

On Isomorphisms of Grassmann Spaces

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1. Let $\Pi = (\mathcal{P}, \mathcal{G})$ be a projective space with \mathcal{P} and \mathcal{G} denoting its set of points and lines, respectively; moreover let $\dim \Pi \geq 3$. The **Grassmann space** on \mathcal{G} may be seen as (\mathcal{G}, \sim) , i.e. \mathcal{G} endowed with a binary relation \sim such that

$$a \sim b \Leftrightarrow a \cap b \neq \emptyset.$$

Cf. [6] for an axiomatic approach and [10] for a different view on (even more general) Grassmann spaces. We remark that (\mathcal{G}, \sim) is the classical example of a **Plücker space** [1,p.199].

Now assume that (\mathcal{G}, \sim) and (\mathcal{G}', \sim') are two Grassmann spaces and that $\beta : \mathcal{G} \rightarrow \mathcal{G}'$ is a bijection. In [3], Satz 2, it is claimed that

$$a \sim b \Rightarrow a^\beta \sim' b^\beta \text{ for all } a, b \in \mathcal{G} \quad (1)$$

already characterizes β as being an isomorphism (i.e. β^{-1} is also relation-preserving). The aim of this short communication is to point out a gap in step (d) of the proof of that result and to state several conditions allowing to close this gap. We leave it to the reader to reformulate Satz 3 in [3] as we do here for Satz 2. On the other hand, Satz 1 and Satz 4 in [3] are true without any modification. For easy reference we shall stick to the notations used in [3].

2. Throughout this article $\Pi = (\mathcal{P}, \mathcal{G})$ and $\Pi' = (\mathcal{P}', \mathcal{G}')$ will be projective spaces. We shall be concerned with mappings $\lambda : \mathcal{P} \rightarrow \mathcal{P}'$ sharing some of the following properties:

- (I) λ is injective.
- (II) λ is surjective.
- (III) λ is preserving collinearity of points.
- (IV) λ is preserving non-collinearity of points.

If λ is satisfying (I), (III) and (IV) then it is called **embedding** (cf. [7], [9]). Provided that (I), (II) and (III) hold true λ is called **semicollineaton** (cf. [2], [4]) and if, moreover, (IV) holds true then λ is called **collineation**.

Suppose that $\beta : \mathcal{G} \rightarrow \mathcal{G}'$ is a mapping of lines. We shall say that β is induced by a mapping $\lambda : \mathcal{P} \rightarrow \mathcal{P}'$ of points if

$$(AB)^\beta = A^\lambda B^\lambda \text{ for all } A, B \in \mathcal{P}, A \neq B.$$

Here AB, \dots denotes the unique line joining two distinct points.

THEOREM 1. Let $\beta : \mathcal{G} \rightarrow \mathcal{G}'$ be a bijection from the set of lines of a projective space Π onto the set of lines of a projective space Π' with $\dim \Pi' \geq 3$ such that under β intersecting lines go over to intersecting lines. Then the following assertions are true¹:

- (a) If $\dim \Pi' \geq 4$ then β is induced by an embedding κ of Π in Π' .
- (b) If $\dim \Pi' = 3$ then β is induced by an embedding κ of Π in Π' or an embedding κ of Π in the dual space of Π' .
- (c) $\dim \Pi \geq \dim \Pi'$.
- (d) If Q is a point of Π then the restriction of β to the star $\mathcal{G}[Q]$ (i.e. the set of all lines running through Q) is a semicollineation of the quotient space Π/Q onto Π'/Q^κ .

Proof. (a) and (b) are shown in [3,pp.328-329], steps (a), (b) and (c).

(c) If \mathcal{B} is a basis of Π then it is easily seen that $\text{span}(\mathcal{P}^\kappa) = \text{span}(\mathcal{B}^\kappa)$, since κ is satisfying conditions (I) and (III); cf., e.g., the proof of Hilfssatz 1.5 in [8,p.102]. But $\text{span}(\mathcal{P}^\kappa)$ has to coincide with \mathcal{P}' , since β is surjective. Thus \mathcal{B}^κ is containing a basis of Π' , i.e. $\dim \Pi \geq \dim \Pi'$.

(d) We read off from [3,p.329], step (d), that $\beta | \mathcal{G}[Q] : \mathcal{G}[Q] \rightarrow \mathcal{G}'[Q^\kappa]$ is bijective. A 'line' of the quotient space Π/Q is a pencil of lines with centre Q and it may be written as the set of all lines through Q meeting some line $a \in \mathcal{G} \setminus \mathcal{G}[Q]$. Hence its image under β is a set of 'collinear points' of Π'/Q^κ . \square

The bijection β , as is described in Theorem 1, cannot be induced by any mapping $\mathcal{P} \rightarrow \mathcal{P}'$ other than κ , since

$$\mathcal{G}[Q]^\beta = \mathcal{G}'[Q^\kappa] \text{ for all } Q \in \mathcal{P}. \quad (2)$$

In [3], Satz 2, it is claimed that β is always induced by a collineation κ . Yet, there is a small gap in the 'proof' that κ is surjective: One must not deduce from (2) (and this is actually done in [3,p.329] at the end of step (d)) that β^{-1} takes intersecting lines to intersecting lines. One may only infer from (2) that under β^{-1} a star $\mathcal{G}'[Q'] \subset \mathcal{G}'$ either goes over to a star $\mathcal{G}[Q]$, whence $Q^\kappa = Q'$, or to a set of mutually skew lines of \mathcal{G} , whence Q' is not in the image of κ .

It seems to be a difficult task to prove or disprove that κ is necessarily surjective. Cf. the remarks before Theorem 3.

THEOREM 2. With the settings of Theorem 1, the following assertions are equivalent:

- (a) β is induced by a collineation κ of Π onto Π' .

¹ Replacing Π' by its dual space permits to rule out the second alternative in (b). Thus, without loss of generality, we may assume in the subsequent assertions (c) and (d) as well as in Theorems 2 and 3 that κ is an embedding of Π in Π' .

- (b) β takes skew lines to skew lines (i.e. β is an isomorphism of the Grassmann space (\mathcal{G}, \sim) onto (\mathcal{G}', \sim')).
- (c) For at least one point $Q \in \mathcal{P}$ the restriction $\beta | \mathcal{G}[Q] : \mathcal{G}[Q] \rightarrow \mathcal{G}'[Q^\kappa]$ is a collineation of Π/Q onto Π'/Q^κ .
- (d) β maps at least one pencil of lines onto a pencil of lines.

Proof. (a) \Rightarrow (b): This is obviously true².

(b) \Rightarrow (c): Three distinct lines $a, b, c \in \mathcal{G}[Q]$ are 'non-collinear points' of Π/Q if there exists a line d being skew to c but intersecting a and b . This yields immediately the 'non-collinearity' of $a^\beta, b^\beta, c^\beta \in \mathcal{G}'[Q^\kappa]$ so that $\beta | \mathcal{G}[Q] : \mathcal{G}[Q] \rightarrow \mathcal{G}'[Q^\kappa]$ is a collineation of quotient spaces.

(c) \Rightarrow (d): Choose any pencil of lines with centre Q . Since this is a 'line' of Π/Q , its β -image is a 'line' of Π'/Q^κ or, in other words, a pencil of lines.

(d) \Rightarrow (a): Let $\mathcal{G}[Q, \varepsilon] := \{x \in \mathcal{G}[Q] \mid x \subset \varepsilon, \varepsilon \text{ a plane}\}$ be such a pencil. Choose a line $a \subset \varepsilon$ such that $Q \notin a$. Then

$$(l \cap a)^\kappa = l^\beta \cap a^\beta \text{ for all lines } l \in \mathcal{G}[Q, \varepsilon]$$

so that a^κ is a line of Π' and not only a subset of a line. But an embedding that maps at least one line onto a line is a collineation onto a subspace of Π' ; cf. Hilfssatz 1.3 in [8,p.101]. By the surjectivity of β , this subspace has to be \mathcal{P}' . \square

We infer from Theorem 2 that κ is not surjective if, and only if, $\beta | \mathcal{G}[Q]$ yields a proper semicollineation (violating condition (IV)) for one point $Q \in \mathcal{P}$ or, equivalently, for all points $Q \in \mathcal{P}$. Examples of proper semicollineations of n -dimensional projective spaces ($n \geq 4$, n even) onto non-Desarguesian projective planes are given in [2], [4]. However, those semicollineations cannot serve as a basis for an example of a bijection β with non-surjective κ , since Π/Q and Π'/Q^κ are always Desarguesian projective spaces. The author does not know whether or not there are proper semicollineations between Desarguesian projective spaces.

THEOREM 3. *With the settings of Theorem 1, each of the following conditions is sufficient for κ to be a collineation:*

- (a) $\dim \Pi \leq \dim \Pi' < \infty$.
- (b) Π or Π' is a finite projective space.
- (c) Every monomorphism of an underlying field of Π in an underlying field of Π' is surjective.

² The equivalence of (a) and (b) has already been established in [5].

Proof. Choose any point $Q \in \mathcal{P}$. By Theorem 2 (c) it is sufficient to show that $\beta | \mathcal{G}[Q] : \mathcal{G}[Q] \rightarrow \mathcal{G}'[Q^\kappa]$ is a collineation of Π/Q onto Π'/Q^κ .

(a) We observe

$$\dim(\Pi/Q) = \dim \Pi - 1 = \dim \Pi' - 1 = \dim(\Pi'/Q^\kappa) < \infty$$

by our assumption and Theorem 1 (c). But this forces $\beta | \mathcal{G}[Q]$ to be a collineation; see result 8.4 in [4,p.325].

(b) Since β is bijective, both Π and Π' are finite projective spaces. By proposition 14.2 in [4,p.339], $\beta | \mathcal{G}[Q]$ is a collineation.

(c) Π is Desarguesian by Theorem 1 (c) and $\dim \Pi' \geq 3$. Use proposition 5.3 in [4,p.320] to establish that $\beta | \mathcal{G}[Q]$ is a collineation. \square

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