On linear morphisms of product spaces*

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Abstract

Let χ be a linear morphism of the product of two projective spaces $\mathrm{PG}(n, F)$ and $\mathrm{PG}(m, F)$ into a projective space. Let γ be the Segre embedding of such a product. In this paper we give some sufficient conditions for the existence of an automorphism α of the product space and a linear morphism of projective spaces φ, such that $\gamma\varphi = \alpha\chi$.
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1 Introduction

A *semilinear space* is a pair $\Sigma = (\mathcal{P}, \mathcal{G})$, where $\mathcal{P}$ is a set, whose elements are called *points*, and $\mathcal{G} \subset 2^{\mathcal{P}}$. (In this paper "$A \subset B$" just means that $x \in A$ implies $x \in B$.) The elements of $\mathcal{G}$ are *lines*. The axioms defining a semilinear space are the following: (*i*) $|g| \geq 2$ for every line g; (*ii*) $\cup_{g \in \mathcal{G}} g = \mathcal{P}$; (*iii*) $g, h \in \mathcal{G}$, $g \neq h \Rightarrow |g \cap h| \leq 1$. Two points $X, Y \in \mathcal{P}$ are *collinear*, $X \sim Y$, if a line g exists such that $X, Y \in g$ (for $X \neq Y$ we will also write $XY := g$). An *isomorphism* between the semilinear spaces $(\mathcal{P}, \mathcal{G})$ and $(\mathcal{P}', \mathcal{G}')$ is a bijection $\alpha : \mathcal{P} \to \mathcal{P}'$ such that both α and α^{-1} map lines onto lines.

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The *join* of $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{P}$ is:

$$\mathcal{M}_1 \vee \mathcal{M}_2 := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \Big(\bigcup_{\substack{X_i \in \mathcal{M}_i \\ X_1 \sim X_2,\, X_1 \neq X_2}} X_1 X_2 \Big).$$

If X is a point, we will often write X instead of $\{X\}$. Let $\mathbb{P}' = (\mathcal{P}', \mathcal{G}')$ be a projective space. A *linear morphism* $\chi : \Sigma \to \mathbb{P}'$ is a mapping of a subset $\mathbf{D}(\chi)$ of $\mathcal{P}$ into $\mathcal{P}'$ satisfying the following axioms (L1) and (L2) [1, 2]. Here $X\chi = \emptyset$ for $X \in \mathbf{A}(\chi) := \mathcal{P} \setminus \mathbf{D}(\chi)$.

(L1) $(X \vee Y)\chi = X\chi \vee Y\chi$ for $X, Y \in \mathcal{P}$, $X \sim Y$;

(L2) $X, Y \in \mathcal{P}$, $X\chi = Y\chi$, $X \neq Y$, $X \sim Y \Rightarrow \exists A \in XY$ such that $A\chi = \emptyset$.

$\mathbf{D}(\chi)$ is the *domain* of χ and $\mathbf{A}(\chi)$ is the *exceptional set*. The linear morphism χ is said to be *global* when $\mathbf{D}(\chi) = \mathcal{P}$; is called *embedding* if it is global and injective. It should be noted the last definition is somewhat particular, since for instance the inclusion of an affine space into its projective extension is not an embedding.

Throughout this paper we will deal with the following: a commutative field F; two natural numbers $n \geq 2$ and $m \geq 1$; $\mathbb{P}_1 = \mathrm{PG}(n, F) = (\mathcal{P}_1, \mathcal{G}_1)$, that is the projective space of dimension n over F; $\mathbb{P}_2 = \mathrm{PG}(m, F) = (\mathcal{P}_2, \mathcal{G}_2)$; $\overline{\mathbb{P}} = \mathrm{PG}(nm + n + m, F) = (\overline{\mathcal{P}}, \overline{\mathcal{G}})$; a further projective space $\mathbb{P}' = (\mathcal{P}', \mathcal{G}')$.

The *product* of $\mathbb{P}_1$ and $\mathbb{P}_2$ is the semilinear space $\mathbb{P}_1 \times \mathbb{P}_2 = (\mathcal{P}^*, \mathcal{G}^*)$, where $\mathcal{P}^* = \mathcal{P}_1 \times \mathcal{P}_2$, and the elements of $\mathcal{G}^*$ are of two kinds: $X_1 \times g_2$ with $X_1 \in \mathcal{P}_1$, $g_2 \in \mathcal{G}_2$, and $g_1 \times X_2$ with $g_1 \in \mathcal{G}_1$, $X_2 \in \mathcal{P}_2$.

We shall be concerned with a linear morphism $\chi : \mathbb{P}_1 \times \mathbb{P}_2 \to \mathbb{P}'$, and a *regular* embedding $\gamma : \mathbb{P}_1 \times \mathbb{P}_2 \to \overline{\mathbb{P}}$. The word "regular" means that the projective closure $[(\mathcal{P}_1 \times \mathcal{P}_2)\gamma]$ of $(\mathcal{P}_1 \times \mathcal{P}_2)\gamma$ has dimension $mn + m + n$ (which is the greatest possible one, see [3]). Our purpose is to generalize the main result of [3], which is the following: if χ is an embedding, then there exist an automorphism α' of $\mathbb{P}_2$ and a linear morphism $\varphi : \overline{\mathbb{P}} \to \mathbb{P}'$ such that $\gamma\varphi = (\mathrm{id}_{\mathcal{P}_1}, \alpha')\chi$.

We define the *first* and *second radical* of χ by

$$\begin{aligned} \mathrm{rad}_1\chi &:= \{X \in \mathcal{P}_1 | X \times \mathcal{P}_2 \subset \mathbf{A}(\chi)\}; \\ \mathrm{rad}_2\chi &:= \{Y \in \mathcal{P}_2 | \mathcal{P}_1 \times Y \subset \mathbf{A}(\chi)\}. \end{aligned}$$

χ is a *degenerate* linear morphism if $\mathrm{rad}_1\chi \cup \mathrm{rad}_2\chi \neq \emptyset$.

Clearly each radical $\mathrm{rad}_i\chi$ is a subspace of $\mathbb{P}_i$, $i = 1, 2$. The following proposition can be useful in reducing the investigation to nondegenerate linear morphisms.

Proposition 1.1 *Let* $\mathrm{rad}_i\chi$ *and* D_i *be complementary subspaces of* $\mathbb{P}_i$ *(*$i = 1, 2$*). Let* $\pi_i : \mathbb{P}_i \to D_i$ *be the projection onto* D_i *from* $\mathrm{rad}_i\chi$*, and* $\chi' := \chi_{|D_1 \times D_2}$*. Then, for every* $(X, Y) \in \mathcal{P}_1 \times \mathcal{P}_2$*, it holds:*

$$\{(X, Y)\}\chi = (X\pi_1 \times Y\pi_2)\chi'. \tag{1}$$

Proof. If $X \in \mathrm{rad}_1\chi$ or $Y \in \mathrm{rad}_2\chi$ the statement is trivial. Otherwise, let $X' \in \mathrm{rad}_1\chi$ such that X, X' and $X\pi_1$ are collinear. We have $\{(X, Y)\}\chi \in \{(X', Y)\}\chi \vee \{(X\pi_1, Y)\}\chi = \{(X\pi_1, Y)\}\chi$; since $X \neq X'$, the set $\{(X, Y)\}\chi$ is empty if and only if $\{(X\pi_1, Y)\}\chi$ is. Then $\{(X, Y)\}\chi = (X\pi_1 \times Y)\chi$. A similar argument on Y yields the result.□

We prove next the following result which will be useful later on.

Proposition 1.2 *Let* ℓ_1*,* ℓ_2 *be two lines,* $\mu : \ell_1 \times \ell_2 \to \mathbb{P}'$ *a linear morphism, and* A *a point of* ℓ_1 *such that* $A \times \ell_2$ *is contained in* $\mathbf{D}(\mu)$*. Then the exceptional set* $\mathbf{A}(\mu)$ *is either a line, or* $|\mathbf{A}(\mu)| \leq 2$*. Furthermore:*
(i) If $|\mathbf{A}(\mu)| = 1$ *and* (X_1, P) *is the only exceptional point, then, for* $X \in \ell_1 \setminus \{A, X_1\}$*, one of the following possibilities occurs:*
(a) $(A \times \ell_2)\mu = (X \times \ell_2)\mu$*; in this case* $\{(A, P)\mu\} = (X_1 \times \ell_2)\mu$*;*
(b) $(A \times \ell_2)\mu \cap (X \times \ell_2)\mu$ *is the point* $(A, P)\mu$*.*
(ii) If $|\mathbf{A}(\mu)| = 2$*, and* (X_1, P)*,* (X_1', P') *are the exceptional points, then* $X_1 \neq X_1'$*,* $P \neq P'$*,* $(X_1' \times \ell_2)\mu = \{(A, P)\mu\}$*, and* $(\ell_1 \times \ell_2)\mu = (A \times \ell_2)\mu$*.*

Proof. If $\mathbf{A}(\mu)$ contains a line, then such a line is of type $P \times \ell_2$, $P \in \ell_1$. By the assumption $A \times \ell_2 \subset \mathbf{D}(\mu)$, there exist no more exceptional points.

Now assume that $\mathbf{A}(\mu)$ does not contain lines, and that $(X_1, P), (X_1', P') \in \mathbf{A}(\mu)$. This implies $X_1 \neq X_1'$, $P \neq P'$. Next,

$$(\ell_1 \times P)\mu = \{(A, P)\mu\} \neq \{(A, P')\mu\} = (\ell_1 \times P')\mu, \tag{2}$$

$$\begin{aligned} \{(A, P)\mu\} &= \{(X_1', P)\mu\} = (X_1' \times \ell_2)\mu, \\ \{(A, P')\mu\} &= \{(X_1, P')\mu\} = (X_1 \times \ell_2)\mu. \end{aligned} \tag{3}$$

Let g be a line of $\ell_1 \times \ell_2$. If g is of type $Y \times \ell_2$, then g intersects $\ell_1 \times P$ and $\ell_1 \times P'$, hence $g\mu \subset (A \times \ell_2)\mu$ follows from (2). Otherwise, the same conclusion follows from (3).

Now assume that a further point $(X_1'', P'') \in \mathbf{A}(\mu)$ exists. We have $P \neq P'' \neq P'$, and

$$(X_1'' \times \ell_2)\mu = \{(X_1'', P)\mu\} = \{(X_1'', P')\mu\}.$$

This contradicts $(X_1'', P)\mu = (A, P)\mu \neq (A, P')\mu = (X_1'', P')\mu$.

Now we have to prove (i). By assumption, $(X_1 \times \ell_2)\mu$ is a point Z.

If case (a) occurs, take a $Q \in \ell_2 \setminus P$. Since $(\ell_1 \times Q) \subset \mathbf{D}(\mu)$ and $(X_1, Q)\mu = Z$, it holds $(X, Q)\mu \neq Z$. Furthermore, $(\ell_1 \times Q)\mu = (A \times \ell_2)\mu$ implies $Z \in (A \times \ell_2)\mu$ and $(A, P)\mu = Z$.

If case (a) does not occur, then $(A, P)\mu = (X, P)\mu$ implies (b).□

2 The first decomposition of χ

In this section we will prove a universal property of a regular embedding γ with respect to a given class of linear morphisms. We assume that the linear morphism χ satisfies the following properties:
(i) There exist a basis $\mathcal{B} = \{A_0, A_1, \ldots, A_m\}$ of $\mathbb{P}_2$ and a plane $\mathcal{E} \subset \mathcal{P}_1$, such that for every $i = 1, \ldots, m$, $\chi_{|\mathcal{E} \times \{A_0, A_i\}}$ is global and has rank[1] greater than 2.
(ii) There exists $A \in \mathcal{P}_1$ such that $\dim(A \times \mathcal{P}_2)\chi = m$.

Proposition 2.1 *Assume $g \in \mathcal{G}_1$, $P_1^*, P_2^* \in \mathcal{P}_2$, $P_1^* \neq P_2^*$, and*

$$g_i := (g \times P_i^*)\gamma, \qquad i = 1, 2. \tag{4}$$

Then the mapping $f : g_1 \to g_2$, determined by $(P, P_1^*)\gamma f := (P, P_2^*)\gamma$ for every $P \in g$, is a projectivity.

Proof. Let $P_0^* \in P_1^* P_2^* \setminus \{P_1^*, P_2^*\}$ and $g_0 := (g \times P_0^*)\gamma$. For every $X \in g_1$ it holds $\{Xf\} = (X \vee g_0) \cap g_2$.□

Proposition 2.2 *Let $g \in \mathcal{G}_1$, $g' \in \mathcal{G}_2$, P_0^*, P_1^*, P_2^* three distinct points on g', and*

$$h_i := (g \times P_i^*)\chi, \qquad i = 0, 1, 2. \tag{5}$$

Then, for every permutation (i, j, k) of $(0, 1, 2)$, it holds $h_i \subset h_j \vee h_k$.□

[1]The rank is the projective dimension of the projective closure of $(\mathcal{E} \times \{A_0, A_i\})\chi$.

Proposition 2.3 *With the assumptions of props. 2.1 and 2.2, if h_1 and h_2 are lines of $\mathbb{P}'$ and $\sigma' : h_1 \to h_2$ is the mapping*

$$(P, P_1^*)\chi\sigma' := (P, P_2^*)\chi, \qquad P \in g, \tag{6}$$

then $\sigma : g_1 \to g_2$, defined by

$$\sigma := ((\gamma^{-1}\chi)_{|g_1})\sigma'((\gamma^{-1}\chi)_{|g_2})^{-1}, \tag{7}$$

is a projectivity.

Proof. Indeed σ coincides with the mapping f of prop. 2.1.□

Proposition 2.4 $(\gamma^{-1}\chi)_{|(\mathcal{P}_1 \times \mathcal{B})\gamma}$ *can be extended to a linear morphism $\varphi : \overline{\mathbb{P}} \to \mathbb{P}'$.*

Proof. We show the following statement by induction on $t = 0, \ldots, m$:
$(S)_t$ *There exists a linear morphism $\varphi_t : [(\mathcal{P}_1 \times \{A_0, A_1, \ldots, A_t\})\gamma] \to \mathbb{P}'$ which extends $(\gamma^{-1}\chi)_{|(\mathcal{P}_1 \times \{A_0, A_1, \ldots, A_t\})\gamma}$.*
For $t = 0$ there is nothing to extend. Suppose $(S)_t$ is true for a $t < m$. Let $P_1^* = A_0$, $P_2^* = A_{t+1}$, take $P_0^* \in P_1^*P_2^* \setminus \{P_1^*, P_2^*\}$ so, that the dimension of $U^* := (\mathcal{E} \times \{P_0^*\})\chi$ is minimal. Define:

$$\psi := \gamma^{-1}\chi_{|(\mathcal{P}_1 \times P_2^*)\gamma}.$$

We use theor. 1.6 in [2], case (V4), in order to extend the pair of linear morphisms φ_t and ψ to a linear morphism

$$\varphi_{t+1} : [(\mathcal{P}_1 \times \{A_0, A_1, \ldots, A_{t+1}\})\gamma] \to \mathbb{P}'.$$

By the regularity of γ, it holds $((\mathcal{P}_1 \times P_2^*)\gamma) \cap [(\mathcal{P}_1 \times \{A_0, A_1, \ldots, A_t\})\gamma] = \emptyset$. Next, it is enough to prove the existence of a line $g \subset \mathcal{E}$ such that the mapping σ' defined in (6) is a projectivity: indeed, in this case, the mapping in (7) is $(\varphi_{t|g_1})\sigma'(\psi_{|g_2})^{-1}$.

Case 1: There exists a line $g \subset \mathcal{E}$ such that the lines h_1 and h_2 (cf. (5)) are skew.
By prop. 2.2, any two of h_0, h_1 and h_2 are skew lines. In this case, the

same argument as in prop. 2.1 shows that the mapping $\sigma' : h_1 \to h_2$, which is defined in (6), is a projectivity.

Case 2: $\dim U^* \leq 0$.
There exists a line, say g, lying on $\mathcal{E}$ and such that $g \times P_0^* \subset \mathbf{A}(\chi)$. For every $P \in g$,

$$(P, P_1^*)\chi \in (P, P_0^*)\chi \vee (P, P_2^*)\chi = \{(P, P_2^*)\chi\}.$$

So, the mapping in (6) is the identity.

Case 3: $\dim U^* = 1$.
In this case $\mathbf{A}(\chi) \cap (\mathcal{E} \times P_0^*)$ is a point (Q, P_0^*) of $\mathbb{P}_1 \times \mathbb{P}_2$. If every line h through Q and in $\mathcal{E}$ satisfies $(h \times P_1^*)\chi = (h \times P_2^*)\chi$, then $(\mathcal{E} \times P_1^*)\chi = (\mathcal{E} \times P_2^*)\chi$, and this contradicts the assumptions. Thus there is a line of $\mathcal{E}$ and through Q, say g, such that, according to definitions (5), $h_1 \neq h_2$. The sets h_1, h_2 are lines and h_0 is a point not belonging to $h_1 \cup h_2$ (cf. prop. 2.2). For every $P \in g$ it holds $(P, P_2^*)\chi \in h_0 \vee \{(P, P_1^*)\chi\}$; therefore the mapping in (6) is the perspectivity between h_1 and h_2 with center h_0.

Case 4: $\dim U^* = 2$.
Assume that we are not in case 1. Then for every line h of $\mathcal{E}$ we have $(h \times P_1^*)\chi \cap (h \times P_2^*)\chi \neq \emptyset$. There is a line g in $\mathcal{E}$ such that $((g \times P_1^*)\chi) \cap ((\mathcal{E} \times P_2^*)\chi)$ is a point P^*. It holds:

$$P^* = (P_1, P_1^*)\chi = (P_2, P_2^*)\chi \qquad \text{with } P_1, P_2 \in g. \tag{8}$$

There exists a line g' in $\mathcal{E}$ such that $P_1 \in g'$, $g' \neq g$ and $(g' \times P_1^*)\chi \cap ((\mathcal{E} \times P_2^*)\chi) = P^*$. Next, there is a point $P_2' \in g'$ such that $(P_1, P_1^*)\chi = (P_2', P_2^*)\chi$. It follows $P_2' = P_2 = P_1$. By (8), there is $P_0' \in P_1^* P_2^*$ such that $(P_1, P_0') \in \mathbf{A}(\chi)$. But then the dimension of U^* is not minimal, a contradiction.$\square$

Now we use the assumption (ii) at the beginning of this section:

$$\begin{array}{rcl} (A \times \mathcal{P}_2)\chi & = & (A, A_0)\chi \vee (A, A_1)\chi \vee \ldots \vee (A, A_m)\chi = \\ & = & (A, A_0)\gamma\varphi \vee (A, A_1)\gamma\varphi \vee \ldots \vee (A, A_m)\gamma\varphi = \\ & = & (A \times \mathcal{P}_2)\gamma\varphi. \end{array} \tag{9}$$

Then for every $X \in \mathcal{P}_2$ there is exactly one element of $\mathcal{P}_2$, say Y, such that

$$(A, Y)\chi = (A, X)\gamma\varphi. \tag{10}$$

By defining $X\alpha' := Y$ we have a collineation α' of $\mathbb{P}_2$; $\alpha := (\mathrm{id}_{\mathcal{P}_1}, \alpha')$ is an automorphism of $\mathbb{P}_1 \times \mathbb{P}_2$.

Each line $\ell_2 \in \mathcal{G}_2$ such that
(a) for every $X \in \mathcal{P}_1$, it holds $(X \times \ell_2)\gamma\varphi = (X \times \ell_2)\alpha\chi$, and
(b) the rank of $(\alpha\chi)_{|\mathcal{P}_1 \times \ell_2}$ is greater than one,
will be called a *special line*.

The next result follows directly from prop. 2.4 and (i):

Proposition 2.5 *Every line containing two points of $\mathcal{B}$ is special.*$\square$

Proposition 2.6 *Let ℓ_2 be a special line, $X_1 \in \mathcal{P}_1 \setminus A$, $\mathbf{A}_1 := \mathbf{A}(\gamma\varphi) \cap (AX_1 \times \ell_2)$, and $\mathbf{A}_2 := \mathbf{A}(\alpha\chi) \cap (AX_1 \times \ell_2)$. Then $\mathbf{A}_1 = \mathbf{A}_2$.*

Proof. We will use prop. 1.2. Choose $B \in AX_1$ such that the dimension of $U = (B \times \ell_2)\gamma\varphi = (B \times \ell_2)\alpha\chi$ is minimal. If $U = \emptyset$, prop. 1.2 implies $\mathbf{A}_1 = \mathbf{A}_2 = B\times\ell_2$. Otherwise, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the same cardinality $|\mathbf{A}_1| \leq 2$.

Suppose $|\mathbf{A}_1| = 1$. Take $X \in AB \setminus \{A, B\}$. If $(A \times \ell_2)\gamma\varphi = (X \times \ell_2)\gamma\varphi$, and P is the unique point of ℓ_2 satisfying $\{(A, P)\gamma\varphi\} = (B \times \ell_2)\gamma\varphi$, then $\mathbf{A}_1 = \{(B, P)\}$. By definition of α, $(A, P)\gamma\varphi = (A, P)\alpha\chi$, so $\mathbf{A}_2 = \{(B, P)\}$. If $(A \times \ell_2)\gamma\varphi \cap (X \times \ell_2)\gamma\varphi = \{(A, P_1)\gamma\varphi\}$, we obtain $\mathbf{A}_1 = \mathbf{A}_2 = \{(B, P_1)\}$ in a similar way.

Now suppose $|\mathbf{A}_1| = 2$. Then there is a unique $C \in AB \setminus \{A, B\}$ such that $(C \times \ell_2)\gamma\varphi = (C \times \ell_2)\alpha\chi$ has dimension zero. Let Q and Q' be defined by

$$\{(A, Q)\gamma\varphi\} = (C \times \ell_2)\gamma\varphi, \qquad \{(A, Q')\gamma\varphi\} = (B \times \ell_2)\gamma\varphi,$$

then $\mathbf{A}_1 = \mathbf{A}_2 = \{(B, Q), (C, Q')\}$.$\square$

Proposition 2.7 *If ℓ_2 is a special line, $X_1 \in \mathcal{P}_1$ and $X_2 \in \ell_2$, then* $\{(X_1, X_2)\}\gamma\varphi = \{(X_1, X_2)\}\alpha\chi$.

Proof. In case $X_1 = A$ the statement holds by definition of α. From now on we assume $X_1 \neq A$. Let $r := (A \times \ell_2)\gamma\varphi$, $s := (X_1 \times \ell_2)\gamma\varphi$. Then $\dim r = 1$, $\dim s \leq 1$. Since ℓ_2 is special, it holds $r = (A \times \ell_2)\alpha\chi$, $s = (X_1 \times \ell_2)\alpha\chi$. There are several cases to be considered:

Case 1: $\dim s = -1$. This is a trivial case.

Case 2: $\dim s = 0$.
Since s contains exactly one point, and $\{(X_1, X_2)\}\gamma\varphi$, $\{(X_1, X_2)\}\alpha\chi$ are subsets of s, the statement follows from prop. 2.6.

Case 3: $\dim s = 1$, $r \cap s = \emptyset$.
Let $A' \in AX_1 \setminus \{A, X_1\}$. Any two of the lines r, s and $t := (A' \times \ell_2)\gamma\varphi$ are skew. Then, $(X_1, X_2)\gamma\varphi$ is the intersection of s with the only line through $(A, X_2)\gamma\varphi = (A, X_2)\alpha\chi$ intersecting s and t. Since the same property characterizes also $(X_1, X_2)\alpha\chi$, we have $(X_1, X_2)\gamma\varphi = (X_1, X_2)\alpha\chi$.

Case 4: $\dim s = 1$, *$r \cap s$ is a point Z.*
In this case, by props. 1.2 and 2.6, $\mathbf{A}_1 = \mathbf{A}_2$ either is empty or contains exactly one point.

Case 4.1: $|\mathbf{A}_1| = 1$.
Let $\mathbf{A}_1 = \{(A^*, P)\}$ ($A^* \in AX_1$), then $A \neq A^* \neq X_1$, and $(A^* \times \ell_2)\gamma\varphi = (A^* \times \ell_2)\alpha\chi$ is a point Z'.

The relations $r \vee s = r \vee \{Z'\} = s \vee \{Z'\}$ imply $Z' \notin r \cup s$. Then $(X_1, X_2)\gamma\varphi$ is the intersection point of s and the line containing both Z' and $(A, X_2)\gamma\varphi$. The same characterization holds for $(X_1, X_2)\alpha\chi$.

Case 4.2: $\mathbf{A}_1 = \emptyset$.
Let $P, Q \in \ell_2$ defined by $Z = (A, P)\gamma\varphi = (X_1, Q)\gamma\varphi$; then $P \neq Q$.

Let $g_X := (X \times \ell_2)\gamma\varphi$ for $X \in AX_1$. We now investigate on the following family of lines lying on the plane $r \vee s$:

$$\mathcal{F} := \{g_X | X \in AX_1\}.$$

First, $\mathcal{F}$ is an injective family: indeed, if $X \neq X'$, then $g_X \vee g_{X'} = r \vee s$, hence $g_X \neq g_{X'}$.

Now we prove that *for every $T \in \ell_2$, the line $h_T := (AX_1 \times T)\gamma\varphi$ belongs to $\mathcal{F}$.* Indeed, if $T = Q$, then $h_T = g_A = r$. If $T \neq Q$, then h_T contains $(X_1, T)\gamma\varphi$, that is a point not on r. Let $Z'' := r \cap h_T = (A, T)\gamma\varphi$. There exists $C \in AX_1$ such that $Z'' = (C, Q)\gamma\varphi$. It follows $g_C = (C, Q)\gamma\varphi \vee (C, T)\gamma\varphi = h_T$.

Next we prove that for every $X \in AX_1$, *the line g_X contains exactly one distinguished point W_X which belongs to no line of $\mathcal{F} \setminus \{g_X\}$; if $U \in g_X \setminus W_X$, then U lies on exactly one line of $\mathcal{F} \setminus \{g_X\}$.* [2] Let $X' \in AX_1 \setminus X$. The lines g_X and $g_{X'}$ intersect in exactly one point, say $(X, Y)\gamma\varphi = (X', Y')\gamma\varphi$. Then

$$(AX_1 \times Y')\gamma\varphi = g_X, \tag{11}$$

and every line $g_{X''}$, $X'' \neq X$, meets g_X in exactly one point different from $W_X := (X, Y')\gamma\varphi$; furthermore, distinct lines meet g_X in distinct points.

[2] It should be noted that $\mathcal{F}$ is a dual conic.

Conversely, by (11) every point of $g_X \setminus W_X$ lies on a line $g_{X''}$ with $X'' \in AX_1 \setminus X$.

The structure of $\mathcal{F}$ allows the following considerations. The distinguished points of the lines r and s are $W_A = (A, Q)\gamma\varphi$ and $W_{X_1} = (X_1, P)\gamma\varphi$, respectively. The mapping $f : r \to s$, defined by $(A, X_2)\gamma\varphi f := (X_1, X_2)\gamma\varphi$, can be characterized as follows: $W_A f = Z$; $Zf = W_{X_1}$; for every $U \in r \setminus \{W_A, Z\}$, Uf is the intersection of s with the unique line of $\mathcal{F} \setminus \{r\}$ containing U. Since for every $X \in AX_1$ we have $g_X = (X \times \ell_2)\alpha\chi$, the same geometric characterization holds for the mapping $(A, X_2)\alpha\chi \mapsto (X_1, X_2)\alpha\chi$.

Case 5: $r = s$.
By property (b) for ℓ_2, there is $B \in \mathcal{P}_1$ such that $(B \times \ell_2)\gamma\varphi \setminus r \neq \emptyset$. Then, by prop. 1.2, there exists $B' \in AB$ such that $(B' \times \ell_2)\gamma\varphi$ is a line $r' \neq r$. If $X_2' \in \ell_2$, then $(B', X_2')\gamma\varphi = (B', X_2')\alpha\chi$, because of the results of case 3 and 4. Since $r' \neq s$, we can repeat the arguments in such cases replacing A by B' and r by r': indeed, the only properties of A which are used there are $\gamma\varphi_{|A \times \ell_2} = \alpha\chi_{|A \times \ell_2}$ and $\dim((A \times \ell_2)\gamma\varphi) = 1$.□

Now we can establish the main result of this section.

Theorem 2.8 *If conditions (i) and (ii), stated at the beginning of this section, hold, then there are a collineation α' of $\mathbb{P}_2$ and a linear morphism $\varphi : \overline{\mathbb{P}} \to \mathbb{P}'$ such that, for $\alpha := (\mathrm{id}_{\mathcal{P}_1}, \alpha')$, it holds $\gamma\varphi = \alpha\chi$.*

Proof. Take into account the linear morphism φ of prop. 2.4 and the collineation α' defined by (10). We will prove by induction on $t = 0, 1, \ldots, m$, that

$$\gamma\varphi_{|\mathcal{P}_1 \times [A_0, A_1, \ldots, A_t]} = \alpha\chi_{|\mathcal{P}_1 \times [A_0, A_1, \ldots, A_t]}. \tag{12}$$

This is clear for $t = 0$. Next, assume that (12) is true for some $t < m$. Let $X_1 \in \mathcal{P}_1$, and $X_2 \in [A_0, A_1, \ldots, A_{t+1}] \setminus [A_0, A_1, \ldots, A_t]$. Then there is $X_2' \in [A_0, A_1, \ldots, A_t]$ such that $X_2 \in A_{t+1}X_2' =: \ell_2$.

If $X \in \mathcal{P}_1$, then by the induction hypothesis $\{(X, X_2')\}\gamma\varphi = \{(X, X_2')\}\alpha\chi$, and, by prop. 2.4, $\{(X, A_{t+1})\}\gamma\varphi = \{(X, A_{t+1})\}\alpha\chi$. It follows $(X \times \ell_2)\gamma\varphi = (X \times \ell_2)\alpha\chi$. This implies that ℓ_2 is a special line (see also assumption (ii)). Our statement follows then from prop. 2.7.□

3 The case $\mathrm{rad}_2\chi \neq \emptyset$

For each $P \in \mathcal{P}_1$, let $\chi_P : \mathbb{P}_2 \to \mathbb{P}'$ denote the linear morphism which is defined by $\{X\}\chi_P = \{(P,X)\}\chi$.

Theorem 3.1 *Assume that there exists $A \in \mathcal{P}_1$ such that* $\dim \mathbf{A}(\chi_A) \leq m-2$, *and, for every $P \in \mathcal{P}_1$, $\mathbf{A}(\chi_A) \subset \mathbf{A}(\chi_P)$. Also assume that there is a basis $\{A_0, A_1, \ldots, A_p\}$ of a subspace D_2, which is complementary with $\mathbf{A}(\chi_A)$ in $\mathbb{P}_2$, and a plane $\mathcal{E} \subset \mathcal{P}_1$ such that for every $i = 1, \ldots, p$, $\chi_{|\mathcal{E} \times \{A_0, A_i\}}$ is global and of rank greater than two. Then there exist a linear morphism $\varphi : \overline{\mathbb{P}} \to \mathbb{P}'$ and a collineation α' of $\mathbb{P}_2$ such that, for $\alpha := (\mathrm{id}_{\mathcal{P}_1}, \alpha')$, it holds $\gamma\varphi = \alpha\chi$.*

Proof. There is a subspace $D_1 \subset \mathcal{P}_1$ such that D_1 and $\mathrm{rad}_1\chi$ are complementary subspaces of $\mathbb{P}_1$ and $\mathcal{E} \subset D_1$. Next, $\mathrm{rad}_2\chi = \mathbf{A}(\chi_A)$, and $D_2 \subset \mathbf{D}(\chi_A)$. Then theor. 2.8 applies to the linear morphism χ' that is defined in prop. 1.1. So, there are a collineation β' of D_2 and a linear morphism $\varphi' : [(D_1 \times D_2)\gamma] \to \mathbb{P}'$ such that, for $\beta := (\mathrm{id}_{D_1}, \beta')$, it holds $\beta\chi' = (\gamma\varphi')_{|D_1 \times D_2}$.

It is possible to extend β' to a collineation α' of $\mathbb{P}_2$ such that $(\mathrm{rad}_2\chi)\alpha' = \mathrm{rad}_2\chi$. Since γ is regular,

$$U := [((\mathrm{rad}_1\chi) \times \mathcal{P}_2)\gamma] \vee [(\mathcal{P}_1 \times (\mathrm{rad}_2\chi))\gamma]$$

is a subspace of $\overline{\mathbb{P}}$ which is complementary with $[(D_1 \times D_2)\gamma]$. Let $\varphi : \overline{\mathbb{P}} \to \mathbb{P}'$ be a linear morphism extending φ' such that $U\varphi = \emptyset$. This φ satisfies the required property.□

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