INVARIANT POINTS OF CIRCULAR TRANSFORMATIONS

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Abstract. A geometric construction is given for the invariant points of opposite circular transformations of a real Möbius, Minkowski or Laguerre plane.

1. Introduction

There are three classical circle planes, viz. the real planes of Möbius, Minkowski and Laguerre. We consider them as the conformal closures of a euclidean, pseudo-euclidean or isotropic (galileian) plane, respectively. Cf. e.g. [1,§1,§2,§4]. For further literature on these circle planes and their generalizations we refer to [1], [4], [8], [12], [13]. A bijection of a circle plane which is circle preserving in both directions will be called a circular transformation [1,97]; see [1,§6] for major results on these transformations.

The problem of finding geometric constructions for the invariant points of circular transformations has a long history. It has been dealt with by many authors, but usually attention is paid exclusively to projectivities or, in other words, those transformations which preserve cross ratios. Moreover many authors restrict themselves to involutions. Cf. [3,324], [5], [6], [7], [11,75-77], [14], [15], [16], [19,213], [20].

Only recently Hermann Schaal [9] made a contribution to this subject by establishing a construction for the invariant points of Möbius transformations which are not preserving cross ratios (antiprojectivities). It is based upon the decomposition of such a transformation into a product of inversions.

However the ideas used in [9] cannot be transferred to Minkowski and Laguerre planes: On one hand in a Minkowski plane the subgroup generated by inversions is *not* the full group of circular transformations. This is a corollary to the following well know facts: Let Φ be ruled quadric within a 3-dimensional real projective space. Then Φ is a model for the Minkowski plane and inversions correspond to automorphic harmonic homologies of Φ. Any product of such homologies preserves the two sides of Φ, but there are automorphic collineations of Φ which interchange the two sides of Φ. On the other hand elementary mid-points of circles play a crucial rôle in [9], but circles in a Laguerre plane fail to have mid-points.

In this paper we shall be concerned with opposite circular transformations; they are characterized by the property that cross ratios are subject to conjugation, i.e. the only $\mathbb{R}$-automorphism of the underlying $\mathbb{R}$-algebra (complex numbers, double numbers or dual numbers, respectively) which has order two. A circular transformation is opposite if, and only if, oriented measured angles are transformed by the factor -1. We develop a construction for the invariant points of opposite circular transformations which will work in either of the three classical circle planes thus emphasizing common properties of these geometries rather than particular ones.

2. Invariant points

Let $\overline{\mathcal{E}}$ be the conformal closure of an euclidean, pseudo-euclidean or isotropic affine plane $\mathcal{E}$. Recall that the non-isotropic lines of $\mathcal{E}$ extend to circles of $\overline{\mathcal{E}}$ passing through the ideal point ∞. These circles again will be called lines. On the other hand isotropic lines of $\mathcal{E}$ give rise to generators of $\overline{\mathcal{E}}$.

Let $\alpha : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$ be an opposite circular transformation. If one α-invariant point is known, then all other invariant points may be found in affine terms; cf. [9,172] and the remarks made there.

Now suppose that every point $X \in \overline{\mathcal{E}}$ is parallel (i.e. identical or not cocircular) with its image X^{α}.

If $\overline{\mathcal{E}}$ is a Möbius plane, then α = id, since parallel points always coincide. This contradicts α being opposite.

If $\overline{\mathcal{E}}$ is a Minkowiski plane, then the two families of generators are interchanged, whence α = id, a contradiction.

If $\overline{\mathcal{E}}$ is a Laguerre plane, then $\alpha \neq$ id is easily seen to be possible. Cf. e.g. the examples in [8], [17]. The following construction fails to work in this case[1], but the invariant points of α may be found as follows: The map α induces a projectivity on the family generators of $\overline{\mathcal{E}}$, i.e. the isotropic lines of $\mathcal{E}$ and the ideal generator through ∞. The fixed elements of this projec-

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[1]The same situation arises for projectivities of a Laguerre plane: The construction given in [14,257] cannot be applied when every point is parallel to its image point.

tivity can be constructed by intersecting a circle with a line; see e.g. [2,63]. If fixed generators do exist, then invariant points on them can be constructed in affine terms, since generators are affine lines.

Any opposite circular transformation is uniquely determined by its action on three non-parallel points A,B,C which will be chosen subject to the following restrictions: Assume that none of A,B,C is invariant and that A is not parallel to A^α. Hence we may put $B = A^\alpha$. We shall make assumptions on C and C^α later on. Finally, we may suppose that $B = \infty$. Otherwise $\mathcal{E}$ has to be replaced by the affine plane $\overline{\mathcal{E}}_B$ consisting of all points of $\overline{\mathcal{E}}$ which are not parallel to B. So this last assumption is not really essential.

Given non-isotropic lines k_1,k_2 through A, then $k_1{}^\alpha,k_2{}^\alpha$ are non-isotropic lines again and will pass through B^α. The following equation of oriented measured angles holds true:

$$\sphericalangle(k_1,k_2;A) = -\sphericalangle(k_1{}^\alpha,k_2{}^\alpha;B) = \sphericalangle(k_1{}^\alpha,k_2{}^\alpha;B^\alpha). \tag{1}$$

We denote by δ the restriction of α on the pencil of circles with fundamental points A and B. The set

$$\mathcal{J}_{\text{aff}} := \{X \mid X \in k\cap k^\delta,\ k \text{ is a non-isotropic line through } A\}\setminus\{B\} \tag{2}$$

is a subset of the affine plane $\mathcal{E}$. In the projective closure $\hat{\mathcal{E}}$ of $\mathcal{E}$ this δ gives rise to a projectivity of the pencil of lines through A onto the pencil of lines through B^α and $\mathcal{J}_{\text{aff}}$ is contained in the set of points generated by this projectivity. The actual description of $\mathcal{J}_{\text{aff}}$ will be done in the discussion below.

Let F be a α-invariant point. If F is parallel to B, then F is also parallel to A and B^α, because both α and its inverse α^{-1} preserve parallelism of points. Hence such an F does not exist, when either $\overline{\mathcal{E}}$ is a Möbius or Laguerre plane or $\overline{\mathcal{E}}$ is a Minkowski plane and $A \neq B^\alpha$. If F is not parallel to $B = \infty$, then F is not parallel to A and FAB is a non-isotropic line, whence

$$F \in \mathcal{J}_{\text{aff}} \cap \mathcal{J}_{\text{aff}}{}^\alpha.$$

Now we have to discuss several cases:

Case 1.1. $A \nparallel B^\alpha$ and at least one line k and its image k^δ (cf. (2)) are not touching in B. In affine terms these two lines are parallel. Then there is a uniquely determined circle l such that

$$\mathcal{J}_{\text{aff}} = \{X \in l \mid X \nparallel B\} = l\cap\mathcal{E}.$$

So the set of α-invariant points, $\operatorname{fix}(\alpha)$ say, is a subset of $\mathcal{J}_{\text{aff}}\cap\mathcal{J}_{\text{aff}}{}^\alpha$. The tangent line t of l in A is mapped under α to the circle ABB^α, whence l^α and ABB^α are touching in B. Let C be chosen such that both C and C^α are off the circle ABB^α. Write S for the affine point of intersection of lines ABC and

$BB^\alpha C^\alpha$. Hence $S \in l \setminus \{A,B\}$. Since α is preserving real cross ratios (CR), we obtain

$$\mathrm{CR}(A,C,S,B) = \mathrm{CR}(B,C^\alpha,S^\alpha,B^\alpha) = \mathrm{CR}(B^\alpha,S^\alpha,C^\alpha,B).$$

The cross ratios at either pole of this equation may be interpreted as affine ratios in $\mathcal{E}$, because $B = \infty$. Finding S^α may be done according to figure 1. Then l^α is the unique line touching ABB^α in B and passing through S^α.

If $F \in l \cap l^\alpha$, then $F \in \mathcal{E}$ and

$$\{F\}^\alpha = \big((ABF \cap l) \setminus \{A\}\big)^\alpha = (BB^\alpha F \cap l^\alpha) \setminus \{B\} = \{F\}.$$

So we obtain either two, one or no invariant points in this case.

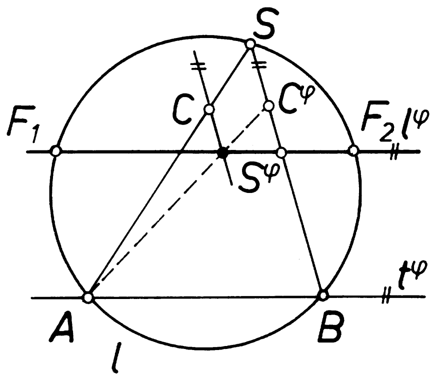

Fig. 1.

Case 1.2. $A \nparallel B^\alpha$ and the circles k, k^δ in formula (2) are always touching in B. Here $\mathrm{fix}(\alpha) = \varnothing$.

Case 2.1. $A \neq B^\alpha$, $A \parallel B^\alpha$ and at least one circle k and its image k^δ are not touching in B. Then $\overline{\mathcal{E}}$ is either a Minkowski or Laguerre plane. There is a uniquely determined generator g of $\overline{\mathcal{E}}$ such that

$$\mathcal{J}_{\mathrm{aff}} = \{X \in g \mid X \nparallel A,\ X \nparallel B^\alpha\} = g \cap \mathcal{E}.$$

In a Minkowski plane $g \cap g^\alpha$ is a single point F_2 and $\mathrm{fix}(\alpha) = \{F_1, F_2\}$ with F_1 being the only point parallel to A,B and B^α. In a Laguerre plane α induces a projectivity of order two on the set of generators. If one point of g is fixed, then g is pointwise invariant by (1). Hence we just have to check if the common point of the circle ABC and g is α-invariant or not.

Case 2.2. $A \neq B^\alpha$, $A \parallel B^\alpha$ and k, k^δ are always touching in B, i.e. in affine terms these lines are always parallel. Then $\mathrm{fix}(\alpha) = \varnothing$, if $\overline{\mathcal{E}}$ is Möbius or Laguerre, and $\mathrm{fix}(\alpha)$ is given by the only point F that is parallel to A,B and B^α in a Minkowski plane.

Case 3.1. $A = B^\alpha$, $\delta \neq \mathrm{id}$. Then $\mathcal{J}_{\mathrm{aff}} = \varnothing$ and thus $\mathrm{fix}(\alpha) = \varnothing$ in a Möbius or Laguerre plane. If however $\overline{\mathcal{E}}$ is a Minkowski plane, then there are exactly two points F_1, F_2 which are parallel to both B and B^α and these two points are

α-invariant, since B and B^α are interchanged.

Case 3.2. $A = B^\alpha$, $\delta = \mathrm{id}$: Now $\mathcal{J}_{\mathrm{aff}} = \mathcal{E}$ fails to give us any information on fix(α). The points A, C, C^α are one a common line k and α restricted to k is an involution. Check the sign of the cross ratio

$$\mathrm{CR}(C, C^\alpha, A, B) \in \mathbb{R}\setminus\{0\}.$$

If this sign is +1, then the projectivity on k is hyperbolic and there are two invariant points F_1, F_2 on k which can be found as usual (cf. e.g. [2,63]). If $\overline{\mathcal{E}}$ is not Laguerre, then fix(α) is given by the circle orthogonal to AF_1F_2 and passing through both F_1 and F_2. Actually α is the inversion at that circle. In a Laguerre plane fix(α) is the union of the two generators passing through F_1 and F_2, respectively. In [17] this transformation too is called an inversion.

If the sign is -1, then the projectivity on k is elliptic. In a Möbius or Laguerre plane there are no invariant points. In a Minkowski plane there are two α-invariant points F_1, F_2 on the line through A which is orthogonal to ABC and fix(α), as before, is a circle[2].

The construction given in case 1.1 is completely independent of the type of circle plane we work in. Unfortunately in the other cases this "common feature" is somehow covered up by the different types of parallelism relation on $\overline{\mathcal{E}}$.

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[2]Here we essentially use the euclidean ordering of $\mathbb{R}$.

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