal{S} and \mathcal{T} . Hence there are two cases:

Case 1. $C \in \mathcal{S} \setminus \mathcal{T}$: We claim that $\mathcal{A} \subseteq \mathcal{S}$. For if there were an $X \in \mathcal{A} \setminus \mathcal{S}$ then we could apply Lemma 2.3 first to A, B, X and then to A, C, X . This would give $X \in \mathcal{G}\langle A + B \rangle \cap \mathcal{G}\langle A + C \rangle = \{A\} \subset \mathcal{S}$, an absurdity.

Case 2. $C \in \mathcal{T} \setminus \mathcal{S}$: Here $\mathcal{A} \subseteq \mathcal{T}$ follows dually to Case 1.

(b) Let $\mathcal{S} = \mathcal{G}[M]$ be any star. By Lemma 2.1.2, the star \mathcal{S} is an adjacency clique. Furthermore, let $\mathcal{A} \subseteq \mathcal{G}$ be an adjacency clique containing \mathcal{S} . We infer from (a) that \mathcal{A} is a subset of a star or a subset of a top. However the latter cannot occur, because there is no top containing \mathcal{S} . So there is a star, say $\mathcal{S}' = \mathcal{G}[M']$, with $\mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{S}'$. There are two distinct elements $A, B \in \mathcal{S}$. From Lemma 2.1.3 we infer $M = A \cap B = M'$. Hence $\mathcal{S} = \mathcal{A} = \mathcal{S}'$ which shows that \mathcal{S} is a maximal adjacency clique.

Dually, any top is a maximal adjacency clique.

(c) Given any maximal adjacency clique $\mathcal{A} \subseteq \mathcal{G}$, it is contained in a star \mathcal{S} or in a top \mathcal{T} by (a). The maximality of \mathcal{A} implies that $\mathcal{A} = \mathcal{S}$ or $\mathcal{A} = \mathcal{T}$. \square

Let M, N be subspaces of V such that

$$\text{there is an } X \in \mathcal{G} \text{ with } M \leq X \leq N \text{ and } \dim(X/M) = \dim(N/X) = 1. \quad (3)$$

Then

$$\mathcal{G}[M, N] := \{X \in \mathcal{G} \mid M < X < N\} \quad (4)$$

is called a *pencil* in \mathcal{G} . If $\dim V = 2n$ is finite then (3) is equivalent to $M \leq N$, $\dim M = n - 1$, and $\dim N = n + 1$.

Any pencil $\mathcal{G}[M, N]$ is contained in the star $\mathcal{G}[M]$. As stars are adjacency cliques of \mathcal{G} by Lemma 2.1.1, so are pencils. Let \mathfrak{P} denote the set of all pencils. Then $(\mathcal{G}, \mathfrak{P})$ can be viewed as a partial linear space, i.e. a point-line incidence geometry with “point set” \mathcal{G} and “line set” \mathfrak{P} such that any two “points” are joined by at most one “line”. This geometry is called the *Grassmann space* on \mathcal{G} (see, for example, [25, 3.1] for the finite-dimensional case).

Two distinct elements X, Y of \mathcal{G} are adjacent if, and only if, they are “collinear” in $(\mathcal{G}, \mathfrak{P})$, i.e., if they belong to a common pencil (which then has to be $\mathcal{G}[X \cap Y, X + Y]$). Stars and tops are the maximal singular subspaces of $(\mathcal{G}, \mathfrak{P})$, i.e., subspaces in which any two distinct points are collinear. More precisely, they are projective spaces. An underlying vector space of a star $\mathcal{G}[M]$ is the quotient space V/M , whereas for a top $\mathcal{G}[N]$ the dual space N^* of N plays this role. Consequently, all “lines” of the Grassmann space $(\mathcal{G}, \mathfrak{P})$ contain at least three (actually $|K| + 1$) “points”.

We saw in the preceding paragraph that the adjacency relation \sim can be defined using the concept of pencil only. The subsequent Theorem 2.5 implies that pencils can be defined in \mathcal{G} by using the relation \sim only.

Theorem 2.5. *The pencils of the Grassmann space $(\mathcal{G}, \mathfrak{P})$ are exactly the sets with more than one element that are intersections of two distinct maximal adjacency cliques.*

Proof. Given a pencil as in (4) we noted already that $|\mathcal{G}[M, N]| \geq 3$. The second required property follows from $\mathcal{G}[M, N] = \mathcal{G}[M] \cap \mathcal{G}[N]$ and Proposition 2.4.

Conversely, let \mathcal{S} with $|\mathcal{S}| \geq 2$ be the intersection of two maximal adjacency cliques. By Proposition 2.4 we are led to the following cases:

Case 1. \mathcal{S} is the intersection of two distinct stars, say $\mathcal{S} = \mathcal{G}[M] \cap \mathcal{G}[M']$. Choose an $E \in \mathcal{S}$. Then $M < M + M' \leq E$, so $E = M + M'$ as $\dim(E/M) = 1$. This

means that E is uniquely determined, a contradiction. Dually, \mathcal{S} cannot be the intersection of two distinct tops.

Case 2. \mathcal{S} is the intersection of a star and a top, say $\mathcal{S} = \mathcal{G}[M] \cap \mathcal{G}[N]$. If $M \not\leq N$ then $\mathcal{G}[M] \cap \mathcal{G}[N] = \emptyset$ which is impossible. So $M \leq N$ and hence $\mathcal{G}[M] \cap \mathcal{G}[N] = \mathcal{G}[M, N]$. \square

Up to here the dimension of V did not play an essential role. Yet there are properties of the Grassmann graph and the Grassmann space on \mathcal{G} which depend on the dimension of V being finite or not.

Remark 2.6. By [7, 2.3], the Grassmann graph (\mathcal{G}, \sim) is connected if, and only if, $\dim V = 2n < \infty$. In this case the diameter of the graph is n . For infinite dimension of V the connected component of $X \in \mathcal{G}$ equals

$$\{E \in \mathcal{G} \mid \dim(E/(E \cap X)) = \dim(X/(E \cap X)) < \infty\} \quad (5)$$

and its diameter is infinite.

3 Distant graph and chain geometries

We say that $X, Y \in \mathcal{G}$ are *distant* (in symbols: $X \triangle Y$) whenever they are complementary, i.e., $X \oplus Y = V$. Also this is an antireflexive and symmetric relation. The *distant graph* on \mathcal{G} is the graph whose vertex set is \mathcal{G} and whose edges are the 2-sets of distant vertices. See [6], [7]. The cliques of the distant graph will be called *distant cliques*.

Remark 3.1. In [7] the following is proved:

1. The relation \sim can be defined by using \triangle only [7, Thm. 3.2]: Two different elements $A, B \in \mathcal{G}$ are adjacent if, and only if, there is a $C \in \mathcal{G} \setminus \{A, B\}$ such that for all $X \in \mathcal{G}$ with $X \triangle C$ also $X \triangle A$ or $X \triangle B$ holds.
2. If $\dim V = 2n < \infty$, the relation \triangle can be defined by \sim only: The elements $X, Y \in \mathcal{G}$ are distant if, and only if, the distance of X and Y in the Grassmann graph on \mathcal{G} equals n (which is the diameter of the Grassmann graph); this follows from formula (3) in [7]. Therefore some authors speak of *opposite* rather than distant vertices of the Grassmann graph. Cf. [25, 3.2.4].
3. If $\dim V = \infty$, the relation \triangle cannot be defined by \sim only: There are permutations of \mathcal{G} leaving \sim invariant but *not* leaving \triangle invariant [7, Ex. 4.3]. (E.g., the κ from Example 5.2 has this property.)

The relation “distant” comes from ring geometry; see, among others, [9, p. 15], [14, Def. 1.2.1], and [27, Def. 3.1]. We therefore recall some definitions and results. For any associative ring S with 1 the general linear group $\mathrm{GL}(2, S)$ acts on the free left module S^2 and on the lattice of its submodules. The *projective line over S* is the orbit

$$\mathbb{P}(S) := S(1, 0)^{\mathrm{GL}(2, S)}$$

of the free cyclic submodule $S(1, 0)$ under this action. On $\mathbb{P}(S)$, the antireflexive and symmetric relation Δ (*distant*) is defined by

$$\Delta := (S(1, 0), S(0, 1))^{\mathrm{GL}(2, S)}$$

See [9] or [14] for a detailed exposition. We now adopt the additional assumption that S contains a field F (with $1_F = 1_S$) as a proper subring. Then $\mathbb{P}(F)$ can be embedded in $\mathbb{P}(S)$ via $F(a, b) \mapsto S(a, b)$. The orbit

$$\mathfrak{C}(F, S) := \mathbb{P}(F)^{\mathrm{GL}(2, S)}$$

is called the set of *F-chains* in $\mathbb{P}(S)$, and the incidence geometry $\Sigma(F, S) := (\mathbb{P}(S), \mathfrak{C}(F, S))$ is called the *chain geometry over (F, S)* .

Originally, chain geometries have been studied in the case that S is an F -algebra, i.e., the field F is contained in the centre of S (see [9], [14]). Then, given three mutually distant points, there is a *unique* chain containing them. If F is not in the centre of S , then, in general, there is more than one chain through three mutually distant points. See [5], where we used the term “generalized chain geometry” in order to emphasise the deviations from the original setting. The crucial observation for us is as follows:

Remark 3.2. Two distinct points of $\mathbb{P}(S)$ are distant if, and only if, they are on a common F -chain [5, Lemma 2.1].

Observe that this characterisation provides a definition of the distant relation in terms of F -chains. It does not depend on the chosen field $F \subset S$.

The set \mathfrak{G} can be interpreted as the projective line over the endomorphism ring of a vector space:

Remark 3.3. Let U and U' be arbitrary distant elements of \mathfrak{G} , and let $R = \mathrm{End}_K(U)$ be the endomorphism ring of U . Furthermore let $\lambda : U \rightarrow U'$ be a linear isomorphism. By [3, Thm. 2.4] the following assertions hold:²

1. The mapping

$$\Phi : \mathbb{P}(R) \rightarrow \mathfrak{G} : R(\alpha, \beta) \mapsto U^{(\alpha, \beta)} = \{u^\alpha + u^{\beta\lambda} \mid u \in U\} \quad (6)$$

is a well defined bijection.

²The results in [3] are stated in terms of $U \times U$. We rephrase them by virtue of the linear isomorphism which maps $(u_0, u_1) \in U \times U$ to $u_0 + u_1^\lambda \in V = U \oplus U'$.

2. Points $p, q \in \mathbb{P}(R)$ are distant if, and only if, their images p^Φ, q^Φ are distant (i.e. complementary) in \mathcal{G} .
3. Φ induces an isomorphism of group actions

$$(\mathbb{P}(R), \mathrm{GL}(2, R)) \rightarrow (\mathcal{G}, \mathrm{Aut}_K(V))$$

as follows: For any $\psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, R)$ let $\hat{\psi} : V \rightarrow V$ be defined by

$$u_0 + u_1^\lambda \mapsto (u_0^\alpha + u_1^\gamma) + (u_0^\beta + u_1^\delta)^\lambda \text{ for all } u_0, u_1 \in U.$$

Then $\mathrm{GL}(2, R) \rightarrow \mathrm{Aut}_K(V) : \psi \mapsto \hat{\psi}$ is an isomorphism of groups satisfying $\psi\Phi = \Phi\hat{\psi}$.

In the case that $\dim V = 2n < \infty$ we can identify R with the ring of $n \times n$ matrices over K . Then (6) shows once more that the Grassmannian \mathcal{G} can be identified with the point set of the projective geometry of square matrices studied in [28], since $U^{(\alpha, \beta)}$ equals the left row space of the $n \times 2n$ matrix (α, β) .

The definition of Φ in (6) relies on the choice of U, U' , and λ . However, this choice is immaterial: For, if we select instead any two distant elements \tilde{U}, \tilde{U}' of \mathcal{G} and a linear isomorphism $\tilde{\lambda} : \tilde{U} \rightarrow \tilde{U}'$ then we obtain a bijection $\tilde{\Phi}$ of the projective line over the endomorphism ring \tilde{R} of \tilde{U} onto \mathcal{G} like the one in (6). There exists a linear isomorphism $\iota : U \rightarrow \tilde{U}$, whence the mapping $R \rightarrow \tilde{R} : \alpha \mapsto \iota^{-1}\alpha\iota =: \tilde{\alpha}$ is an isomorphism of rings, and the bijection $\mathbb{P}(R) \rightarrow \mathbb{P}(\tilde{R}) : R(\alpha, \beta) \mapsto \tilde{R}(\tilde{\alpha}, \tilde{\beta})$ takes distant points to distant points in both directions. Further, the linear automorphism $V \rightarrow V : u_0 + u_1^\lambda \mapsto u_0' + u_1'^{\tilde{\lambda}}$ (with $u_0, u_1 \in U$) will send any $R(\alpha, \beta)^\Phi \in \mathcal{G}$ to $\tilde{R}(\tilde{\alpha}, \tilde{\beta})^{\tilde{\Phi}} \in \mathcal{G}$. (This observation generalises Remark 3.3.3.)

Remark 3.4. By virtue of (6) we obtain the following: The distant graph on \mathcal{G} is connected; it has diameter 3 for $\dim V = \infty$ [6, Thm. 5.3] and diameter 2 for $\dim V < \infty$. The second part of the last assertion is immediate from [14, Prop. 1.1.3] and [26, 2.6].

We shall see below that the projective line $\mathbb{P}(R), R = \mathrm{End}_K(U)$, can be considered as the point set of a chain geometry $\Sigma(F, R)$ in at least one way. As we noted above, the distant relation is then definable in terms of F -chains. Taking into account Remark 3.3, the following converse question arises:

Problem. *Given a subfield F of the endomorphism ring $R = \mathrm{End}_K(U)$ is it possible to define the Φ -images of F -chains in terms of the distant graph (\mathcal{G}, Δ) ?*

A major obstacle in solving this problem is that for arbitrary F an explicit description of the Φ -images of F -chains even in terms of the projective space on V seems

to be unknown. The situation seems less intricate for the following class of examples: Let F be any subfield of K . We embed F in R by fixing a basis $(b_i)_{i \in I}$ of U and mapping $a \in F$ to the unique endomorphism of U with $b_i \mapsto ab_i$ for all $i \in I$. The case $F = K$ was detailed in [4]. Here the Φ -images of K -chains are *reguli*. However the definition of a regulus in [4] is rather involved in its most general form³, i.e., when K is a proper skew field and $\dim U = \infty$. We therefore focus on the case when F equals the centre Z of K . Here the choice of a basis from before is immaterial, since the endomorphism corresponding to $a \in Z$ is simply

$$a \cdot \text{id} \in R = \text{End}_K(U). \quad (7)$$

The subsets of \mathcal{G} that correspond under Φ to Z -chains will be exhibited in the next section.

4 Z -Reguli

We start with a definition of Z -reguli. Their connection with the Φ -images of Z -chains will only be shown in Theorem 4.5. Note that most of the following proofs are considerably easier in the case of finite dimension.

Definition 4.1. A Z -regulus is a subset \mathcal{R} of \mathcal{G} satisfying the following conditions:

- (R1) \mathcal{R} is a distant clique with at least three elements.
- (R2) If three mutually distinct elements of \mathcal{R} meet⁴ a line then all elements of \mathcal{R} meet that line.
- (R3) \mathcal{R} is not properly contained in any subset of \mathcal{G} satisfying conditions (R1) and (R2).

During our investigation we shall frequently come across subsets of \mathcal{G} that satisfy conditions (R1) and (R2), but not necessarily the maximality condition (R3). Such a set will be termed as being a *partial Z -regulus*. A line L that meets all elements of a partial Z -regulus \mathcal{R} is called a *directrix* of \mathcal{R} . Note that this does not necessarily mean that each point of L is on some element of \mathcal{R} .

³In [4, Def. 2.3] the following minor revision has to be made in order to assure the results from [4]: Replace the assumption that $(T_i)_{i \in I}$ is a minimal set of lines generating the vector space V by the stronger assumption that $(T_i)_{i \in I}$ is a family of lines such that $V = \bigoplus_{i \in I} T_i$.

⁴We say that two subspaces of V *meet* each other if they have a common point.

Remark 4.2. Let U and U' be distant elements of \mathcal{G} and let $\lambda : U \rightarrow U'$ be a linear isomorphism. There are two distinguished families of subspaces of V which are entirely contained in the set

$$Q := \{ru + su^\lambda \mid u \in U, r, s \in K\}.$$

The first family \mathcal{S}_I comprises all subspaces of the form

$$L_u := \{ru + su^\lambda \mid r, s \in K\} \text{ with } u \in U \setminus \{0\}, \quad (8)$$

and we call them subspaces of *first kind*. The second family \mathcal{S}_{II} is formed by the subspaces of *second kind*. They are given as

$$T^{(x,y)} := \{xu + yu^\lambda \mid u \in U\} \text{ with } (x, y) \in Z^2 \setminus \{(0, 0)\}. \quad (9)$$

Up to minor notational differences the following was shown in [12, Thm. 1]:

1. The subspaces of second kind are precisely the *transversal subspaces* of \mathcal{S}_I , i.e., those subspaces $T \leq V$ for which a bijection of \mathcal{S}_I to the point set of T is given by the assignment $L (\in \mathcal{S}_I) \mapsto L \cap T$.
2. If $L \subseteq Q$ is a line then either L is a subspace of first kind or L is contained in a subspace of second kind.

The first result can be rephrased as follows: Any two subspaces of different kind have a unique point in common. Each point which is on some subspace of second kind is on a unique subspace of first kind. (Compare also with [9, Satz 10.1.4], where similar results are derived under stronger assumptions.) We add in passing that in [12] the set of all points p with $p \subseteq Q$ is called a *Segre manifold*. However, we shall not be concerned with this notion.

In order to prove that \mathcal{S}_{II} is a Z -regulus we need an auxiliary result.

Lemma 4.3. *Let E be any element of a partial Z -regulus \mathcal{R} . Then a bijection from the set of directrices of \mathcal{R} to the point set of E is given by the assignment $L \mapsto L \cap E$. Consequently, each point of E is on a unique directrix of \mathcal{R} .*

Proof. There are $E', E'' \in \mathcal{R} \setminus \{E\}$ with $E' \neq E''$ and hence $E' \triangle E''$.

First, let L be a directrix of \mathcal{R} . As L meets the distant subspaces E and E' the intersection $L \cap E$ is a point. Thus $L \mapsto L \cap E$ gives a well-defined mapping from the set of directrices of \mathcal{R} to the point set of E .

Next, let $p \leq E$ be a point. We have $V = E' \oplus E''$, $p \not\leq E'$, and $p \not\leq E''$. So there is a unique line L' through p meeting E' and E'' . Since \mathcal{R} is a partial Z -regulus, this line L' is a directrix of \mathcal{R} and, by the uniqueness of L' , no other directrix of \mathcal{R} can pass through p . Hence our mapping is bijective. \square

From now on we assume the bijection $\Phi : \mathbb{P}(R) \rightarrow \mathcal{G}$ to be given in terms of U , U' , and λ as in Remark 3.3. We use the same U , U' , and λ to define the notions from Remark 4.2.

Proposition 4.4. *The set \mathcal{S}_{II} comprising all subspaces $T^{(x,y)}$ from (9) is, on the one hand, the Φ -image of a Z -chain and, on the other hand, a Z -regulus.*

Proof. (a) We consider the Z -chain \mathcal{C} which arises by embedding $\mathbb{P}(Z)$ in $\mathbb{P}(R)$ via $Z(x, y) \mapsto R(x, y)$ (cf. Remark 3.1) and obtain

$$R(x, y)^\Phi = U^{(x \cdot \text{id}, y \cdot \text{id})} \quad \text{for all } (x, y) \in Z^2 \setminus \{(0, 0)\}.$$

Comparing (6) and (7) with (9) yields

$$U^{(x \cdot \text{id}, y \cdot \text{id})} = T^{(x,y)} \quad \text{for all } (x, y) \in Z^2 \setminus \{(0, 0)\},$$

whence $\mathcal{S}_{\text{II}} = \mathcal{C}^\Phi$.

(b) Taking into account Remark 3.3.2 and the fact that the points of the Z -chain \mathcal{C} are mutually distant (see Remark 3.2), the elements of \mathcal{S}_{II} turn out to be mutually distant. Together with $3 \leq |\mathcal{C}| = |\mathcal{S}_{\text{II}}|$ this shows that \mathcal{S}_{II} satisfies condition (R1).

Suppose now that a line L meets three distinct elements E_0 , E_1 , and E of \mathcal{S}_{II} . Then $L \cap E$ is a point and, by Remark 4.2.1, there is a unique line $L_u \in \mathcal{S}_{\text{I}}$ through $L \cap E$. The line L_u meets all elements of \mathcal{S}_{II} . Since there is a unique line through p which meets E_0 and E_1 , we get $L = L_u$, and from this \mathcal{S}_{II} is seen to satisfy condition (R2).

Finally, let \mathcal{R} be a partial Z -regulus which contains \mathcal{S}_{II} . The partial Z -reguli \mathcal{S}_{II} and \mathcal{R} have the same directrices, namely all lines that meet three arbitrarily chosen elements of \mathcal{S}_{II} or, said differently, all lines from \mathcal{S}_{I} . We deduce from Lemma 4.3, applied to an arbitrarily chosen $X \in \mathcal{R}$, that X is a transversal subspace of \mathcal{S}_{I} . Now Remark 4.2.1 gives $X \in \mathcal{S}_{\text{II}}$, whence $\mathcal{R} = \mathcal{S}_{\text{II}}$. This verifies that \mathcal{S}_{II} fulfills condition (R3). \square

Theorem 4.5. *The Φ -images of the Z -chains in $\Sigma(Z, R)$, $R = \text{End}_K(U)$, are exactly the Z -reguli in \mathcal{G} .*

Proof. (a) By definition, all Z -chains comprise an orbit under the action of $\text{GL}(2, R)$. Because of Remark 3.3.3, the Φ -images of Z -chains comprise an orbit under the action of the group $\text{Aut}_K(V)$. Clearly, all $f \in \text{Aut}_K(V)$ map Z -reguli to Z -reguli. Hence Proposition 4.4 implies that the Φ -image of any Z -chain of $\mathbb{P}(R)$ is a Z -regulus in \mathcal{G} .

(b) The group $\text{GL}(2, R)$ acts transitively on the set of mutually distant triplets of points of $\mathbb{P}(R)$ (see [9, Satz 1.3.8]). Due to Remark 3.3.3, we have a similar

action of $\text{Aut}_K(V)$ on \mathcal{G} . So, if we are given any Z -regulus \mathcal{R} then there exists an $f \in \text{Aut}_K(V)$ which takes three distinct (arbitrarily chosen) elements of \mathcal{R} to $U = T^{(1,0)}$, $U' = T^{(0,1)}$, and $T^{(1,1)}$ (cf. also [22, Lemma 2.1]). Clearly, \mathcal{R}^f is a Z -regulus. According to Proposition 4.4, the set \mathcal{S}_{II} is a Z -regulus, too. The reguli \mathcal{R}^f and \mathcal{S}_{II} have $T^{(1,0)}$, $T^{(0,1)}$, and $T^{(1,1)}$ in common. This implies that \mathcal{R}^f and \mathcal{S}_{II} have the same set of directrices, namely \mathcal{S}_{I} . By Lemma 4.3, any $X \in \mathcal{R}^f$ is a transversal subspace of \mathcal{S}_{I} , so that Remark 4.2.1 implies $X \in \mathcal{S}_{\text{II}}$. Now $\mathcal{R}^f \subseteq \mathcal{S}_{\text{II}}$ together with the maximality of \mathcal{R}^f yields $\mathcal{R}^f = \mathcal{S}_{\text{II}}$. By Proposition 4.4, the regulus \mathcal{S}_{II} is the image of a Z -chain and, by virtue of f^{-1} , the same property holds for \mathcal{R} according to (a). \square

Corollary 4.6. *All Z -reguli of \mathcal{G} comprise an orbit under the action of $\text{Aut}_K(V)$. Given any three mutually distant elements of \mathcal{G} there is a unique Z -regulus containing them.*

By Proposition 4.4 and Corollary 4.6 the projectively invariant properties of any Z -regulus \mathcal{R} can be read off from the Z -regulus \mathcal{S}_{II} . Below we state one such property. It is immediate from (8) and (9) for the directrices of \mathcal{S}_{II} , since these are precisely the lines from \mathcal{S}_{I} .

Corollary 4.7. *Let L be any directrix of a Z -regulus \mathcal{R} . All points of L which are contained in an element of \mathcal{R} form a Z -subline or, in other words, a Z -chain of L .*

A notion of *regulus* is introduced for any projective space over a (not necessarily commutative) field K in [19]. Furthermore it is pointed out that according to [13] the existence of such a regulus implies K being equal to its centre Z (see also [11]). As a matter of fact, our conditions (R1) and (R2) mean the same as the identically named conditions in [19, p. 55] together with a richness condition stated there. Also, our directrices are precisely the *transversals* in the sense of [19]. Corollary 4.7 shows that our directrices satisfy the remaining condition (R3) in [19] if, and only if, $K = Z$. Hence for a commutative field K the reguli in the sense of [19] are precisely our Z -reguli. Likewise, the reguli from [4] coincide with our Z -reguli in this particular case, but fail to have this property in case of a non-commutative ground field K . The last assertion follows immediately from [4, Lemma 4.1].

We proceed with two lemmas which will be needed in order to show that Z -reguli can be defined in terms of the distant graph (\mathcal{G}, Δ) .

Lemma 4.8. *Let $W, E_0, E \in \mathcal{G}$ with $W \sim E_0$, $E \Delta E_0$, $W \not\Delta E$. Then $W \cap E$ is a point.*

Proof. Assume that $W \cap E$ contains a line L . Then $L \cap E_0 = 0$, because of $E \Delta E_0$. This implies that $\dim(W/(W \cap E_0)) > 1$, a contradiction to $W \sim E_0$.

Assume now that $W \cap E = 0$. Since $W \sim E_0$, we have that $W = (W \cap E_0) + p$ for some point $p \leq W$ with $p \not\leq E_0$. Then $p \not\leq E$, since $W \cap E = 0$. So there is a unique line L through p meeting E_0 and E . Let $q_0 = E_0 \cap L$ and $q = E \cap L$. Then $q \not\leq W$, since $W \cap E = 0$. So also $q_0 \not\leq W$, whence $E_0 = (W \cap E_0) + q_0$. Moreover, $q_0 \leq L = p + q \leq W + E$, and we get $V = E_0 + E = (W \cap E_0) + q_0 + E \leq W + E$, a contradiction to $W \not\leq E$.

Consequently, $W \cap E$ has to be a point. \square

Lemma 4.9. *Let $E_0, E_1, E_2 \in \mathcal{G}$ be mutually distant, and let $W \in \mathcal{G}$ satisfy $W \sim E_0$, $W \not\leq E_1, E_2$. Let $p_i = E_i \cap W$, $i \in \{1, 2\}$, be the unique intersection points according to Lemma 4.8. Then the line $L = p_1 + p_2$ meets E_0 , i.e., L is the unique line through p_1 meeting E_0 and E_2 .*

Proof. By definition, the line L belongs to W . Since $W \sim E_0$, we have that $W \cap E_0$ is a hyperplane in W . So L must meet $W \cap E_0$. \square

Theorem 4.10. *A subset \mathcal{R} of \mathcal{G} is a Z-regulus if, and only if, the following conditions are satisfied:*

- ($\Delta 1$) \mathcal{R} is a distant clique with at least three elements.
- ($\Delta 2$) If three mutually distinct elements $E_0, E_1, E_2 \in \mathcal{R}$ and any $W \in \mathcal{G}$ satisfy $W \sim E_0$ and $W \not\leq E_1, E_2$ then $W \not\leq E$ for all $E \in \mathcal{R}$.
- ($\Delta 3$) \mathcal{R} is not properly contained in any subset of \mathcal{G} satisfying conditions ($\Delta 1$) and ($\Delta 2$).

Proof. It suffices to show that the partial Z-reguli are precisely those subsets \mathcal{R} of \mathcal{G} which satisfy ($\Delta 1$) and ($\Delta 2$).

First, let \mathcal{R} be a partial Z-regulus. Consider E_0, E_1, E_2, W as in ($\Delta 2$). By Lemma 4.8, the subspace $W \cap E_1$ is a point, say p_1 , and by Lemma 4.9, W contains the unique line L through p_1 meeting E_0 and E_2 . Due to (R2), each $E \in \mathcal{R}$ meets $L \leq W$, whence $E \not\leq W$. So \mathcal{R} satisfies ($\Delta 2$) and clearly also ($\Delta 1$), since this condition coincides literally with (R1).

Conversely, let \mathcal{R} be a subset of \mathcal{G} satisfying ($\Delta 1$) and ($\Delta 2$). Given any line L that meets three distinct elements of \mathcal{R} , say E_0, E_1, E_2 , we let $p_i = E_i \cap L$ for $i \in \{0, 1, 2\}$. Consider the set

$$\mathcal{H} := \{H \mid p_0 \leq H \leq E_0, \dim(E_0/H) = 1\}.$$

This is the set of all hyperplanes of E_0 containing p_0 . For each $H \in \mathcal{H}$ let $W_H = H + L$. Each W_H belongs to the star $\mathcal{G}[H]$. Consequently, W_H satisfies $W_H \sim E_0$. Furthermore, for $i \in \{1, 2\}$ we have $W_H \not\leq E_i$, since $p_i = L \cap E_i \leq W_H \cap E_i$.

Let now E be an arbitrary element of \mathcal{R} different from E_0, E_1 . As \mathcal{R} satisfies $(\Delta 2)$, we have that $W_H \not\subseteq E$ holds for all W_H with $H \in \mathcal{H}$. So by Lemma 4.9, applied to E_0, E_1, E , we obtain that all W_H also contain the unique line L' through p_1 meeting E_0 and E . Since $p_0 = \bigcap_{H \in \mathcal{H}} H$, we have that $L' = L$. This implies that \mathcal{R} satisfies (R2), and clearly (R1) is satisfied, too. \square

Theorem 4.11. *The Φ -images of Z-chains or, said differently, the Z-reguli can be defined in terms of the distant graph (\mathcal{G}, Δ) .*

Proof. This is immediate from Theorems 4.5 and 4.10, since the formulation of $(\Delta 2)$ only uses the relations Δ and \sim , the latter of which can be described with Δ alone according to Remark 3.1. \square

5 Consequences

This final section is devoted to the automorphism groups of the various structures on \mathcal{G} . The following corollaries are based on the observation that two notions on \mathcal{G} give rise to the same automorphisms on \mathcal{G} if, and only if, each of these notions is definable in terms of the other. Our first result is a consequence of Theorem 2.5 and the remarks preceding that theorem:

Corollary 5.1. *The automorphisms of the Grassmann graph (\mathcal{G}, \sim) are precisely the collineations of the Grassmann space $(\mathcal{G}, \mathfrak{F})$.*

This corollary, which is part of the “dimension-free” theory, allows us to draw several conclusions: Under any automorphism of the Grassmann graph maximal adjacency cliques are preserved in both directions and, from Corollary 5.1, so are pencils. As mentioned in Section 2, any star $\mathcal{G}\langle M \rangle$ and any top $\mathcal{G}\langle N \rangle$ is a singular subspace of $(\mathcal{G}, \mathfrak{F})$ which is isomorphic to the projective space on V/M and N^* (the dual of N), respectively. However, for a closer analysis we have to distinguish two cases:

In the finite-dimensional case the automorphisms of the Grassmann graph are precisely those bijections of \mathcal{G} onto itself which stem from semilinear isomorphisms of V onto itself or onto its dual V^* (provided that K admits an antiautomorphism). The two possibilities can be distinguished by exhibiting the images of stars and tops: In the first case stars and tops are preserved, in the second case they are interchanged. This is part of the celebrated theorem of W. L. CHOW [10]. See also [15], [16], [20], [24], [25, 3.2.1], [28, Thm. 3.52], [29], and the references therein for proofs of Chow’s initial result and various generalisations.

For infinite dimension the situation is different though: If we are given any star $\mathcal{G}[M]$ and any top $\mathcal{G}\langle N \rangle$ then, due to $\dim V = \infty$, both M and N belong to \mathcal{G} . Consequently, $N \cong M \cong V/M$ (as vector spaces). However, $\dim(V/M) = \dim N < \dim N^*$. Hence the projective spaces on $\mathcal{G}[M]$ and $\mathcal{G}\langle N \rangle$ are non-isomorphic. Given any automorphism $\kappa : \mathcal{G} \rightarrow \mathcal{G}$ of the Grassmann graph the Fundamental Theorem of Projective Geometry (see, among others, [25, 1.4]) implies that the restriction of κ to any star and any top arises from a semilinear isomorphism of the underlying vector spaces. This in turn allows us to deduce that *stars have to go over to stars and tops must go over to tops*. However, an analogue of Chow's theorem fails to hold, as follows from the subsequent example:

Example 5.2. Choose $f \in \text{Aut}_K(V)$ such that some $A \in \mathcal{G}$ is mapped to $A^f \sim A$. Define $\kappa : \mathcal{G} \rightarrow \mathcal{G}$ by $X^\kappa = X^f$ for all X in the connected component of A and $X^\kappa := X$ otherwise. This κ is an automorphism of the Grassmann graph, but it does not stem from any semilinear automorphism, say g , of V . For, if there were such a g then, on the one hand, we would have $A^g = A^f \neq A$. On the other hand, all subspaces $Y \leq A$ with $\dim(A/Y) = 1$ belong to \mathcal{G} , but not to the connected component of A by (5). Therefore we would have $Y^g = Y$ and, A being the union of all such Y s, this would imply $A^g = A$, an absurdity. As a matter of fact our example shows even more: The given κ cannot be induced by any bijection $g : V \rightarrow V$ such that g and g^{-1} preserve subspaces belonging to \mathcal{G} , let alone g being semilinear.

The previous example is based on the fact that the Grassmann graph is disconnected precisely when $\dim V = \infty$. (See Remark 2.6.) Without going into details let us just mention that the related algebraic result about $R = \text{End}_K(U)$ is as follows: All linear endomorphisms of U with finite rank comprise a proper two-sided ideal of R precisely when $\dim U = \infty$. See, among others, [1, p. 164], [2, pp. 197–199], and [23].

In the infinite-dimensional case an explicit description of all automorphisms of the Grassmann graph (\mathcal{G}, \sim) seems to be unknown. On the other hand, an analogue of Chow's theorem holds for the automorphism group of the Grassmann graph formed by all subspaces of a fixed *finite* dimension of an infinite-dimensional vector space [21].

We now turn to the distant graph (\mathcal{G}, Δ) . Let us write \mathfrak{R} for the set of all Z -reguli in \mathcal{G} . Then the (non-linear) incidence geometry $(\mathcal{G}, \mathfrak{R})$ is a model of the chain geometry $\Sigma(Z, R)$, and we call it the *space of Z -reguli* on \mathcal{G} . This generalises a notion from [9, Kap. 10]. By specialising Remark 3.2 to $F = Z$, we see that two distinct elements of $\mathbb{P}(R)$ are distant if, and only if, they are on a common Z -chain. Together with Theorem 4.5 and Theorem 4.11 we therefore obtain:

Corollary 5.3. *The automorphisms of the distant graph (\mathcal{G}, Δ) are precisely the automorphisms of the space $(\mathcal{G}, \mathfrak{R})$ of Z -reguli.*

Corollary 5.4. *The automorphisms of the distant graph $(\mathbb{P}(R), \Delta)$ are precisely the automorphisms of the chain geometry $\Sigma(Z, R)$.*

According to Remark 3.1.1 any automorphism of the distant graph is also an automorphism of the Grassmann graph on \mathcal{G} . There are two cases:

In the finite-dimensional case a complete description of the automorphisms of (\mathcal{G}, Δ) can be derived from Chow's theorem: See [7, Thm. 4.4] and cf. [15], [17], [18], [21] for generalisations. Furthermore, the results from [8, Thm. 5.4] provide an explicit description of the automorphisms of the chain geometry $\Sigma(Z, R)$, thereby avoiding the richness condition appearing in the related result from [9, Kor. 4.3.10].

For infinite dimension of V the only known automorphisms of the distant graph on \mathcal{G} seem to be those which stem from semilinear automorphisms of V . Thus in this case there remains the problem of finding all automorphisms of (\mathcal{G}, Δ) .

Acknowledgements

The first author was partially supported by a grant from Vienna University of Technology (International Office). She thanks the members of the Research Group Differential Geometry and Geometric Structures at TU Vienna for their hospitality.

References

- [1] F.W. Anderson and K.R. Fuller. *Rings and categories of modules*. Springer-Verlag, New York, 1974. Graduate Texts in Mathematics, Vol. 13.
- [2] R. Baer. *Linear Algebra and Projective Geometry*. Academic Press, New York, 1952.
- [3] A. Blunck. Regular spreads and chain geometries. *Bull. Belg. Math. Soc. Simon Stevin*, 6:589–603, 1999.
- [4] A. Blunck. Reguli and chains over skew fields. *Beiträge Algebra Geom.*, 41:7–21, 2000.
- [5] A. Blunck and H. Havlicek. Extending the concept of chain geometry. *Geom. Dedicata*, 83:119–130, 2000.
- [6] A. Blunck and H. Havlicek. The connected components of the projective line over a ring. *Adv. Geom.*, 1:107–117, 2001.

- [7] A. Blunck and H. Havlicek. On bijections that preserve complementarity of subspaces. *Discrete Math.*, 301:46–56, 2005.
- [8] A. Blunck and H. Havlicek. On distant-isomorphisms of projective lines. *Aequationes Math.*, 69:146–163, 2005.
- [9] A. Blunck and A. Herzer. *Kettengeometrien*. Shaker, Aachen, 2005.
- [10] W.-L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. of Math.*, 50(1):32–67, 1949.
- [11] T. Grundhöfer. Reguli in Faserungen projektiver Räume. *Geom. Dedicata*, 11:227–237, 1981.
- [12] H. Havlicek. Baer subspaces within Segre manifolds. *Resultate Math.*, 23:322–329, 1993.
- [13] A. Herzer. Eine Verallgemeinerung des Satzes von Dandelin. *Elem. Math.*, 27:52–56, 1972.
- [14] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.
- [15] L.-P. Huang. Diameter preserving bijections between Grassmann spaces over Bezout domains. *Geom. Dedicata*, 138:1–12, 2009.
- [16] W.-l. Huang. Adjacency preserving transformations of Grassmann spaces. *Abh. Math. Sem. Univ. Hamburg*, 68:65–77, 1998.
- [17] W.-l. Huang. Bounded distance preserving surjections in the geometry of matrices. *Linear Algebra Appl.*, 433(11-12):1973–1987, 2010.
- [18] W.-l. Huang and H. Havlicek. Diameter preserving surjections in the geometry of matrices. *Linear Algebra Appl.*, 429(1):376–386, 2008.
- [19] N. Knarr. *Translation Planes*, volume 1611 of *Lecture Notes in Mathematics*. Springer, Berlin, 1995.
- [20] J. Kosiorek, A. Matraś, and M. Pankov. Distance preserving mappings of Grassmann graphs. *Beiträge Algebra Geom.*, 49(1):233–242, 2008.
- [21] M.-H. Lim. Surjections on Grassmannians preserving pairs of elements with bounded distance. *Linear Algebra Appl.*, 432(7):1703–1707, 2010.
- [22] H. Lüneburg. *Translation Planes*. Springer, Berlin, 1980.
- [23] A. Orsatti and N. Rodinò. On the endomorphism ring of an infinite-dimensional vector space. In *Abelian groups and modules (Padova, 1994)*, volume 343 of *Math. Appl.*, pages 395–417. Kluwer Acad. Publ., Dordrecht, 1995.
- [24] M. Pankov. Chow’s theorem and projective polarities. *Geom. Dedicata*, 107:17–24, 2004.
- [25] M. Pankov. *Grassmannians of Classical Buildings*. World Scientific, Singapore, 2010.

- [26] F.D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*. D. Reidel, Dordrecht, 1985.
- [27] F.D. Veldkamp. Geometry over rings. In F. Buekenhout, editor, *Handbook of Incidence Geometry*. Elsevier, Amsterdam, 1995.
- [28] Z.-X. Wan. *Geometry of Matrices*. World Scientific, Singapore, 1996.
- [29] R. Westwick. On adjacency preserving maps. *Canad. Math. Bull.*, 17:403–405, 1974. Correction, *ibid.* 623.

Andrea Blunck
Fachbereich Mathematik
Universität Hamburg
Bundesstraße 55
D-20146 Hamburg
Germany
andrea.blunck@math.uni-hamburg.de

Hans Havlicek
Institut für Diskrete Mathematik und Geometrie
Technische Universität
Wiedner Hauptstraße 8–10/104
A-1040 Wien
Austria
havlicek@geometrie.tuwien.ac.at