Lifting of divisible designs

Andrea Blunck Hans Havlicek Corrado Zanella

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Dedicated to Walter Benz on the occasion of his 75th birthday

Abstract

The aim of this paper is to present a construction of t-divisible designs for $t > 3$, because such divisible designs seem to be missing in the literature. To this end, tools such as finite projective spaces and their algebraic varieties are employed. More precisely, in a first step an abstract construction, called t-lifting, is developed. It starts from a set X containing a t-divisible design and a group G acting on X. Then several explicit examples are given, where X is a subset of $\mathrm{PG}(n,q)$ and G is a subgroup of $\mathrm{GL}_{n+1}(q)$. In some cases X is obtained from a cone with a Veronesean or an h-sphere as its basis. In other examples X arises from a projective embedding of a Witt design. As a result, for any integer $t \geq 2$ infinitely many non-isomorphic t-divisible designs are found.

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1 Introduction

1.1 This paper is concerned with the construction t-divisible designs; see Definition 2.2. We shall frequently use the shorthand "DD" for "divisible design". A well known construction of a t-DD is due to A. G. Spera [27, Proposition 4.6]. It uses a finite set X of points which is endowed with an equivalence relation $\mathcal{R}$, a group G acting on X, and a subset B of X called the 'base block'. Then, under certain conditions, the action of G on X gives rise to a t-divisible design with point set X, equivalence relation $\mathcal{R}$, and the G-orbit of B as set of blocks. If all equivalence classes are singletons then Spera's construction turns into a construction of t-designs due to D. R. Hughes [19, Theorem 3.4].
C. Cerroni, S. Giese, R. H. Schulz, A. G. Spera, and others successfully made use of Spera's construction and obtained examples of 2- and 3-DDs. See [5], [6], [7], [8], [11], [12], [24], [25], [28], and [29]. We refer also to [11, 3.1] for a detailed survey. It seems, however, that no examples of t-DDs for $t > 3$ were constructed in this way.

1.2 One of the results in the thesis of S. Giese is a construction of a 2-DD which it is called "Konstruktion (A)" in [11, p. 64]: It starts with a given 2-DD, say $\mathcal{D}$, a finite projective space $\mathrm{PG}(n+1,q)$ with a distinguished hyperplane $H = \mathrm{PG}(n,q)$ and a distinguished point $O \in$

$\mathrm{PG}(n+1,q) \setminus H$, called the *origin*. Assuming that the dimension n and the prime power q are sufficiently large, the point set of the given 2-DD can be mapped bijectively onto a set of $n-1$-spaces of H subject to certain technical properties. Then each of these subspaces is joined with the origin. This gives an isomorphic copy of the given 2-DD whose "point set" consists of hyperplanes of $\mathrm{PG}(n+1,q)$ through the origin. Then a new 2-DD, say $\mathcal{D}'$, can be obtained from the action of the translation group (with respect to H) on this model of the given 2-DD. See [11, Satz 3.2.4]. Consequently, the "points" of $\mathcal{D}'$ are also hyperplanes of $\mathrm{PG}(n+1,q)$, but not all through the origin. It turns out that this construction can be repeated by embedding $\mathrm{PG}(n+1,q)$ as a hyperplane in $\mathrm{PG}(n+2,q)$, choosing a new origin in $\mathrm{PG}(n+2,q) \setminus \mathrm{PG}(n+1,q)$, and so on. In this way infinite series of 2-DDs can be obtained from any given 2-DD.

Of course, there is also the possibility to start the construction of Giese when $\mathcal{D}$ is a t-DD ($t \geq 2$), since such a structure is also a 2-DD. In [11, Lemma 3.2.18] necessary and sufficient conditions are given for $\mathcal{D}'$ to be a t-DD. However, those conditions are in terms of the new structure $\mathcal{D}'$ rather than the initial structure $\mathcal{D}$, whence they cannot be checked at the very beginning.

1.3 The aim of the present note is to present a construction of a t-DD which generalizes the ideas from [11]. We start with an abstract group acting G on some set X, and a t-DD embedded in X. Then, under certain conditions which can be read off from Theorem 2.5, a new t-DD is obtained via the action of G on X. This process will be called a *t-lifting*.

Several explicit examples for t-liftings are presented in Section 3. We choose X to be a cone (without its vertex) in a finite projective space $\mathrm{PG}(n,q)$, and G to be a certain group of matrices. This approach is still very general, since there are many possibilities for X. In particular, when the base of the cone is chosen to be a Veronese variety, infinitely many non-isomorphic t-divisible designs can be found for any $t \geq 2$; see Theorem 3.8. The construction of Giese, even after a finite number of iterations, is just a particular case of our construction of a 2-lifting in a finite projective space. However, in order to get Giese's results in their original form, one has to adopt a dual point of view. Cf. the remarks in 3.2.

2 Construction of t-liftings

2.1 Assume that X is a finite set of *points*, endowed with an equivalence relation $\mathcal{R}$; its equivalence classes are called *point classes*. A subset Y of X is called *$\mathcal{R}$-transversal* if for each point class C we have $\#(C \cap Y) \leq 1$. Let us recall the following:

Definition 2.2 A triple $\mathcal{D} = (X, \mathcal{B}, \mathcal{R})$ is called a *t-(s,k,λ_t)-divisible design* if there exist positive integers t, s, k, λ_t such that the following axioms hold:

(A) $\mathcal{B}$ is a set of $\mathcal{R}$-transversal subsets of X, called *blocks*, with $\#B = k$ for all $B \in \mathcal{B}$.

(B) Each point class has size s.

(C) For each $\mathcal{R}$-transversal t-subset $Y \subset X$ there exist exactly λ_t blocks containing Y.

(D) $t \leq \frac{v}{s}$, where $v := \#X$.

Observe that (D) is necessary to avoid the trivial case where no $\mathcal{R}$-transversal t-subset exists.

2.3 Sometimes we shall speak of a t-DD without explicitly mentioning the remaining *parameters* s, k, and λ_t. According to our definition, a block is merely a subset of X. Hence the DDs which we are going to discuss are *simple*, i.e., we do not take into account the possibility of "repeated blocks". Cf. [1, p. 2] for that concept.
A divisible design with $s = 1$ is called a *design*; we refer to the two volumes [1] and [2]. In design theory the parameter s is not taken into account, and a t-$(1, k, \lambda_t)$-DD with v points is often called a t-(v, k, λ_t)-design.

2.4 One possibility to construct divisible designs is given by the following theorem. The ingredients for this construction are a finite set X, a finite group G acting on X, and a so-called *base divisible design*, say $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$. Its orbit under the action of G will then yield a DD. More precisely, we can show the following:

Theorem 2.5 (t-Lifting) *Let X be a finite set, let t be a fixed positive integer, let $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$, where $\overline{X} \subset X$, be a t-$(\overline{s}, k, \overline{\lambda}_t)$-divisible design, and let G be a group acting on X. Suppose, furthermore, that the following properties hold:*

(a) *For each $x \in X$ there is a unique element of $\overline{X}$, say $\widehat{x}$, such that $x^G = \widehat{x}^G$.*
(b) *All orbits $\overline{x}^G$, where $\overline{x} \in \overline{X}$, have the same cardinality.*
(c) *Given any subset $Y = \{y_1, y_2, \ldots, y_t\}$ of X, for which $\widehat{Y} := \{\widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_t\}$ is an $\overline{\mathcal{R}}$-transversal t-subset of $\overline{X}$, there exists at least one $g \in G$ such that $Y^g = \widehat{Y}$.*
(d) *All setwise stabilizers $G_{\overline{Y}}$, where $\overline{Y} \subset \overline{X}$ is any $\overline{\mathcal{R}}$-transversal t-subset, have the same cardinality.*
(e) *All setwise stabilizers $G_{\overline{B}}$, where $\overline{B} \in \overline{\mathcal{B}}$ is any block, have the same cardinality.*

Then $(X, \mathcal{B}, \mathcal{R})$ with

$$\mathcal{B} := \overline{\mathcal{B}}^G = \{\overline{B}^g \mid \overline{B} \in \overline{\mathcal{B}}, g \in G\}, \quad \mathcal{R} := \{(x, x') \in X \times X \mid (\widehat{x}, \widehat{x}') \in \overline{\mathcal{R}}\}, \tag{1}$$

is a t-(s, k, λ_t)-divisible design, where

$$s = (\#\overline{x}^G)\overline{s}, \quad \lambda_t := \overline{\lambda}_t \frac{\#G_{\overline{Y}}}{\#G_{\overline{B}}} \tag{2}$$

with arbitrary $\overline{x}$, $\overline{Y}$, and $\overline{B}$ as above.

Proof. It is clear from (a) that $\mathcal{R}$ is a well-defined equivalence relation. Due to (a) and (b), all its equivalence classes have cardinality $(\#\overline{x}^G)\overline{s}$, where $\overline{x} \in \overline{X}$ can be chosen arbitrarily. This establishes the first equation in (2).
Next, we show that

$$\forall\, \overline{Z} \subset \overline{X},\ \forall\, g \in G,\ \text{and}\ \forall\, \overline{x} \in \overline{Z} \cap \overline{Z}^g : \overline{x}^g = \overline{x}. \tag{3}$$

To prove this assertion consider $\overline{z} := \overline{x}^{g^{-1}}$. From $\overline{x} \in \overline{Z}^g$ follows $\overline{z} \in \overline{Z} \subset \overline{X}$, whence (a) yields $\overline{z} \in \overline{x}^G \cap \overline{X} = \{\overline{x}\}$. Thus $\overline{z} = \overline{x}$ which of course means $\overline{x}^g = \overline{x}$.
Now let $\overline{Y}$ be an $\overline{\mathcal{R}}$-transversal t-subset of $\overline{X}$. Denote by $\overline{B}$ one of the $\overline{\lambda}_t \geq 1$ blocks of the DD $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ containing the point set $\overline{Y}$. We claim that

$$\forall\, g \in G : \overline{Y} \subset \overline{B}^g \Leftrightarrow g \in G_{\overline{Y}}. \tag{4}$$

If $\overline{Y} \subset \overline{B}^g$ then $\overline{Y} \subset \overline{B} \cap \overline{B}^g$. We infer from (3), applied to $\overline{B} \subset \overline{X}$, that all elements of $\overline{B} \cap \overline{B}^g$ remain fixed under the action of g, whence $g \in G_{\overline{Y}}$; the converse is trivial. Next we describe the stabilizer of the subset $\overline{B}$ in the subgroup $G_{\overline{Y}}$. Taking into account that all our stabilizers are in fact pointwise stabilizers we read off from $\overline{Y} \subset \overline{B}$ that $G_{\overline{B}} \subset G_{\overline{Y}}$. This shows

$$G_{\overline{Y}} \cap G_{\overline{B}} = G_{\overline{B}}. \tag{5}$$

By combining (4) with (5) we see that the orbit $\overline{B}^G$ contains precisely $(\#G_{\overline{Y}})/(\#G_{\overline{B}})$ distinct subsets $\overline{B}^g$ passing through $\overline{Y}$.

If $\overline{B}' \neq \overline{B}$ is another block of $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ through $\overline{Y}$ then, by $\#\overline{B} = \#\overline{B}'$, there are elements $\overline{x} \in \overline{B} \setminus \overline{B}'$ and $\overline{x}' \in \overline{B}' \setminus \overline{B}$. As the G-orbits of $\overline{x}$ and $\overline{x}'$ are disjoint due to (a), so are the G-orbits of $\overline{B}$ and $\overline{B}'$. Consequently, the number of blocks in $\mathcal{B}$ containing $\overline{Y}$ equals the integer λ_t as defined in (2).

Finally, let $Y = \{y_1, y_2, \ldots, y_t\} \subset X$ be any $\mathcal{R}$-transversal t-subset. Define the t-subset $\widehat{Y} \subset \overline{X}$ as in (c). By the definition of $\mathcal{R}$, this $\widehat{Y}$ is an $\overline{\mathcal{R}}$-transversal t-subset of $\overline{X}$. So there is a $g \in G$ with $Y^g = \widehat{Y}$. Hence the number of blocks in $\mathcal{B}$ containing Y is λ_t, as required. $\square$

We shall refer to the t-DD $(X, \mathcal{B}, \mathcal{R})$ as a *t-lifting* of the t-DD $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ under the action of G. Clearly, $v := \#X = (\#x^G)\overline{v}$, where $\overline{v} := \#\overline{X}$ and $x \in X$ can be chosen arbitrarily. Note that we did not exclude the case $k = \overline{v}$ in the previous theorem. In this case the t-DD $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ is trivial, since $\overline{X}$ is its only block, and the lifted t-DD is transversal.

By construction, the group G acts as a group of automorphisms of the t-DD $(X, \mathcal{B}, \mathcal{R})$. The group G acts transitively on the set of blocks if, and only if, the base DD has a unique block.

As has been noted, (3) implies that for all sets $\overline{Z} \subset \overline{X}$ the *setwise* stabilizer $G_{\overline{Z}}$ coincides with the *pointwise* stabilizer of $\overline{Z}$ in G. It is therefore unambiguous to call $G_{\overline{Z}}$ just the *stabilizer* of $\overline{Z}$ in G, a terminology which is adopted below.

We recall from [27] that a t-DD can be obtained with Spera's construction if, and only if, it admits a group of automorphisms which acts transitively on the set of blocks and transitively on the set of transversal t-subsets of points. The following theorem states that under one additional condition the procedure of t-lifting preserves the property that a t-DD can be obtained with Spera's construction.

Theorem 2.6 *Let $\mathcal{D} = (X, \mathcal{B}, \mathcal{R})$ be the t-lifting of a t-divisible design $\overline{\mathcal{D}} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ under the action of G. Assume that there is a group $\overline{H}$ of automorphisms of $\overline{\mathcal{D}}$ which acts transitively on $\overline{\mathcal{B}}$ and transitively on the set of $\overline{\mathcal{R}}$-transversal t-subsets of $\overline{X}$. If each $\overline{h} \in \overline{H}$ can be extended to an automorphism of $\mathcal{D}$, then $\mathcal{D}$ admits a group of automorphisms which acts transitively on $\mathcal{B}$ and transitively on the set of $\mathcal{R}$-transversal t-subsets of X. Hence $\mathcal{D}$ can also be obtained with the construction of Spera* [27, Proposition 4.6].

Proof. Let $B_1, B_2 \in \mathcal{B}$ be blocks. So, by the definition of $\mathcal{B}$, there exist $g_1, g_2 \in G$ and $\overline{B}_1, \overline{B}_2 \in \overline{\mathcal{B}}$ with $B_i = \overline{B}_i^{g_i}$ for $i \in \{1, 2\}$. The assumption on $\overline{H}$ gives the existence of an automorphism h of $\mathcal{D}$ such that $\overline{B}_1^h = \overline{B}_2$. Hence $B_1^{g_1^{-1}hg_2} = B_2$, i.e., the automorphism group of $\mathcal{D}$ acts transitively on $\mathcal{B}$.

The transitivity of the automorphism group of $\mathcal{D}$ on the set of $\mathcal{R}$-transversal t-subsets of X can be shown similarly. $\square$

The following lemma gives a sufficient condition for an extension of an automorphism of $\overline{\mathcal{D}}$ to be an automorphism of $\mathcal{D}$. We shall use it in Theorem 3.4.

Lemma 2.7 *Let $\mathcal{D} = (X, \mathcal{B}, \mathcal{R})$ be the t-lifting of a t-divisible design $\overline{\mathcal{D}} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ under the action of G. Assume that an automorphism $\overline{h}$ of $\overline{\mathcal{D}}$ can be extended to a permutation h of X which normalizes the group of automorphisms of $\mathcal{D}$ induced by G. Then h is an automorphism of $\mathcal{D}$.*

Proof. Since h normalizes the automorphism group induced by G, the following holds: For each $g \in G$ there exists $g' \in G$ with $x^{gh} = x^{hg'}$ for all $x \in X$.
Let $B \in \mathcal{B}$ be a block. Hence $B = \overline{B}^g$ for some $g \in G$ and some block $\overline{B} \in \overline{\mathcal{B}}$. As $\overline{B}^h = \overline{B}^{\overline{h}}$ is a block, so is $B^h = \overline{B}^{gh} = \overline{B}^{hg'}$.
Suppose that C is a point class of $\mathcal{D}$. Hence $C = \bigcup_{g \in G} \overline{C}^g$ for some point class $\overline{C}$ of $\overline{\mathcal{D}}$. Therefore

$$C^h = \bigcup_{g \in G} \overline{C}^{gh} = \bigcup_{g' \in G} \overline{C}^{hg'} = \bigcup_{g' \in G} \overline{C}^{\overline{h}g'}$$

is also a point class of $\mathcal{D}$. □

The question arises, whether *proper t-liftings* (i.e. $\overline{X} \neq X$) do exist. The next theorem gives an answer.

Theorem 2.8 *Each t-divisible design $\overline{\mathcal{D}} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ can be used as base for a proper t-lifting.*

Proof. We may assume that $\overline{X} = \{1, 2, \ldots, \overline{v}\}$ is a set of integers. We fix an integer $w \geq 1$ and write $W := \{1, 2, \ldots, w\}$. Let $(G_i)_{i \in \overline{X}}$ be a family of subgroups (not necessarily distinct) of the symmetric group of W. Assume, furthermore, that each G_i acts transitively on W. We now define $X := \overline{X} \times W$, and then we identify $i \in \overline{X}$ with the pair $(i, 1) \in X$. Let G be the direct product $\prod_{i=1}^{\overline{v}} G_i$. An action of G on X is given by defining the image of (i, j) under $(g_1, g_2, \ldots, g_{\overline{v}})$ as (i, j^{g_i}). Obviously, conditions (a), (b), and (c) in Theorem 2.5 hold. Given an $\overline{\mathcal{R}}$-transversal u-subset $\overline{Z}$ we obtain that $\#\overline{Z}^G = w^u$. Therefore

$$\#G_{\overline{Z}} = \frac{\#G}{w^u},$$

whence also the remaining two conditions (d) and (e) are satisfied. So Theorem 2.5 can be applied. For $w > 1$ this yields a proper t-lifting. □

It should be noted that the lifted DD from the proof above allows an alternative description without referring to the group G: A subset of X is a block if, and only if, its projection on $\overline{X}$ is a block of $\overline{\mathcal{D}}$. The point classes of the lifted DD are the cartesian products of the point classes of $\overline{\mathcal{D}}$ with W.
We shall present other, less trivial, general constructions for proper t-liftings of an arbitrary t-DD in 3.10.

2.9 Let s be a positive integer and $\mathcal{D} = (X, \mathcal{B}, \mathcal{R})$ a t-DD. Given $Y \subset X$ denote by Y^* the set of all $x \in X$ for which there exists an $y \in Y$ with $x \mathrel{\mathcal{R}} y$. Then $\mathcal{D}$ is called *s-hypersimple* if for every block B and for every $\mathcal{R}$-transversal t-subset Y contained in B^* there exist exactly s blocks $B_1, B_2, \ldots, B_s$ containing Y and such that $B_i^* = B^*$ for each $i \in \{1, 2, \ldots, s\}$; see [28]. The t-liftings described in Theorem 2.5 are s-hypersimple with $s = \#G_Y / \#G_B$. It seems to be an open problem to find regular t-divisible designs with $t > 3$ and which are not s-hypersimple for any s.

3 Geometric examples of t-divisible designs for any t

In this chapter we focus our attention on t-DDs which arise from point sets in a finite projective or affine space.

Theorem 3.1 *Let t be a fixed positive integer and let $\overline{\mathcal{D}} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ be a t-$(\overline{s}, k, \overline{\lambda}_t)$ divisible design with the following properties:*

(i) *$\overline{X}$ is a set of $\overline{v}$ points generating a finite projective space* $\mathrm{PG}(d, q)$.

(ii) *All $\overline{\mathcal{R}}$-transversal t-subsets of $\overline{X}$ are independent in* $\mathrm{PG}(d, q)$.

(iii) *All blocks in $\overline{\mathcal{B}}$ generate subspaces of* $\mathrm{PG}(d, q)$ *with the same dimension $\beta - 1$.*

Then for each non-negative integer c there exists a t-$(q^c\overline{s}, k, q^{c(\beta-t)}\overline{\lambda}_t)$-divisible design with $q^c\overline{v}$ points.

Proof. Let c be a non-negative integer, $n := d + c$, and identify $\mathrm{PG}(d, q)$ with the subspace of $\mathrm{PG}(n, q)$ given by the linear system

$$x_{d+1} = x_{d+2} = \cdots = x_n = 0.$$

Furthermore, choose $S \subset \mathrm{PG}(n, q)$ to be the $(c - 1)$-dimensional subspace

$$x_0 = x_1 = \cdots = x_d = 0.$$

Next, let G be the multiplicative group formed by all upper triangular matrices of the form

$$\begin{pmatrix} I_{d+1} & M \\ 0 & I_c \end{pmatrix} \in \mathrm{GL}_{n+1}(q), \tag{6}$$

where M is any $(d + 1) \times c$ matrix with entries in $\mathbb{F}_q = \mathrm{GF}(q)$, I_* stands for an identity matrix of the indicated size, and 0 denotes a zero matrix of the appropriate size. The group G is elementary abelian, since it is isomorphic to the additive group of $(d+1) \times c$ matrices over $\mathbb{F}_q$. By writing the coordinates of points as row vectors, the group G acts in a natural way (from the right hand side) on $\mathrm{PG}(n, q)$ as a group of projective collineations. The subspace S is fixed pointwise, and every subspace of $\mathrm{PG}(n, q)$ containing S remains invariant, as a set of points. We obtain

$$\forall\, x \in \mathrm{PG}(n, q) \setminus S : x^G = (\{x\} \vee S) \setminus S, \tag{7}$$

i.e., the orbit of a point x not in S is the c-dimensional affine space which arises from the projective space $\{x\} \vee S$ by removing the subspace S. We define $\pi : \mathrm{PG}(n, q) \setminus S \to \mathrm{PG}(d, q)$ to be the projection through the centre S onto $\mathrm{PG}(d, q)$. By (7), two points of $\mathrm{PG}(n, q) \setminus S$ are in the same G-orbit if, and only if, their images under π coincide.

We shall frequently make use of the following *auxiliary result*. Let Q be an independent $(d + 1)$-subset of $\mathrm{PG}(n, q)$ which together with S generates $\mathrm{PG}(n, q)$. We claim that there is a unique matrix in G taking each element of Q to its image under π. In order to show this assertion, we choose a $(d + 1) \times (d + 1)$ matrix L and a $(d + 1) \times c$ matrix M in such a way that the rows of $(L\ M)$ represent the points of Q (written in some fixed order). Consequently, the rows of the matrix $(L\ 0)$ represent the $(d+1)$ points of Q^π (ordered accordingly). By the exchange lemma, the points of Q^π are also independent, whence L is invertible. We infer from

$$\begin{pmatrix} L & M \end{pmatrix} \underbrace{\begin{pmatrix} I_{d+1} & -L^{-1}M \\ 0 & I_c \end{pmatrix}}_{:=g} = \begin{pmatrix} L & 0 \end{pmatrix} \tag{8}$$

that $g \in G$ takes each point $x \in Q$ to $x^\pi \in Q^\pi$. Conversely, if a matrix $\tilde{g} \in G$ takes Q to Q^π then $(L\ M) \cdot \tilde{g} = (L\ 0)$, so $\tilde{g} = g$.

Finally, we define X as the union of all orbits $\overline{x}^G$, where $\overline{x}$ ranges in $\overline{X}$, and proceed by showing that the assumptions (a)–(e) of Theorem 2.5 are satisfied:

Ad (a): By (7), the projection π maps each $x \in X$ to the only element $\widehat{x} \in \overline{X}$ with the required property.

Ad (b): All orbits $\overline{x}^G$, where $\overline{x} \in \overline{X}$, have size q^c according to (7).

Ad (c): Let Y be a subset of X, such that $\widehat{Y}$ is an $\overline{\mathcal{R}}$-transversal t-subset of $\overline{X}$. Due to our assumption (ii), the projected t-subset $Y^\pi = \widehat{Y}$ of $\overline{X}$ is independent. Thus it can be extended to a basis of $\mathrm{PG}(d,q)$ by adding a $(d-t+1)$-subset P. The set Y is independent because its projection is independent. Moreover, $Q := Y \cup P$ meets the requirement from our auxiliary result. Now the matrix g from (8) takes Y to $\widehat{Y}$.

Ad (d): First, let $Y' \subset \mathrm{PG}(d,q)$ be the t-set of points given by the first t vectors of the canonical basis of $\mathbb{F}_q^{d+1}$. So the pointwise stabilizer of Y' in G consists of all matrices

$$\begin{pmatrix} I_t & 0 & 0 \\ 0 & I_{d-t+1} & K \\ 0 & 0 & I_c \end{pmatrix}, \tag{9}$$

with an arbitrary $(d-t+1) \times c$ submatrix K over $\mathbb{F}_q$. Obviously, the pointwise and the setwise stabilizers of Y' in G coincide.

Next, suppose that $\overline{Y} \subset \overline{X}$ is an $\overline{\mathcal{R}}$-transversal t-subset, whence $\overline{Y}$ is independent. So $\overline{Y}$ can be extended to a basis of $\mathrm{PG}(d,q)$. There exists a $(d+1) \times (n+1)$ matrix of the form $(L\ 0)$ whose rows represent the points of the chosen basis. Thereby it can be assumed that the first t rows are representatives for $\overline{Y}$. We read off from

$$\begin{pmatrix} L^{-1} & 0 \\ 0 & I_c \end{pmatrix} \begin{pmatrix} I_{d+1} & M \\ 0 & I_c \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & I_c \end{pmatrix} = \begin{pmatrix} I_{d+1} & L^{-1}M \\ 0 & I_c \end{pmatrix},$$

where M is arbitrary, that

$$G = \begin{pmatrix} L^{-1} & 0 \\ 0 & I_c \end{pmatrix} G \begin{pmatrix} L & 0 \\ 0 & I_c \end{pmatrix} \quad \text{and} \quad G_{\overline{Y}} = \begin{pmatrix} L^{-1} & 0 \\ 0 & I_c \end{pmatrix} G_{Y'} \begin{pmatrix} L & 0 \\ 0 & I_c \end{pmatrix}.$$

Hence $\#G_{\overline{Y}}$ does not depend on the choice of $\overline{Y}$, and (9) shows that

$$\#G_{\overline{Y}} = q^{c(d-t+1)}. \tag{10}$$

Ad (e): Choose any block $\overline{B} \in \overline{\mathcal{B}}$. There exists an independent β-subset $\overline{Z} \subset \overline{B}$. The setwise and the pointwise stabilizers of $\overline{Z}$ and $\overline{B}$ in G are all the same. We may now proceed as in the proof of (d), with t, Y', and $\overline{Y}$ to be replaced by β, an adequate β-set Z', and $\overline{Z}$, respectively. Then (10) gives that

$$\#G_{\overline{B}} = q^{c(d-\beta+1)} \tag{11}$$

has a constant value.

Now $\lambda_t = q^{c(\beta-t)}\overline{\lambda}_t$ is immediate from (2), (10), and (11). □

Let us add some remarks on Theorem 3.1.

3.2 The only reason for including condition (i) is to simplify matters. We could also drop it and carry out our construction in the join of S and the subspace generated by $\overline{X}$.

It is easily seen that the t-lifting process of Theorem 3.1 can be iterated. Given a base t-DD we may first apply a t-lifting for some fixed integer $c_1 > 0$. This gives a second t-DD which can be used as the base DD for a second t-lifting for some fixed integer $c_2 > 0$. The t-DD obtained in this way may also be reached in a single step from the initial base DD by applying a t-lifting with the integer $c := c_1 + c_2$.

Suppose that $t = 2$, $c = 1$. By removing the assumption (i), we obtain a variation of Theorem 3.1 which yields once more results from [11, Theorem 3.2.7]. In order illustrate how the settings in [11] (hyperplanes of an affine space, translation group) correspond to our settings, we merely have to adopt a dual point of view: Each point p of $\mathrm{PG}(n, q)$ gives rise to the star of hyperplanes of $\mathrm{PG}(n, q)$ with vertex p or, said differently, a single hyperplane of $\mathrm{PG}(n, q)^*$. In this way we obtain a bijective correspondence of $\mathrm{PG}(n, q)$ (as a set of points) with the set of hyperplanes of its dual space $\mathrm{PG}(n, q)^*$. Due to $c = 1$ the subspace S corresponds to a hyperplane of $\mathrm{PG}(n, q)^*$ which can be considered as being at infinity. The group G acts on the dual space as the corresponding translation group. For an arbitrary t and $c = 1$ our Theorem improves [11, Proposition 3.2.9].

There is a particular case, where we can give an alternative description of the divisible design $(X, \mathcal{B}, \mathcal{R})$ from Theorem 3.1.

Corollary 3.3 *Let t be any positive integer and let $\overline{X}$ be a k-set of points generating the projective space $\mathrm{PG}(d, q)$, such that each t-subset of $\overline{X}$ is independent, where $t \le k$. We embed $\mathrm{PG}(d, q)$ as a subspace in $\mathrm{PG}(n, q)$, where $n = d + c$ for some positive integer c, and choose any subspace S of $\mathrm{PG}(n, q)$ complementary with $\mathrm{PG}(d, q)$. Define $(X, \mathcal{B}, \mathcal{R})$ as follows.*

(i) *X is the cone with basis $\overline{X}$ and vertex S, but without its vertex S.*

(ii) *$\mathcal{B}$ is the set of all sections $X \cap D$, where D is complementary with S.*

(iii) *$\mathcal{R} := \{(x, x') \in X \times X \mid \{x\} \vee S = \{x'\} \vee S\}$.*

This $(X, \mathcal{B}, \mathcal{R})$ is a transversal t-$(q^c, k, q^{c(d-t+1)})$-divisible design.

Proof. Let $\overline{\mathcal{B}} := \{\overline{X}\}$ and let $\overline{\mathcal{R}}$ be the diagonal relation on $\overline{X}$. The triple $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ is a trivial transversal t-$(1, k, 1)$-DD with $\overline{v} = k$ points and just one block. Define $(X, \mathcal{B}, \mathcal{R})$ as in the proof of Theorem 3.1, where $\beta = d + 1$. By (7), the point set X and the equivalence relation $\mathcal{R}$ can be described as in (i) and (iii), respectively. The auxiliary result in the proof of Theorem 3.1 shows that G acts transitively on the set of complements of S, whence (ii) characterizes the set of blocks. □

Next, we compare the lifting from the proof of Theorem 3.1 with Spera's construction.

Theorem 3.4 *Under the assumptions of Theorem* 3.1 *suppose that there exists a group* $\overline{\Gamma}$ *of collineations of* $\mathrm{PG}(d,q)$ *which acts on* $\overline{X}$ *as an automorphism group of the base* t*-DD* $\overline{\mathcal{D}}$*. Furthermore, we assume that* $\overline{\Gamma}$ *acts transitively on the set* $\overline{\mathcal{B}}$ *of blocks and transitively on the set of* $\overline{\mathcal{R}}$*-transversal* t*-subsets of* $\overline{X}$*. Then the* t*-lifting from the proof of Theorem* 3.1 *yields* t*-divisible designs which can also be obtained with Spera's construction* [27, Proposition 4.6].

Proof. Let $\overline{J} \subset \Gamma\mathrm{L}_{d+1}(q)$ be the group of those semilinear bijections which give rise to collineations in $\overline{\Gamma}$. (In our setting $\Gamma\mathrm{L}_{d+1}(q) = \mathrm{GL}_{d+1}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$, i.e., a semilinear transformation appears as a pair consisting of a regular matrix and an automorphism of $\mathbb{F}_q$.) Then

$$J := \{(\mathrm{diag}(P, I_c), \zeta) \mid (P,\zeta) \in \overline{J}\} \subset \Gamma\mathrm{L}_{n+1}(q)$$

is a group of semilinear transformations which yields a collineation group of $\mathrm{PG}(n,q)$, say Γ. For each $\overline{\gamma} \in \overline{\Gamma}$ there is at least one extension in Γ. Since $\overline{X}$ and S remain invariant under the collineations in Γ, so does the set X. A straightforward computation shows that

$$j^{-1}Gj = G \text{ for all } j \in J; \tag{12}$$

here we identify each $g \in G$ with $(g, \mathrm{id}_{\mathbb{F}_q}) \in \Gamma\mathrm{L}_{n+1}(q)$. We infer from Lemma 2.7 that Γ acts on X as an automorphism group of the lifted t-DD $\mathcal{D}$. Thus Theorem 2.6 can be applied to the automorphism group of $\overline{\mathcal{D}}$ given by $\overline{\Gamma}$. Altogether, we obtain the required result: Spera's construction can be applied to X, $\mathcal{R}$, an arbitrarily chosen $\overline{B} \in \overline{\mathcal{B}}$ as base block, and the group $\langle G, J\rangle$ of semilinear transformations generated by G and J. □

If the collineation group $\overline{\Gamma}$ from the above has the additional property to act transitively on the set of $\overline{\mathcal{R}}$-transversal t-tuples of $\overline{X}$ then $\langle G, J\rangle$ will even act transitively on the set of $\mathcal{R}$-transversal t-tuples of X. For, if $(y_1, y_2, \ldots, y_t)$ is such a t-tuple then there is an element $g \in G$ taking $(y_1, y_2, \ldots, y_t)$ to the $\overline{\mathcal{R}}$-transversal t-tuple $(y_1^g, y_2^g, \ldots, y_t^g)$ according to assumption (c) in Theorem 2.5.

Examples 3.5 (a) The *small Witt design* $W_{12} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ is a 5-$(1,6,1)$-DD (i.e. a design) with $\overline{v} = 12$ points. By a result of H. S. M. Coxeter [10], W_{12} can be embedded in $\mathrm{PG}(5,3)$ in such a way that the following properties hold: (i) $\overline{X}$ generates $\mathrm{PG}(5,3)$. (ii) All 5-subsets of $\overline{X}$ are independent. (iii) All blocks span hyperplanes of $\mathrm{PG}(5,3)$. In fact, the blocks are those 132 hyperplane sections of $\overline{X}$ which contain more than three points of $\overline{X}$. We refer to [13], [22], [31], and [32] for further properties of this model of W_{12}.

We can apply Theorem 3.1 to construct 5-$(3^c,6,1)$-DDs with $12 \cdot 3^c$ points from W_{12}.

By [10], each automorphism of W_{12} can be extended in a unique way to a a collineation of $\mathrm{PG}(5,3)$ leaving invariant the set $\overline{X}$. The automorphism group of W_{12} is the Mathieu group M_{12}. So we have a collineation group $\overline{\Gamma}$ which acts sharply 5-transitively on $\overline{X}$. Since each block is uniquely determined by five of its points, all blocks are in one orbit of $\overline{\Gamma}$. By Theorem 3.4, this implies that the lifted 5-DDs could also be obtained with the construction of Spera.

(b) Let $\overline{X}$ be as in (a). Corollary 3.3, applied to the set $\overline{X}$, yields the existence of 5-$(3^c, 12, 3^c)$-DDs with the same set of points and the same point classes as in (a), but with a different set of blocks. As before, the lifted DDs could also be obtained with the construction of Spera.

(c) The *large Witt design* $W_{24} = (\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ is a 5-$(1,8,1)$-DD (i.e. a design) with $\overline{v} = 24$ points and 758 blocks. An embedding in $\mathrm{PG}(11,2)$ is due to J. A. Todd [31]. It has the following properties: (i) $\overline{X}$ generates $\mathrm{PG}(11,2)$. (ii) All 5-subsets of $\overline{X}$ are independent. (iii) All blocks span 6-dimensional subspaces of $\mathrm{PG}(11,2)$. The automorphism group of W_{24} is the Mathieu group M_{24} which acts 5-transitively on the point set of W_{24}. Each automorphism of W_{24} extends to a unique collineation of $\mathrm{PG}(11,2)$; see [31]. Mutatis mutandis, it is now possible to proceed as in (a) and (b).

(d) Any field extension $\mathbb{F}_{q^h}/\mathbb{F}_q$, $h > 1$, gives rise to a *chain geometry* $\Sigma(\mathbb{F}_q, \mathbb{F}_{q^h})$; see, for example, [3, pp. 40–41] ("Möbiusraum") or [17]. Such a chain geometry is a 3-$(1, q+1, 1)$-DD (i.e. a design) with $q^h + 1$ points. We speak of chains rather than blocks in this context. The following is due to G. Lunardon [21, p. 307]: This design can be embedded in $\mathrm{PG}(2^h - 1, q)$ as an algebraic variety, say $\overline{X}$, called an h-*sphere*. Any three distinct points of $\overline{X}$ are independent. Furthermore, all its chains span subspaces with a constant dimension $\min\{q, h\}$. (The chains on the h-sphere are normal rational curves; see 3.6 below.) Hence Theorem 3.1 can be applied to construct 3-DDs from this embedded chain geometry. Observe that it remains open from [21] whether or not $\overline{X}$ will always generate $\mathrm{PG}(2^h - 1, q)$.

Each semilinear automorphism of this chain geometry extends to a collineation of $\mathrm{PG}(2^h - 1, q)$. The group of these collineations meets the conditions from Theorem 3.4, whence one could also apply Spera's construction to obtain the lifted 3-DDs.

We add in passing that for $h = 2$ an h-sphere is just an elliptic quadric in $\mathrm{PG}(3, q)$ and the associated design is a miquelian Möbius plane. Cf. also [11, pp. 48–50], where the case $h = 2$, $c = 1$, q odd is treated from a dual point of view.

If we disregard the chains on the h-sphere then Corollary 3.3 gives a 3-DD with block size $q^h + 1$.

(e) Any generating set $\overline{X}$ of $\mathrm{PG}(d, q)$ yields a 2-DD according to Corollary 3.3.

3.6 We proceed by showing that the assumptions of Corollary 3.3 can be realized for each integer $t \geq 2$ if $\overline{X}$ is chosen as an appropriate *Veronese variety*.

Suppose that three integers $c, m \geq 1$, $t \geq 2$, and a finite field $\mathbb{F}_q$ are given. We let $d = \binom{m+t-1}{m} - 1$ and consider the projective space $\mathrm{PG}(d, q)$. Its $d+1$ coordinates will be indexed by the set $E_{m,t-1}$ of all sequences $e = (e_0, e_1, \ldots, e_m)$ of non-negative integers satisfying $e_0 + e_1 + \cdots + e_m = t-1$; the coordinates are written in some fixed order. The *Veronese mapping* is given by

$$v_{m,t-1} : \mathrm{PG}(m, q) \to \mathrm{PG}(d, q) : \mathbb{F}_q(x_0, x_1, \ldots, x_m) \mapsto \mathbb{F}_q(\ldots, y_{e_0,e_1,\ldots,e_m}, \ldots), \tag{13}$$

where $y_{e_0,e_1,\ldots,e_m} := x_0^{e_0} x_1^{e_1} \cdots x_m^{e_m}$. Its image is known as a *Veronese variety* (or, for short a *Veronesean*) $\mathcal{V}_{m,t-1}(q)$. A $\mathcal{V}_{1,t-1}$ is also called a *normal rational curve*.

There is a widespread literature on Veronese varieties. We refer to [16] for a coordinate-free definition of the Veronese mapping which allows to derive its essential properties in a very elegant way. See also [15]. The case of a finite ground field is presented in [18, Chapter 25] for $t = 3$, and in [9] for arbitrary t. Many references, in particular to the older literature (over the real and complex numbers), can also be found in [14].

For the reader's convenience we present now two results together with their short proofs. The first coincides with [9, Corollary 2.6], the second seems to be part of the folklore.

Lemma 3.7 *The following assertions hold:*

(a) *The Veronesean $\mathcal{V}_{m,t-1}(q)$ spans $\mathrm{PG}(d,q)$ if, and only if, $t \le q+1$.*

(b) *The Veronese mapping* (13) *maps any $t \ge 2$ distinct points of $\mathrm{PG}(m,q)$ to t independent points of $\mathrm{PG}(d,q)$.*

Proof. Ad (a): Each family $(a_e)_{e\in E_{m,t-1}}$ with entries in $\mathbb{F}_q$, but not all zero, corresponds in $\mathrm{PG}(d,q)$ to a hyperplane, say H, with equation $\sum_{e\in E_{m,t-1}} a_e y_e = 0$, and in $\mathrm{PG}(m,q)$ to an algebraic hypersurface, say $\mathcal{F}$, with degree $t-1$ which is given by

$$\sum_{e\in E_{m,t-1}} a_{e_0,e_1,\dots,e_m} x_0^{e_0} x_1^{e_1} \cdots x_m^{e_m} = 0.$$

A point p of $\mathrm{PG}(m,q)$ is in $\mathcal{F}$ if, and only if, its Veronese image is in H. Clearly, all hyperplanes of $\mathrm{PG}(d,q)$ and all hypersurfaces with degree $t-1$ of $\mathrm{PG}(m,q)$ arise in this way.

By a result of G. Tallini [30, p. 433–434] there are hypersurfaces of any degree $\ge q+1$ containing *all* points of $\mathrm{PG}(m,q)$, but no such hypersurfaces of degree less than $q+1$. By the above, this means that $\mathcal{V}_{m,t-1}(q)$ does not span $\mathrm{PG}(d,q)$ precisely when $t-1 \ge q+1$.

Ad (b): Let $p_1, p_2, \dots, p_t$ be $t \ge 2$ distinct points of $\mathrm{PG}(m,q)$. Choose one of them, say p_t. There exist (not necessarily distinct) hyperplanes Z_i of $\mathrm{PG}(m,q)$, such that $p_i \in Z_i$ and $p_t \notin Z_i$ for all $i \in \{1,2,\dots,t-1\}$. If $\sum_j c_{ij}x_j = 0$ are equations for the Z_is then $\prod_{i=1}^{t-1}(\sum_j c_{ij}x_j) = 0$ gives a hypersurface $\mathcal{F}$ of degree $t-1$ which contains $p_1, p_2, \dots p_{t-1}$, but not p_t. We infer from the the proof of (a) that there is a hyperplane H of $\mathrm{PG}(d,q)$ which contains the Veronese images of $p_1, p_2, \dots p_{t-1}$, but not the image of p_t. Thus the image of p_t is not in the span of the remaining image points. □

Theorem 3.8 *For any integer $t \ge 2$ there exist infinitely many non-isomorphic transversal t-divisible designs.*

Proof. Fix any $t \ge 2$ and choose any integer $m \ge 1$. There is a prime power q such that $t \le q+1$. The Veronesean $\mathcal{V}_{m,t-1}$ has $k := q^m + q^{m-1} + \dots + 1 \ge q+1 \ge t$ points, and it spans $\mathrm{PG}(d,q)$ by Lemma 3.7 (a). We read off from Lemma 3.7 (b) that any t points of $\mathcal{V}_{m,t-1} =: \overline{X}$ are independent. So the assumptions of Corollary 3.3 are satisfied. As c runs in the set of non-negative integers, we obtain infinitely many non-isomorphic transversal t-$(q^c, k, q^{c(d-t+1)})$-DDs. □

Letting $m = c = 1$ in the above proof yields a DD which is contained in a cone with a one-point vertex over a normal rational curve $\mathcal{V}_{1,t-1}$ in $\mathrm{PG}(t-1,q)$. These DDs are finite analogues of *tubular circle planes* [23, p. 398]. We refer also to [7] (dual point of view) and [12] for the case when $m = c = 1$ and $t = 3$.

An alternative proof of Theorem 3.8 is provided by the construction from Theorem 2.8. One may start there with a trivial t-DD with point set $\overline{X} := \{1,2,\dots,\overline{v}\}$, $\overline{\mathcal{B}} := \{\overline{X}\}$, and the diagonal relation as $\overline{\mathcal{R}}$. Then, as w varies in the set of non-negative integers, infinitely many non-isomorphic t-DDs are obtained. However, this approach gives trivial t-DDs, because *every* $\mathcal{R}$-transversal $\overline{v}$-subset of such a t-DD turns out to be a block. The DDs which arise from the proof of 3.8 are trivial if, and only if, the Veronesean $\mathcal{V}_{m,t-1}$ is a basis of $\mathrm{PG}(d,q)$, i.e. for $k = d+1$.

In the previous proof we could also choose $\overline{X}$ to be a *subset* of $\mathcal{V}_{m-1,t}$ with at least t elements. This would also give a t-DD by applying the construction of Corollary 3.3 to the subspace generated by $\overline{X}$. We confine our attention to one particular case.

Example 3.9 In $\mathrm{PG}(d,q)$, i.e. the ambient space of the Veronesean $\mathcal{V}_{m,t-1}$, let us arrange the coordinates in such a way that the first $m+1$ coordinates belong to the sequences

$$(t-1,0,0,\dots 0),(t-2,1,0,\dots 0),\dots,(t-2,0,\dots,0,1)\in E_{m,t-1}.$$

The order of the remaining coordinates is immaterial. As before, we embed $\mathrm{PG}(m,q)$ via the Veronese mapping (13) in $\mathrm{PG}(d,q)$, and then $\mathrm{PG}(d,q)$ in $\mathrm{PG}(n,q)$ via the canonical embedding (cf. the proof of Theorem 3.1). Furthermore, we turn $\mathrm{PG}(m,q)$ into an affine space by considering $x_0=0$ as its *hyperplane at infinity*. The Veronese image of an affine point $\mathbb{F}_q(1,x_1,x_2,\dots x_m)$ is

$$\mathbb{F}_q(1,x_1,x_2,\dots x_m,\underbrace{*,\dots,*}_{d-m},\underbrace{0,0,\dots,0}_{c}).$$

Here the entries marked with an asterisk are polynomials in $x_1,x_2,\dots,x_m$. Let $\overline{X}$ be the set of all such points.
The minimum degree of a hypersurface in $\mathrm{AG}(m,q)$ containing *all* points of $\mathrm{AG}(m,q)$ is q. The proof is similar to the one for the projective case [30]. So, provided that $t\le q$, the set $\overline{X}$ spans $\mathrm{PG}(d,q)$; see also Lemma 3.7 (a). Hence, for $t\le q$ we obtain a t-$(q^c,q^m,q^{c(d-t+1)})$-DD by applying Corollary 3.3.
The action of G on $X=\overline{X}^G$ is as follows: Any matrix $g:=\left(\begin{smallmatrix} I_{d+1} & M\\ 0 & I_c\end{smallmatrix}\right)$ as in (6) takes

$$\mathbb{F}_q(1,x_1,x_2,\dots x_m,\underbrace{*,\dots,*}_{d-m},y_1,y_2,\dots,y_c), \tag{14}$$

to

$$\mathbb{F}_q(1,x_1,x_2,\dots x_m,\underbrace{*,\dots,*}_{d-m},y_1+P_1,y_2+P_2,\dots,y_c+P_c), \tag{15}$$

where each P_j, $j\in\{1,2,\dots,c\}$, denotes a polynomial in $x_1,x_2,\dots,x_m$ with degree $\le t-1$. The coefficients of P_j are the entries in the jth column of M.
However, this DD admits an alternative description which avoids Veroneseans and projective spaces. We simply delete the block of $d-m$ coordinates and go over to inhomogeneous coordinates in (14) and (15). This amounts to applying a projection which maps X bijectively onto $\mathrm{AG}(m+c,q)$. We use this bijection to obtain an isomorphic DD and an isomorphic action of the group G on $\mathrm{AG}(m+c,q)$. It is given by

$$(x_1,x_2,\dots x_m,y_1,y_2,\dots,y_c)\xmapsto{g}(x_1,x_2,\dots x_m,y_1+P_1,y_2+P_2,\dots,y_c+P_c).$$

Hence the blocks of $\mathrm{AG}(m+c,q)$ are precisely the graphs of all the c-tuples of polynomial functions $\mathbb{F}_q^m\to\mathbb{F}_q$ with degree $\le t-1$, whereas the point classes are the cosets of the subspace $x_1=x_2=\dots=x_m=0$ in $\mathbb{F}_q^{m+c}$. In particular, when $m=c=1$ then the unique block through an $\mathcal{R}$-transversal t-subset of $\mathrm{AG}(2,q)$ is just the graph of the polynomial function with degree $\le t-1$ which is obtained by the interpolation formula of Lagrange. Compare with [23, p. 399–400] for similar results over the real numbers. See also [20] for a detailed investigation of this “geometry of polynomials”.

Example 3.10 Let $(\overline{X}, \overline{\mathcal{B}}, \overline{\mathcal{R}})$ be any t-DD with $\overline{v}$ points, $t \geq 2$. There is a prime power q such that $q+1 \geq \overline{v} \geq t$. We consider the normal rational curve $\mathcal{V}_{1,t-1}$ in $\mathrm{PG}(t-1, q)$; it has $q+1$ points. So we can identify $\overline{X}$ with a subset of $\mathcal{V}_{1,t-1}$. Now it is easy to verify the conditions from Theorem 3.1, because any t distinct points of $\overline{X}$ form a basis of $\mathrm{PG}(t-1, q)$.
When $t = 2$ then $\mathcal{V}_{1,t-1} = \mathrm{PG}(1, q)$ is a projective line. In this particular case the result can be found in [11, Bemerkung 3.2.2].

Example 3.11 Let $\mathcal{C}$ be a $[\nu, \kappa]$-linear code on $\mathbb{F}_q$ of minimum weight $t+1 \geq 3$. It is well known (cf. for example [4]) that $\mathcal{C}$ is associated with a ν-set, say $\overline{X}$, of points in $\mathrm{PG}(\nu-\kappa-1, q)$, such that every t-subset of $\overline{X}$ is independent and there exists a dependent $(t+1)$-subset of $\overline{X}$. By Corollary 3.3, for each $c \geq 1$ we obtain a transversal t-$(q^c, \nu, q^{c(\nu-\kappa-t)})$-DD.
On the other hand, each t-DD determines a constant weight code. See [26] and the references given there. Thus, according to our construction, we can link two concepts from coding theory and it would be interesting to know more about this connection.

3.12 In order to apply the construction of DDs according to Theorem 3.1 with an appropriate t one could also embed a given DD in an arc, an oval, a hyperoval, an ovoid, a cap of kind $t-1$ (any t points are independent), etc. Thus many more DDs can be constructed.
The group G used in the proof of Theorem 3.1 is elementary abelian and it yields a so-called *dual translation group* of the lifted DD. See [11, Chapter 5], where characterizations of DDs admitting such a group can also be found.
Another promising setting for a 3-lifting (according to Theorem 2.5) could be to use the projective line over a finite (not necessarily commutative) local ring as X, and a suitable subgroup of the general linear group $\mathrm{GL}_2(R)$ as G. Such a group need not be elementary abelian. Here some overlap with the work of Spera [28], who considered the projective line over a finite local algebra and the full group $\mathrm{GL}_2(R)$, is to be expected.

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Author's addresses:

Andrea Blunck, Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.
`andrea.blunck@math.uni-hamburg.de`

Hans Havlicek, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria.
`havlicek@geometrie.tuwien.ac.at`

Corrado Zanella, Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università di Padova, Stradella S. Nicola, 3, I-36100 Vicenza, Italy.
`corrado.zanella@unipd.it`