Linear sets in the projective line over the endomorphism ring of a finite field*

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Abstract

Let $\mathrm{PG}(1,E)$ be the projective line over the endomorphism ring $E = \mathrm{End}_q(\mathbb{F}_{q^t})$ of the $\mathbb{F}_q$-vector space $\mathbb{F}_{q^t}$. As is well known there is a bijection $\Psi : \mathrm{PG}(1,E) \to \mathcal{G}_{2t,t,q}$ with the Grassmannian of the $(t-1)$-subspaces in $\mathrm{PG}(2t-1,q)$. In this paper along with any $\mathbb{F}_q$-linear set L of rank t in $\mathrm{PG}(1,q^t)$, determined by a $(t-1)$-dimensional subspace T^Ψ of $\mathrm{PG}(2t-1,q)$, a subset L_T of $\mathrm{PG}(1,E)$ is investigated. Some properties of linear sets are expressed in terms of the projective line over the ring E. In particular the attention is focused on the relationship between L_T and the set L'_T, corresponding via Ψ to a collection of pairwise skew $(t-1)$-dimensional subspaces, with $T \in L'_T$, each of which determine L. This leads among other things to a characterization of the linear sets of pseudoregulus type. It is proved that a scattered linear set L related to $T \in \mathrm{PG}(1,E)$ is of pseudoregulus type if and only if there exists a projectivity φ of $\mathrm{PG}(1,E)$ such that $L_T^\varphi = L'_T$.

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1 Introduction

1.1 Motivation

In this paper linear sets of rank t in the projective line $\mathrm{PG}(1,q^t)$ are investigated, where q is a power of a prime p. Such linear sets can be described by means of the *field reduction map* $\mathcal{F} = \mathcal{F}_{2,t,q}$ [15] mapping any point $\langle (a,b)\rangle_{q^t} \in \mathrm{PG}_{q^t}(\mathbb{F}_{q^t}^2) \cong \mathrm{PG}(1,q^t)$ to the $(t-1)$-subspace[1] of

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[1]Abbreviation for $(t-1)$-dimensional subspace.

$\mathrm{PG}_q(\mathbb{F}_{q^t}^2) \cong \mathrm{PG}(2t-1,q)$ associated with $\langle(a,b)\rangle_{q^t}$ (considered here as a t-dimensional $\mathbb{F}_q$-vector subspace). A point set $L \subseteq \mathrm{PG}(1,q^t)$ is said to be $\mathbb{F}_q$*-linear* (or just *linear*) *of rank* n if $L = \mathcal{B}(T')$, where T' is an $(n-1)$-subspace of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$, and

$$\mathcal{B}(T') = \left\{\langle(u,v)\rangle_{q^t} \mid \langle(u,v)\rangle_q \in T'\right\} = \left\{P \in \mathrm{PG}(1,q^t) \mid P^{\mathcal{F}} \cap T' \neq \emptyset\right\}. \quad (1)$$

Additionally, each such T' gives rise to the set $\mathcal{U}(T') = \mathcal{B}(T')^{\mathcal{F}} = L^{\mathcal{F}}$, which is a collection of $(t-1)$-subspaces belonging to the *standard Desarguesian spread* $\mathcal{D} = \mathrm{PG}(1,q^t)^{\mathcal{F}}$ of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$.

If a linear set L of rank n in $\mathrm{PG}(1,q^t)$ has size $\theta_{n-1} = (q^n-1)/(q-1)$ (which is the maximum size for a linear set of rank n), then L is a *scattered* linear set. For generalities on the linear sets the reader is referred to [14], [15], [16], [17], and [20].

As it has been pointed out in [13, Prop. 2], if $L = \mathcal{B}(T')$ is a scattered linear set of rank t in $\mathrm{PG}(1,q^t)$, then the union of all subspaces in $\mathcal{U}(T') = L^{\mathcal{F}}$ is a hypersurface $\mathcal{Q}$ of degree t in $\mathrm{PG}(2t-1,q)$, and an embedded product space isomorphic to $\mathrm{PG}(t-1,q) \times \mathrm{PG}(t-1,q)$. So, $\mathcal{Q}$ has two partitions in $(t-1)$-subspaces. The first one is $\mathcal{U}(T')$, the second one is $\mathcal{U}'(T') = \{T'h \mid h \in \mathbb{F}_{q^t}^*\}$, where $T'h = \{\langle(hu,hv)\rangle_q \mid \langle(u,v)\rangle_q \in T'\}$. By Prop. 3.2, the family $\mathcal{U}'(T')$ can be recovered uniquely from $\mathcal{U}(T')$ and T' (disregarding that $\mathbb{F}_{q^t}^2$ is the underlying vector space of our $\mathrm{PG}(2t-1,q)$).

For $t = n$ there is an alternative approach to $\mathcal{B}(T')$ and $\mathcal{U}(T')$ irrespective of whether T' is scattered or not. It is based on the $\mathbb{F}_q$-endomorphism ring E of $\mathbb{F}_{q^t}$ and the projective line $\mathrm{PG}(1,E)$ over this ring. On the one hand, there is a bijection Ψ between the projective line $\mathrm{PG}(1,E)$ and the Grassmannian $\mathcal{G}_{2t,t,q}$ of $(t-1)$-subspaces of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$. So, instead of T' we may consider its image under Ψ^{-1}, which is *a point* T *of* $\mathrm{PG}(1,E)$. On the other hand, we have a natural embedding $\iota : \mathrm{PG}(1,q^t) \to \mathrm{PG}(1,E)$. It maps the linear set $\mathcal{B}(T')$ to *a subset* $\mathcal{B}(T')^{\iota} =: L_T$ *of* $\mathrm{PG}(1,E)$, which in turn is the preimage under Ψ of $\mathcal{U}(T')$. In Section 2, we take up these ideas, but we start with an equivalent definition, which is in terms of $\mathrm{PG}(1,E)$ only, of the set L_T. There we also define a second set $L'_T \subset \mathrm{PG}(1,E)$ in such a way that $(L'_T)^{\Psi}$ equals the set $\mathcal{U}'(T^{\Psi})$ from above in the scattered case. Furthermore, since T will play a predominant role, $\mathcal{B}(T') = \mathcal{B}(T^{\Psi})$ will frequently also be denoted by $\mathcal{B}(T)$; *mutatis mutandis* this applies also to $\mathcal{U}(T')$ and $\mathcal{U}'(T')$.

A special example of a scattered linear set $L = \mathcal{B}(T)$ in $\mathrm{PG}(1,q^t)$ is a *linear set of pseudoregulus type*, defined in [6], [18], and further investigated in [5]. In our setting it is obtained by taking $T = E(\mathbb{1},\tau)$, where τ is a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$. The related hypersurface $\mathcal{Q}$ in

$\mathrm{PG}(2t-1,q)$ has been studied in [13], revealing a high degree of symmetry. As a matter of fact there are t families $\mathcal{S}_0$, $\mathcal{S}_1$, …, $\mathcal{S}_{t-1}$ of $(t-1)$-subspaces partitioning $\mathcal{Q}$ [13, Thm. 6] where $\mathcal{S}_0 = \mathcal{U}(T) = L_T^\Psi$ and $\mathcal{S}_1 = \mathcal{U}'(T) = (L'_T)^\Psi$ are defined above. Furthermore, in [13, Cor. 18] it is proved that the stabilizer of $\mathcal{Q}$ inside $\mathrm{P\Gamma L}_{2t}(q)$ contains a dihedral subgroup of order $2t$ acting on such t families of $(t-1)$-subspaces. A consequence thereof is the following result, for which we give a short direct proof in Prop. 3.1: *There is a projectivity of* $\mathrm{PG}(1,E)$ *mapping* L_T *onto* L'_T. For $t \geq 3$ this turns out to be a characteristic property of the linear sets of pseudoregulus type (Thm. 3.5). As the projectivities of $\mathrm{PG}(1,E)$ and the projectivities of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2) \cong \mathrm{PG}(2t-1,q)$ are in one-to-one correspondence, this leads to the following:

Theorem. *Let* $L = \mathcal{B}(T')$ *be a scattered linear set of rank* t *in* $\mathrm{PG}(1,q^t)$*, with* T' *a* $(t-1)$*-dimensional subspace of* $\mathrm{PG}(2t-1,q)$*, and* $t \geq 3$*. Then* L *is a linear set of pseudoregulus type if, and only if, a projectivity of* $\mathrm{PG}(2t-1,q)$ *exists mapping the first family* $\mathcal{U}(T')$ *of subspaces of the related embedded product space to the second one* $\mathcal{U}'(T')$*.*

Finally, let us mention our incentive for choosing the projective line $\mathrm{PG}(1,E)$ as an algebraic description of the Grassmannian $\mathcal{G}_{2t,t,q}$ of $(t-1)$-subspaces of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$. There are several instances in recent work about linear sets, for example in [13], where this approach already has been used successfully, but without explicit mention of $\mathrm{PG}(1,E)$. In particular, each $\alpha \in E$ gives rise to the point $E(\mathbb{1},\alpha) \in \mathrm{PG}(1,E)$ and, consequently, to an element of the Grassmannian $\mathcal{G}_{2t,t,q}$. This link between E and a certain subset of $\mathcal{G}_{2t,t,q}$ is a versatile tool, which is well known from the representation of translation planes in terms of *spread sets* [11, Def. 1.10]. As we sketched above, this link reappears in our setting: An algebraic counterpart of a scattered linear set of pseudoregulus type is a point $E(\mathbb{1},\tau) \in \mathrm{PG}(1,E)$ with the additional property that τ is a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$. Last, but not least, the possibility to describe the action of the projective group of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ on the Grassmannian $\mathcal{G}_{2t,t,q}$ via projectivities of $\mathrm{PG}(1,E)$ or, said differently, via invertible 2×2 matrices with entries in E allows us to accomplish necessary computations in a concise way.

1.2 Notation

Let $E = \mathrm{End}_q(\mathbb{F}_{q^t})$ with $t \geq 2$ be the ring of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{F}_{q^t}$. The ring E has the identity $\mathbb{1} \in E$ as its unit element. The multiplicative group comprising all invertible elements of E will be denoted as E^*.

Let us briefly recall the definition of the *projective line over the ring* E, which will be denoted by $\mathrm{PG}(1,E)$, and several basic notions; see [3, 1.3], [8, 3.2], and [9, 1.3]. We start with E^2, which is regarded as a *left* module over E in the usual way. Elements of E^2 are written as rows. This module has the standard basis $((\mathbb{1},0),(0,\mathbb{1}))$, and so it is a free module of rank 2. All invertible 2×2 matrices with entries in E constitute the general linear group $\mathrm{GL}_2(E)$, which acts in a natural way on the elements of E^2 from the *right hand side*. Now $\mathrm{PG}(1,E)$, whose elements will be called *points*, is defined as the orbit of the cyclic submodule $E(\mathbb{1},0)$ (the "starter point") under the action of the group $\mathrm{GL}_2(E)$ on E^2. Therefore, any point of $\mathrm{PG}(1,E)$ can be written in the form $E(\alpha,\beta)$, where the pair $(\alpha,\beta)\in E^2$ is *admissible*, i.e., it is the first row of a matrix from $\mathrm{GL}_2(E)$. Furthermore, if (α',β') is any element of E^2 then $E(\alpha',\beta')=E(\alpha,\beta)$ holds precisely when there is an element $\gamma\in E^*$ such that $(\alpha',\beta')=(\gamma\alpha,\gamma\beta)$. In this case (α',β') is admissible too.

The projective line $\mathrm{PG}(1,E)$ is endowed with a binary *distant relation* $\triangle$ as follows: The relation $\triangle$ is the orbit of the pair $((\mathbb{1},0),(0,\mathbb{1}))$ under the (componentwise) action of $\mathrm{GL}_2(E)$. Thus $E(\alpha,\beta)\ \triangle\ E(\gamma,\delta)$ holds if, and only if, $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\mathrm{GL}_2(E)$.

The map

$$\Psi:\mathrm{PG}(1,E)\to\mathcal{G}_{2t,t,q}:E(\alpha,\beta)\mapsto\left\{\langle(u^\alpha,u^\beta)\rangle_q\mid u\in\mathbb{F}_{q^t}^*\right\}\tag{2}$$

is a bijection of $\mathrm{PG}(1,E)$ onto the Grassmannian $\mathcal{G}_{2t,t,q}$ of $(t-1)$-subspaces of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)\cong\mathrm{PG}(2t-1,q)$. Any two points of $\mathrm{PG}(1,E)$ are distant if, and only if, their images under Ψ are disjoint (or, said differently, complementary) [1, Thm. 2.4]. For versions of the previous results in terms of matrix rings we refer to [3, 10.2], [8, 5.2.3], [9, 4.5], and [10, 500]. See also [21, 123ff.], even though the terminology used there is quite different from ours.

Let φ denote a projectivity of $\mathrm{PG}(1,E)$, i.e., φ is given by a matrix

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{GL}_2(E)\tag{3}$$

acting on E^2. Then the mapping

$$\hat\varphi:\mathrm{PG}_q(\mathbb{F}_{q^t}^2)\to\mathrm{PG}_q(\mathbb{F}_{q^t}^2):\langle(u,v)\rangle_q\mapsto\langle(u^\alpha+v^\gamma,u^\beta+v^\delta)\rangle_q\tag{4}$$

is a projective collineation. The action of $\hat\varphi$ on the Grassmannian $\mathcal{G}_{2t,t,q}$ is given by $\Psi^{-1}\varphi\Psi$. By [12, 642–643], every projective collineation of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ can be written as in (4) for some matrix from $\mathrm{GL}_2(E)$.

Under any projectivity of $\mathrm{PG}(1,E)$ the distant relation $\triangle$ is preserved. The obvious counterpart of this observation is the fact that under any projective collineation of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ the complementarity of subspaces from $\mathcal{G}_{2t,t,q}$ is preserved.

If $a \in \mathbb{F}_{q^t}$ then $\rho_a \in E$ is defined by $x^{\rho_a} = ax$ for all $x \in \mathbb{F}_{q^t}$. The mapping

$$\mathbb{F}_{q^t} \to E : a \mapsto \rho_a$$

is a monomorphism of rings taking $1 \in \mathbb{F}_{q^t}$ to the identity $\mathbb{1} \in E$. The image of this monomorphism will be denoted by F. We now consider $\mathrm{PG}(1,F)$ as a subset of $\mathrm{PG}(1,E)$ by identifying $F(\rho_a,\rho_b)$ with $E(\rho_a,\rho_b)$ for all $(a,b) \in \mathbb{F}_{q^t}^2 \setminus \{(0,0)\}$. This allows us to embed the projective line $\mathrm{PG}(1,q^t)$ in the projective line $\mathrm{PG}(1,E)$ as follows:

$$\iota : \mathrm{PG}(1,q^t) \to \mathrm{PG}(1,E) : \langle (a,b) \rangle_{q^t} \mapsto E(\rho_a,\rho_b). \tag{5}$$

Following [2], the image of $\mathrm{PG}(1,F)$ under any projectivity of $\mathrm{PG}(1,E)$ is called an *F-chain*[2] of $\mathrm{PG}(1,E)$. In particular, $\mathrm{PG}(1,q^t)^\iota$ is an F-chain of $\mathrm{PG}(1,E)$.

Any two distinct points of $\mathrm{PG}(1,E)$ are distant precisely when they belong to a common F-chain [2, Lemma 2.1]. From this we obtain the following result [8, Thm. 3.4.7], which is a slightly modified version of [2, Thm. 2.3]:

Proposition 1.1. *Given three distinct points P_1, Q_1, R_1 on an F-chain C_1 and three distinct points P_2, Q_2, R_2 on an F-chain C_2 there is at least one projectivity π of $\mathrm{PG}(1,E)$ with $P_1^\pi = P_2$, $Q_1^\pi = Q_2$, $R_1^\pi = R_2$ and $C_1^\pi = C_2$.*

2 Scattered points

Definition 2.1. For any point $T = E(\alpha,\beta) \in \mathrm{PG}(1,E)$ define:

$$\begin{aligned} L_T &= \left\{ E(\rho_a,\rho_b) \mid (a,b) \in (\mathbb{F}_{q^t}^2)^* \text{ s.t. } E(\rho_a,\rho_b) \not\triangle T \right\}; \\ L'_T &= \left\{ T \cdot \mathrm{diag}(\rho_h,\rho_h) \mid h \in \mathbb{F}_{q^t}^* \right\}. \end{aligned}$$

Also, we introduce the shorthand $Th := T \cdot \mathrm{diag}(\rho_h,\rho_h)$, where h is as above. By the proof of Prop. 2.11 below, the point set L'_T is the orbit of T under the group of all projectivities of $\mathrm{PG}(1,E)$ that fix $\mathrm{PG}(1,F)$ pointwise.

[2] Our F-chains are different from the chains in [3] and [9], since F is not contained in the centre of E.

The following diagram describes the relationships involving some objects defined so far. (Note that the right hand side of (1) gives $\mathcal{B}(T)^{\mathcal{F}} = L_T^{\Psi}$.)

$$\begin{tikzcd}
\mathcal{B}(T) \subset \mathrm{PG}(1,q^t) \arrow[r, "\iota"] \arrow[d, "\mathcal{F}"'] & L_T \subset \mathrm{PG}(1,E) \arrow[d, "\Psi"] \\
\mathcal{B}(T)^{\mathcal{F}} \subset \mathcal{D} \arrow[r, hook] & L_T^{\Psi} \subset \mathcal{G}_{2t,t,q}
\end{tikzcd}$$

Definition 2.2. A *scattered point* of $\mathrm{PG}(1,E)$ is a point T such that $\#L_T = \theta_{t-1}$.

A point $T \in \mathrm{PG}(1,E)$ is scattered if, and only if, $\mathcal{B}(T)$ is a scattered linear set. A point $X = E(\rho_a, \rho_b)$ is distant from T if, and only if, the $(t-1)$-subspace X^{Ψ} defined by the vector subspace $\langle (a,b)\rangle_{q^t}$ is disjoint from T^{Ψ}.

Example 2.3. If τ is a generator of $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and $T_0 = E(\mathbb{1}, \tau)$, then $\mathcal{B}(T_0)$ is a scattered linear set of pseudoregulus type [18]. Hence T_0 is a scattered point of $\mathrm{PG}(1,E)$.

Definition 2.4. For $T \in \mathrm{PG}(1,E)$, the set L_T will be said *of pseudoregulus type* when $L_T^{\iota^{-1}}$ is a linear set of pseudoregulus type.

Any two linear sets of pseudoregulus type are projectively equivalent [6], [18]. So L_T is of pseudoregulus type if and only if $L_T = L_{E(\mathbb{1},\tau)}^{\pi}$ where τ is a generator of $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and π is a projectivity of $\mathrm{PG}(1,F)$.

Example 2.5. The point $T_1 = E(\mathbb{1}, \sigma\rho_g + \sigma^{t-1})$ with $\sigma : u \mapsto u^q$, $g \in \mathbb{F}_{q^t}^*$, and $g^{\theta_{t-1}} \neq 1$ is scattered [19, Thm. 2].

Proposition 2.6. *Let $T \in \mathrm{PG}(1,E) \setminus \mathrm{PG}(1,F)$. Then $Th = Tk$ for any $h, k \in \mathbb{F}_{q^t}^*$ such that $h^{-1}k \in \mathbb{F}_q$. Furthermore, if T is scattered, then $Th \triangle Tk$ for any $h, k \in \mathbb{F}_{q^t}^*$ such that $h^{-1}k \notin \mathbb{F}_q$.*

Proof. If $h^{-1}k \in \mathbb{F}_q^*$ and $T = E(\alpha, \beta)$, then

$$Th = E(\alpha\rho_h, \beta\rho_h) = E(\rho_{h^{-1}k}\alpha\rho_h, \rho_{h^{-1}k}\beta\rho_h) = E(\alpha\rho_k, \beta\rho_k) = Tk.$$

Let $\mathcal{P}'$ be the set of all points of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2) \cong \mathrm{PG}(2t-1,q)$ belonging to the $(t-1)$-subspaces of ${L'_T}^{\Psi}$. By the previous paragraph, it follows $\#\mathcal{P}' \leq \theta_{t-1}^2$, and the equality holds if, and only if, for any h, k the relation $h^{-1}k \in \mathbb{F}_{q^t} \setminus \mathbb{F}_q$ implies $Th \triangle Tk$.

Let $\mathcal{P}$ be the set of all points of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ belonging to the $(t-1)$-subspaces in L_T^{Ψ}. Then $\mathcal{P} \subseteq \mathcal{P}'$.

If T is scattered, then $\#\mathcal{P} = \theta_{t-1}^2$. $\square$

Proposition 2.7. *Let $T = E(\mathbb{1}, \beta)$ be a scattered point of* $\mathrm{PG}(1, E)$. *Then the following assertions hold:*

(*i*) *A point $P \in \mathrm{PG}(1, E)$ belongs to L_T if, and only if, an element $u \in \mathbb{F}_{q^t}^*$ exists such that $P = E(\mathbb{1}, \rho_{u^\beta/u})$;*

(*ii*) *the size of the set $I = \{u^\beta/u \mid u \in \mathbb{F}_{q^t}^*\}$ is θ_{t-1};*

(*iii*) *for any $u, v \in \mathbb{F}_{q^t}^*$, u and v are $\mathbb{F}_q$-linearly dependent if, and only if, $u^\beta/u = v^\beta/v$;*

(*iv*) *the dimension of* $\ker \beta$ *is at most one;*

(*v*) *β is a singular endomorphism if, and only if, $E(\mathbb{1}, 0) \in L_T$.*

Proof. Let $P = E(\rho_a, \rho_b)$ be a point. Then $P \in L_T$ holds precisely when the $(t-1)$-subspaces P^Ψ and T^Ψ are not disjoint; that is, there are two nonzero elements of $\mathbb{F}_{q^t}$, say u and v, such that $u = va$ and $u^\beta = vb$. This is equivalent to $a \neq 0$ and $u^\beta = a^{-1}bu$. This implies (i), and consequently (v).

The size of I equals the size of L_T, and this implies (ii).

If $r \in \mathbb{F}_q^*$ and $u \in \mathbb{F}_{q^t}^*$, then $(ru)^\beta/(ru) = u^\beta/u$. This implies that the size of the image of the map $u \in \mathbb{F}_{q^t}^* \mapsto u^\beta/u \in \mathbb{F}_{q^t}$ is at most θ_{t-1}, and the equality holds only if condition (iii) is satisfied. The last condition implies (iv). □

Take notice that (i) and (v) hold irrespective of whether the point $T \in \mathrm{PG}(1, E)$ is scattered or not.

The following result is merely a reformulation of [13, Prop. 2], with $\beta \in E$ playing the role of the matrix A from there.

Proposition 2.8. *Let $T = E(\mathbb{1}, \beta)$ be a scattered point. For each $h \in \mathbb{F}_{q^t}^*$, the map*

$$\varepsilon : \Big(\langle h\rangle_q, \langle (u, u^\beta)\rangle_q\Big) \mapsto \langle (hu, hu^\beta)\rangle_q \tag{6}$$

is a projective embedding of the product space $\mathrm{PG}_q(\mathbb{F}_{q^t}) \times T^\Psi$ into $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$, that is, is an injective mapping such that the image of any line of the product space is a line of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$.

Remark 2.9. In the case of non-scattered linear sets, the map ε is not an embedding, but the image of ε is still a non-injective projection of a Segre variety [17].

In [7, Thm. 1] a result similar to the following one is proved in terms of the matrix group $\mathrm{GL}_2(q^t)$.

Proposition 2.10. *Let $\kappa \in \mathrm{P\Gamma L}_2(q^t)$ be a collineation of $\mathrm{PG}(1,q^t)$ whose accompanying automorphism η is in $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$. After embedding $\mathrm{PG}(1,q^t)$ in $\mathrm{PG}(1,E)$ according to (5), the collineation $\iota^{-1}\kappa\iota$ of $\mathrm{PG}(1,F)$ can be extended to at least one projectivity of $\mathrm{PG}(1,E)$. Conversely, the restriction to $\mathrm{PG}(1,F)$ of any projectivity of $\mathrm{PG}(1,E)$ that fixes $\mathrm{PG}(1,F)$ as a set is a collineation with accompanying automorphism in $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$.*

Proof. There is a matrix $(m_{ij}) \in \mathrm{GL}_2(q^t)$ such that

$$\langle (a,b)\rangle_{q^t} \overset{\kappa}{\mapsto} \langle (a^\eta, b^\eta)\rangle_{q^t} \cdot (m_{ij}) \text{ for all } \langle (a,b)\rangle_{q^t} \in \mathrm{PG}(1,q^t).$$

For all $x, a \in \mathbb{F}_{q^t}$ we have $xa^\eta = (x^{\eta^{-1}}a)^\eta$ and so $\rho_{a^\eta} = \eta^{-1}\rho_a\eta$. The permutation of $\mathrm{PG}(1,E)$ given by

$$E(\alpha,\beta) \mapsto E(\eta^{-1}\alpha\eta, \eta^{-1}\beta\eta)\cdot(\rho_{m_{ij}}) \text{ for all } E(\alpha,\beta) \in \mathrm{PG}(1,E) \tag{7}$$

is a projectivity, since the automorphism of E acting on α and β is inner. By construction, this projectivity extends the collineation $\iota^{-1}\kappa\iota$ of $\mathrm{PG}(1,F)$.

Conversely, let π be a projectivity of $\mathrm{PG}(1,E)$ that fixes $\mathrm{PG}(1,F)$ as a set. Since $\mathrm{PGL}(2,q^t)$ acts (sharply) 3-transitively on $\mathrm{PG}(1,q^t)$ there is a (unique) projectivity λ of $\mathrm{PG}(1,q^t)$ such that the images of $E(\mathbb{1},0)$, $E(0,\mathbb{1})$, $E(\mathbb{1},\mathbb{1})$ under $\iota^{-1}\lambda\iota$ and π are the same. We choose matrices $(\pi_{ij}) \in \mathrm{GL}_2(E)$ and $(c_{ij}) \in \mathrm{GL}_2(q^t)$ that describe π and λ, respectively. Then $(\pi_{ij})\cdot(\rho_{c_{ij}})^{-1}$ induces a projectivity of $\mathrm{PG}(1,E)$ that fixes each of the points $E(\mathbb{1},0)$, $E(0,\mathbb{1})$, $E(\mathbb{1},\mathbb{1})$ and also the F-chain $\mathrm{PG}(1,F)$. So there is a $\delta \in E^*$ with

$$(\pi_{ij}) = \mathrm{diag}(\delta,\delta)\cdot(\rho_{c_{ij}})$$

and for each $b \in \mathbb{F}_{q^t}$ there is a unique $b' \in \mathbb{F}_{q^t}$ such that

$$E(\mathbb{1},\rho_b)\cdot\mathrm{diag}(\delta,\delta) = E(\delta,\rho_b\delta) = E(\mathbb{1},\rho_{b'}) = E(\delta,\delta\rho_{b'}).$$

This leads us to $\delta^{-1}\rho_b\delta = \rho_{b'}$ for all $b \in \mathbb{F}_{q^t}$. Thus the inner automorphism of E given by δ restricts to an automorphism of the field F. Going back to $\mathbb{F}_{q^t}$ shows that $\eta : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} : b \mapsto b'$ is an automorphism of $\mathbb{F}_{q^t}$. Furthermore, we read off $\eta \in \mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ from ρ_b being in the centre of E for all $b \in \mathbb{F}_q$.

Let $d := 1^\delta \in \mathbb{F}_{q^t}^*$ and choose any $b \in \mathbb{F}_{q^t}$. Calculating the image of d under $\delta^{-1}\rho_b\delta = \rho_{b^\eta}$ in two ways gives $b^\delta = db^\eta$, whence $\delta = \eta\rho_d$. This leads us finally to

$$(\pi_{ij}) = \mathrm{diag}(\eta,\eta)\cdot(\rho_d\rho_{c_{ij}}) = \mathrm{diag}(\eta,\eta)\cdot(\rho_{dc_{ij}}). \tag{8}$$

We now repeat the first part of the proof with (dc_{ij}) instead of (m_{ij}). This gives the projectivity

$$E(\alpha,\beta) \mapsto E(\eta^{-1}\alpha\eta, \eta^{-1}\beta\eta) \cdot (\rho_{dc_{ij}}) \text{ for all } E(\alpha,\beta) \in \mathrm{PG}(1,E), \quad (9)$$

which obviously coincides with π. $\square$

The collineation κ from the previous proposition can be extended in precisely θ_{t-1} different ways to a projectivity of $\mathrm{PG}(1,E)$. Even though this could be derived easily from well known results about spreads [11, Sect. 1], we give a short direct proof.

Proposition 2.11. *There are precisely θ_{t-1} projectivities of $\mathrm{PG}(1,E)$ that fix $\mathrm{PG}(1,F)$ pointwise.*

Proof. Let π be a projectivity of $\mathrm{PG}(1,E)$ that fixes $\mathrm{PG}(1,F)$ pointwise. We repeat the second part of the proof of Prop. 2.10 under this stronger assumption, while maintaining all notations from there. However, in our current setting we may choose $(c_{ij}) = \mathrm{diag}(1,1) \in \mathrm{GL}_2(q^t)$. We are thus led to $(\pi_{ij}) = \mathrm{diag}(\delta,\delta)$ for some $\delta \in E^*$. This δ has to satisfy now $\delta^{-1}\rho_b\delta = \rho_b$ for all $b \in \mathbb{F}_{q^t}$, which in turn gives that η is the trivial automorphism of $\mathbb{F}_{q^t}$. Letting $d := 1^\delta$, as we did before, gives therefore that π is given by the matrix

$$(\pi_{ij}) = \mathrm{diag}(\rho_d, \rho_d) \text{ with } d \in \mathbb{F}_{q^t}^*.$$

Conversely, for all $d \in \mathbb{F}_{q^t}^*$ the matrix $\mathrm{diag}(\rho_d,\rho_d)$ determines a projectivity of $\mathrm{PG}(1,E)$ that fixes $\mathrm{PG}(1,F)$ pointwise. Two matrices of this kind give rise to the same projectivity if, and only if, they differ by a factor $\mathrm{diag}(\gamma,\gamma)$, where γ is an invertible element from the centre of E. This condition for γ is equivalent to $\gamma = \rho_f$ with $f \in \mathbb{F}_q^*$, whence the assertion follows from $\theta_{t-1} = (q^t-1)/(q-1) = (\#\mathbb{F}_{q^t}^*)/(\#\mathbb{F}_q^*)$. $\square$

Proposition 2.12. *If $\pi \in \mathrm{PGL}_2(E)$ stabilizes $\mathrm{PG}(1,F)$, then $L_{T^\pi} = (L_T)^\pi$ and $L'_{T^\pi} = (L'_T)^\pi$ for each $T \in \mathrm{PG}(1,E)$.*

Proof. The first equation can be derived as follows:

$$\begin{aligned} L_{T^\pi} &= \{P \in \mathrm{PG}(1,F) \text{ s.t. } P \not\triangle T^\pi\} = \{P^\pi \text{ s.t. } P \in \mathrm{PG}(1,F),\ P^\pi \not\triangle T^\pi\} \\ &= \{P^\pi \text{ s.t. } P \in \mathrm{PG}(1,F),\ P \not\triangle T\} = L_T^\pi. \end{aligned}$$

Let $h \in \mathbb{F}_{q^t}^*$. Taking into account the structure of the projectivity π, described in (9), $T^\pi h = (Th^{\eta^{-1}})^\pi$, whence $L'_{T^\pi} = (L'_T)^\pi$. $\square$

Remark 2.13. The investigation of scattered points can be restricted taking into account that if T is a scattered point of $\mathrm{PG}(1,E)$, then there exist a projectivity π of $\mathrm{PG}(1,E)$ and an element $\beta \in E^*$, such that $\mathrm{PG}(1,F)^\pi = \mathrm{PG}(1,F)$, $1 \in \mathrm{Spec}(\beta)$, and $T^\pi = E(\mathbb{1},\beta)$ (cf. [18, Rem. 4.2]).

Lemma 2.14. *Let s be the greatest element of $\{1,2,\dots,t-1\}$ that divides t. Then any two distinct F-chains of $\mathrm{PG}(1,E)$ have at most q^s+1 common points.*

Proof. Due to $3 = 2^1 + 1 \leq q^s + 1$ it is enough to consider two distinct F-chains C_1 and C_2 with at least three distinct common points. By Prop. 1.1, we may assume these points to be $E(\mathbb{1},0)$, $E(0,\mathbb{1})$, $E(\mathbb{1},\mathbb{1})$ and $C_1 = \mathrm{PG}(1,F)$. Applying Prop. 1.1 once more shows that there is a projectivity π of $\mathrm{PG}(1,E)$ that fixes each of the points $E(\mathbb{1},0)$, $E(0,\mathbb{1})$, $E(\mathbb{1},\mathbb{1})$ and takes C_1 to C_2. Consequently, there is a $\delta \in E^*$ such that

$$E(\alpha,\beta)^\pi = E(\alpha,\beta)\,\mathrm{diag}(\delta,\delta) = E(\delta^{-1}\alpha\delta,\delta^{-1}\beta\delta) \text{ for all } E(\alpha,\beta) \in \mathrm{PG}(1,E).$$

This gives

$$C_1 \cap C_1^\pi = \{E(\mathbb{1},\rho) \mid \rho \in F \cap (\delta^{-1}F\delta)\} \cup \{E(0,\mathbb{1})\}.$$

The intersection $F \cap \delta^{-1}F\delta$ is a proper subfield of F, since $\delta^{-1}F\delta$ is an isomorphic copy of F in E and $C_1 \neq C_2$ implies $F \neq \delta^{-1}F\delta$. This gives $\#(C_1 \cap C_2) = \#(F \cap \delta^{-1}F\delta) + 1 \leq q^s + 1$. □

Proposition 2.15. *Let T be a scattered point of $\mathrm{PG}(1,E)$ and assume that $t \geq 3$. Then L_T is contained in no F-chain other than $\mathrm{PG}(1,F)$.*

Proof. From $t \geq 3$ follows $\#L_T = \theta_{t-1} > q^{t-1} + 1$. The assertion is now immediate from Lemma 2.14. □

Remark 2.16. For $t = 2$ the set L_T^Ψ is a regulus in $\mathrm{PG}(3,q)$ and the F-chains are precisely the regular spreads in $\mathrm{PG}(3,q)$. Choose a point that is off the hyperbolic quadric $\mathcal{H}$ that carries L_T^Ψ. Then there are as many regular spreads through L_T^Ψ as there are external lines to $\mathcal{H}$ through the chosen point. A straightforward counting shows that the number of these lines is $\frac{1}{2}(q^2-q)$. Thus, unless $q = 2$, there is more than one F-chain through L_T.

Theorem 2.17. *Let T and U be points of $\mathrm{PG}(1,E)$, with T a scattered point. Then the following assertions are equivalent:*

(*i*) *A collineation* $\kappa \in \mathrm{P\Gamma L}_2(q^t)$ *with companion automorphism* $\eta \in \mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ *exists, such that* $\mathcal{B}(T)^\kappa = \mathcal{B}(U)$*;*

(*ii*) L_T *and* L_U *are projectively equivalent in* $\mathrm{PG}(1, E)$.

Proof. If κ is given as in (i) then, by the first part of Prop. 2.10, there is a projectivity of $\mathrm{PG}(1, E)$ that takes $\mathcal{B}(T)^\iota = L_T$ to $\mathcal{B}(U)^\iota = L_U$.

Conversely, let π be a projectivity of $\mathrm{PG}(1, E)$ that takes L_T to L_U. There are two cases:

Case 1: $t \geq 3$. From $\# L_T = \# L_U$ the point U is scattered. By Prop. 2.15, each of the sets L_T and L_U is contained in no F-chain other than $\mathrm{PG}(1, F)$. Hence $\mathrm{PG}(1, F)$ is invariant under π so that the second part of Prop. 2.10 shows the existence of a collineation of $\mathrm{PG}(1, q^t)$ with the required properties.

Case 2: $t = 2$. The sets $\mathcal{B}(T)$ and $\mathcal{B}(U)$ are two linear sets of rank 2 and cardinality $q + 1$, i.e. two Baer sublines of $\mathrm{PG}(1, q^2)$. These are well known to be projectively equivalent. □

Proposition 2.18. *If* $g \neq 0$ *and* $t, q > 3$*, the sets* L_{T_0} *and* L_{T_1}*, where* T_0 *and* T_1 *are defined in Examples* 2.3 *and* 2.5*, are not projectively equivalent in* $\mathrm{PG}(1, E)$.

Proof. The sets $\mathcal{B}(T_0)$ and $\mathcal{B}(T_1)$ are not projectively equivalent [18, Example 4.6]. Since any linear set in the $\mathrm{P\Gamma L}_2(q^t)$-orbit of a linear set of pseudoregulus type is again of pseudoregulus type, $\mathcal{B}(T_0)$ and $\mathcal{B}(T_1)$ also are not equivalent up to collineations. Then the assertion follows from Thm. 2.17. □

3 Characterization of the linear sets of pseudoregulus type

Proposition 3.1. [13] *Let* $T \in \mathrm{PG}(1, E)$ *be such that* L_T *is a set of pseudoregulus type. Then there is a* $\varphi \in \mathrm{PGL}_2(E)$ *such that* $L_T^\varphi = L'_T$.

Proof. Since up to projectivities in $\mathrm{PG}(1, q^t)$ there is a unique linear set of pseudoregulus type [6, 18], it may be assumed that $T = E(\mathbb{1}, \tau)$ with τ a generator of $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$. A projectivity φ satisfying the thesis is given by the matrix $\mathrm{diag}(\mathbb{1}, \tau) \in \mathrm{GL}_2(E)$. As a matter of fact, for any $u \in \mathbb{F}_{q^t}^*$,

$$E(\rho_{1/u^\tau}, \tau\rho_{1/u^\tau}) = E(\mathbb{1}, \rho_{u^\tau}\tau\rho_{1/u^\tau}) = E(\mathbb{1}, \tau\rho_{u^{\tau^2}/u^\tau}) = E(\mathbb{1}, \rho_{u^\tau/u})^\varphi,$$

and by Prop. 2.7 this implies that φ maps L_T onto L'_T. □

The goal of this section is to prove the converse of Prop. 3.1.

Proposition 3.2. *Let $\varepsilon : \Pi_1 \times \Pi_2 \to \Pi_3$ be a projective embedding, where Π_j is a projective space of finite dimension $d_j \geq 1$ for $j = 1, 2, 3$. Let $\mathcal{U}_1$ be the set of all d_1-subspaces of type $(\Pi_1 \times Q)^\varepsilon$ for Q a point in Π_2, and $\mathcal{U}_2$ the set of all d_2-subspaces of type $(P \times \Pi_2)^\varepsilon$ for P a point in Π_1. Fix any point $S \in \Pi_2$ and consider the subspace $(\Pi_1 \times S)^\varepsilon \in \mathcal{U}_1$. Under these assumptions the following assertions hold:*

1. *A line y of Π_3 is contained in some subspace belonging to $\mathcal{U}_1$ if, and only if, there is a d_2-regulus $\mathcal{R}_y$ in Π_3 subject to the following three conditions:*
   - *(a) $\mathcal{R}_y$ has a transversal line in $(\Pi_1 \times S)^\varepsilon$.*
   - *(b) $\mathcal{R}_y$ is contained in $\mathcal{U}_2$.*
   - *(c) y is a transversal line of $\mathcal{R}_y$.*
2. *A subspace V of Π_3 belongs to $\mathcal{U}_1$ if, and only if, V has dimension d_1 and for any line y of V there is a d_2-regulus $\mathcal{R}_y$ subject to the conditions (a), (b), and (c) from above.*

Proof. Our reasoning will be based on the following three facts: (i) due to the injectivity of ε, each point in the image of ε is incident with a unique subspace from $\mathcal{U}_1$ and a unique subspace from $\mathcal{U}_2$; (ii) for any line $\ell \subset \Pi_1$ the set

$$\{(P \times \Pi_2)^\varepsilon \mid P \in \ell\} \subset \mathcal{U}_2 \tag{10}$$

is a d_2-regulus; (iii) the transversal lines of this regulus are precisely the lines of the form $(\ell \times Q)^\varepsilon$ with Q varying in Π_2.

Ad 1. Let y be a line such that $y \subset (\Pi_1 \times R)^\varepsilon \in \mathcal{U}_1$, where $R \in \Pi_2$. Since the restriction of ε to $\Pi_1 \times R$ is a collineation onto the subspace $(\Pi_1 \times R)^\varepsilon$, there is a unique line $\ell_y \subset \Pi_1$ such that $y = (\ell_y \times R)^\varepsilon$. By (10), the d_2-regulus $\mathcal{R}_y := \{(P \times \Pi_2)^\varepsilon \mid P \in \ell_y\}$ satisfies *(b)*. Furthermore, both y and $(\ell_y \times S)^\varepsilon \subset (\Pi_1 \times S)^\varepsilon$ are transversal lines of $\mathcal{R}_y$, whence conditions *(a)* and *(c)* are satisfied too.

Conversely, assume that for a line $y \subset \Pi_3$ there is a regulus $\mathcal{R}_y$ satisfying the three conditions from above. By *(a)*, there is a line $\ell_y \subset \Pi_1$ for which $(\ell_y \times S)^\varepsilon$ is a transversal line of $\mathcal{R}_y$. Now *(b)* implies that $\mathcal{R}_y$ can be written as in (10) with ℓ to be replaced with ℓ_y. Thus *(c)* shows that there is a point $R \in \Pi_2$ such that $y = (\ell_y \times R)^\varepsilon$. This in turn gives $y \subset \Pi_1 \times R \in \mathcal{U}_1$.

Ad 2. Let $V \in \mathcal{U}_1$. The dimension of V obviously is d_1. Given any line $y \subset V$ there is a regulus $\mathcal{R}_y$ with the required properties by the first part of the proposition.

For a proof of the converse, we fix a point $Y \in V$ and consider an arbitrary line $y \subset V$ through Y. By the first part of the proposition, there is at least one point $R \in \Pi_2$ such that $y \subset (\Pi_1 \times R)^\varepsilon \in \mathcal{U}_1$. This implies $Y = (X, R)^\varepsilon$ for some point $X \in \Pi_1$. The point R does not depend on the choice of the line y on Y. Consequently, $V \subset (\Pi_1 \times R)^\varepsilon$ and, due to d_1 being the dimension of V, these two subspaces are identical. □

Clearly, analogous statements hold, where the roles of $\mathcal{U}_1$ and $\mathcal{U}_2$ are interchanged. We will refer to $\mathcal{U}_1$ and $\mathcal{U}_2$ in Prop. 3.2 as to the *maximal subspaces of the embedded product space* $(\Pi_1 \times \Pi_2)^\varepsilon$. Notice that such subspaces are defined with respect to the embedding, since $(\Pi_1 \times \Pi_2)^\varepsilon$ may contain further maximal subspaces. Prop. 3.2 implies:

Proposition 3.3. *Let $\mathcal{U}_1$ and V be a collection of d_1-subspaces and a d_2-subspace in a projective space Π, respectively, where d_1 and d_2 are positive integers. There exists at most one collection $\mathcal{U}_2$ of d_2-subspaces of Π with $V \in \mathcal{U}_2$, and such that $\mathcal{U}_1$ and $\mathcal{U}_2$ are the collections of maximal subspaces of an embedded product space.*

Remark 3.4. By Prop. 3.2, a point $P \in \mathrm{PG}(1, E)$ is in L'_T if, and only if, no line of P^Ψ is irregular as defined in [14] with respect to the scattered subspace T^Ψ.

Theorem 3.5. *Let $T = E(\mathbb{1}, \beta)$ be a scattered point, where $\beta \in E^*$, let $t \geq 3$ and suppose that there exists a projectivity $\varphi \in \mathrm{PGL}_2(E)$ such that $L_T^\varphi = L'_T$. Then L_T is of pseudoregulus type.*

Proof. We split the proof into four steps.

Step 1. First we fix some matrix $(\varphi_{ij}) \in \mathrm{GL}_2(E)$ that describes φ. Then we choose any point $U \in \mathrm{PG}(1, E)$ such that $U^\varphi \in L_T$. The point U^φ is non-distant to all points of L'_T, whence U is non-distant to all (namely θ_{t-1}) points of L_T. So U is scattered and $L_T = L_U$. Also, there is a $\gamma \in E^*$ satisfying

$$U = E(\mathbb{1}, \gamma).$$

According to (4), the matrix (φ_{ij}) describes also that projective collineation $\hat{\varphi}$ of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ whose action on the Grassmannian $\mathrm{PG}(1, E)^\Psi$ coincides with $\Psi^{-1}\varphi\Psi$. It can be deduced from Prop. 2.8 that for any scattered point $X \in \mathrm{PG}(1, E)$, L_X^Ψ and $(L'_X)^\Psi$ are the two collections of maximal subspaces of an embedded product space. So, a repeated application of Prop. 3.3 implies:

(a) L_T^Ψ is the unique collection of maximal subspaces of an embedded product space, containing $U^{\varphi\Psi}$, the other collection being $(L'_T)^\Psi$.

(b) $(L'_U)^\Psi$ is the unique collection of maximal subspaces of an embedded product space, containing U^Ψ, the other one being $L_U^\Psi = L_T^\Psi$.

By applying $\hat{\varphi}$ to (b), one obtains:

(c) $(L'_U)^{\varphi\Psi}$ is the unique collection of maximal subspaces of an embedded product space, containing $U^{\varphi\Psi}$, the other one being $L_T^{\varphi\Psi} = (L'_T)^\Psi$.

By (a) and (c)

$$(L'_U)^\varphi = L_T = L_U. \tag{11}$$

Step 2. Let H be the subgroup $\mathrm{GL}_2(E)$ formed by all matrices $\mathrm{diag}(\rho_h, \rho_h)$ with $h \in \mathbb{F}_{q^t}^*$. Also, let Λ be the associated group of projectivities of $\mathrm{PG}(1, E)$. Both $H \cong \mathbb{F}_{q^t}^*$ and $\Lambda \cong \mathbb{F}_{q^t}^*/\mathbb{F}_q^*$ are cyclic. Now, for all $h, k \in \mathbb{F}_{q^t}^*$ the equality $E(\rho_h, \gamma\rho_h) = E(\rho_k, \gamma\rho_k)$ holds precisely when h and k are $\mathbb{F}_q$-linearly dependent (cf. Prop. 2.6). Consequently we have shown: the cyclic group Λ acts regularly on L'_U and fixes $L_U = L_T$ pointwise.

Consider now the subgroup

$$H' := (\varphi_{ij})^{-1} \cdot H \cdot (\varphi_{ij})$$

of $\mathrm{GL}_2(E)$ and the corresponding group $\Lambda' := \varphi^{-1}\Lambda\varphi$ of projectivities. By the above, the group Λ' is cyclic, acts regularly on L_T, and fixes L'_T pointwise. From $t \geq 3$ and Prop. 2.15 the F-chain $\mathrm{PG}(1, F)$ is invariant under the action of Λ'. Since Λ' has order θ_{t-1}, which is the size of the orbit L_T, the group Λ' acts faithfully on $\mathrm{PG}(1, F)$.

Step 3. Let us choose any element $h \in \mathbb{F}_{q^t}^*$. The projectivity π given by the matrix

$$(\pi_{ij}) := (\varphi_{ij})^{-1} \cdot \mathrm{diag}(\rho_h, \rho_h) \cdot (\varphi_{ij}) \in \mathrm{GL}_2(E)$$

fixes $\mathrm{PG}(1, F)$, as a set. We therefore can repeat the second part of the proof of Prop. 2.10 up to (8). By this formula[3], there exists an automorphism $\eta \in \mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and an invertible matrix $(c_{ij}) \in \mathrm{GL}_2(q^t)$ such that

$$(\pi_{ij}) = \mathrm{diag}(\eta, \eta) \cdot (\rho_{c_{ij}}) \in H'. \tag{12}$$

We claim that $\eta = \mathbb{1}$. In order to verify this assertion, we fix any element $u \in \mathbb{F}_{q^t}^*$. The point $E(\rho_u, \rho_{u^\beta})$ is in L_T, which is invariant under $\pi \in \Lambda'$

[3] Take notice that the elements c_{ij} that are used now play the role of the elements dc_{ij} that appear in (8).

by Step 2. This implies that there is a $v \in \mathbb{F}_{q^t}^*$ such that $E(\rho_u, \rho_{u^\beta})^\pi = E(\rho_v, \rho_{v^\beta})$. Furthermore, L_T' is fixed pointwise under π. So, the associated projective collineation $\hat{\pi}$ of $\mathrm{PG}_q(\mathbb{F}_{q^t}^2)$ sends for all $k \in \mathbb{F}_{q^t}^*$ the point

$$\langle (uk, u^\beta k)\rangle_q = E(\rho_u, \rho_{u^\beta})^\Psi \cap E(\rho_k, \beta\rho_k)^\Psi$$

to the point

$$E(\rho_v, \rho_{v^\beta})^\Psi \cap E(\rho_k, \beta\rho_k)^\Psi = \langle (vk, v^\beta k)\rangle_q.$$

Therefore, for each $k \in \mathbb{F}_{q^t}^*$ there is an element $x_k \in \mathbb{F}_q^*$ such that

$$((uk)^\eta, (u^\beta k)^\eta) \cdot (c_{ij}) = x_k(vk, v^\beta k).$$

As $\mathbb{F}_{q^t}^2$ is also a vector space over $\mathbb{F}_{q^t}$, this can be rewritten in the form

$$k^\eta (u^\eta, u^{\beta\eta}) \cdot (c_{ij}) = x_k k(v, v^\beta).$$

Letting $k := 1$ gives $(u^\eta, u^{\beta\eta}) \cdot (c_{ij}) = x_1(v, v^\beta)$ so that $k^\eta x_1(v, v^\beta) = x_k k(v, v^\beta)$. We thus have arrived at

$$k^\eta = (x_k/x_1)k \text{ for all } k \in \mathbb{F}_{q^t}^*.$$

This means that each $k \in \mathbb{F}_{q^t}^*$ is an eigenvector of the $\mathbb{F}_q$-linear mapping $\eta \in E$. Thus η has a unique eigenvalue, i.e., x_k/x_1 is a constant that does not depend on k. Finally, $1^\eta = 1$ shows that this constant is equal to 1.

Step 4. We maintain all notions from the previous step (with $\eta = \mathbb{1}$), under the additional assumption that h is a generator of the multiplicative group $\mathbb{F}_{q^t}^*$. There are three possibilities for the matrix $(c_{ij}) \in \mathrm{GL}_2(q^t)$:

First, assume that (c_{ij}) has a single eigenvalue in $\mathbb{F}_{q^t}$. Then, since (c_{ij}) cannot be a scalar multiple of the identity matrix, it is similar to a matrix of the form

$$a\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with } a, b \in \mathbb{F}_{q^t}^*.$$

Due to $\mathrm{Char}\,\mathbb{F}_{q^t} = p$, the p-th power of this matrix is $\mathrm{diag}(a^p, a^p)$, which gives that Λ' acts on $\mathrm{PG}(1, F)$ as a cyclic permutation group of order p. Since L_T is an orbit under this action, we obtain the contradiction $\#L_T = \theta_{t-1} \le p$.

Next, assume that (c_{ij}) has no eigenvalue in $\mathbb{F}_{q^t}$, whence it has two distinct eigenvalues $z \neq \overline{z}$ in $\mathbb{F}_{q^{2t}} \supset \mathbb{F}_{q^t}$. Here $^-$ denotes the unique non-trivial automorphism in $\mathrm{Gal}(\mathbb{F}_{q^{2t}}/\mathbb{F}_{q^t})$. So, over $\mathbb{F}_{q^{2t}}$, the matrix (c_{ij}) is similar to $\mathrm{diag}(z, \overline{z})$. Let w be a generator of the multiplicative group $\mathbb{F}_{q^{2t}}^*$. We consider the matrix group $\{\mathrm{diag}(w^s, \overline{w}^s) \mid s = 1, \ldots, q^{2t} - 1\}$ of order

$q^{2t}-1$. It acts as a group Ω of projectivities on $\mathrm{PG}(1,q^{2t})$. The kernel of this action comprises all matrices $\mathrm{diag}(w^s,\overline{w}^s)$ with $w^s=\overline{w}^s=\overline{w^s}$ or, equivalently, with $w^s\in\mathbb{F}_{q^t}^*$. Hence we have

$$\#\Omega=\frac{q^{2t}-1}{q^t-1}=q^t+1.$$

The powers of $\mathrm{diag}(z,\overline{z})$ constitute a matrix group and give rise to a subgroup of Ω. This subgroup is isomorphic to Λ', and therefore its order is θ_{t-1}. This gives that θ_{t-1} divides q^t+1, which is impossible due to

$$(q-1)\,\theta_{t-1}=q^t-1<q^t+1<q^t+q^{t-1}+\cdots+q=q\,\theta_{t-1}.$$

By the above, (c_{ij}) has two distinct eigenvalues a,b in $\mathbb{F}_{q^t}^*$ and so there is a matrix $(m_{ij})\in\mathrm{GL}_2(q^t)$ such that

$$\mathrm{diag}(a,b)=(m_{ij})^{-1}\cdot(c_{ij})\cdot(m_{ij}).$$

Let $(\mu_{ij})=(\rho_{m_{ij}})\in\mathrm{GL}_2(E)$ and let $\mu\in\mathrm{PGL}_2(E)$ be the projectivity given by (μ_{ij}). It fixes the F-chain $\mathrm{PG}(1,F)$ as a set. We therefore obtain that T^μ is a scattered point and the equality $L_T^\mu=L_{T^\mu}$ (cf. Prop. 2.12). The group

$$H'':=(\mu_{ij})^{-1}\cdot H'\cdot(\mu_{ij})=\{\mathrm{diag}(\rho_a,\rho_b)^s\mid s=1,2,\ldots,q^t-1\}$$

induces the cyclic group $\Lambda'':=\mu^{-1}\Lambda'\mu$ of projectivities, which acts regularly on L_T^μ and fixes the points $E(\mathbb{1},0)$ and $E(0,\mathbb{1})$. So L_T^μ contains none of these points, whence we have

$$T^\mu=E(\mathbb{1},\delta)\text{ with }\delta\in E^*.$$

The matrix group generated by $\mathrm{diag}(\mathbb{1},\rho_{b/a})$ also induces the group Λ''. So the order of b/a in the multiplicative group $\mathbb{F}_{q^t}^*$ is θ_{t-1}. There is an element $e\in\mathbb{F}_{q^t}^*$ such that

$$b/a=e^{q-1}.$$

This allows us to give another group of matrices that induces Λ'', namely the group generated by

$$\rho_e\,\mathrm{diag}(\mathbb{1},\rho_{b/a})=\mathrm{diag}(\rho_e,\rho_{e^\sigma}),\tag{13}$$

where $\sigma\in\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ is given by $x\mapsto x^q$.

Finally, let $d := 1^\delta$. Then $E(\mathbb{1}, \rho_d) \in L_T^\mu$ and, by writing the action of Λ'' in terms of the powers of the matrix from (13), we obtain

$$L_T^\mu = \{E(\mathbb{1}, \rho_d)\,\mathrm{diag}(\rho_u, \rho_{u^\sigma}) = E(\rho_u, \rho_{u^\sigma d}) \mid u = e^s,\ s = 1, 2, \ldots, \theta_{t-1}\}.$$

Thus L_T^μ is contained in L_W, where

$$W := E(\mathbb{1}, \sigma\rho_d).$$

This implies, due to $\#L_T^\mu = \#L_W$, that L_T is projectively equivalent to L_W, which is of pseudoregulus type. □

It should be noted that in Step 4 in the proof above the choice of a generator h of the multiplicative group $\mathbb{F}_{q^t}^*$ serves the sole purpose that the projectivity related to the matrix $(\rho_{c_{ij}})$ is a generator of the group Λ' acting regularly on L_T. Bearing this in mind, the following proposition can be extrapolated:

Proposition 3.6. *Let L be a scattered linear set of rank t in $\mathrm{PG}(1, q^t)$. If there exists a cyclic subgroup of $\mathrm{PGL}_2(q^t)$ acting regularly on L, then L is of pseudoregulus type.*

On the other hand, any linear set of pseudoregulus type is projectively equivalent to $\{\langle(1, u^{q-1})\rangle_{q^t} \mid u \in \mathbb{F}_{q^t}^*\}$ [6], [18]. Hence the converse of Prop. 3.6 can be proved directly.

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