

ON THE MATRICES OF CENTRAL LINEAR MAPPINGS

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Summary. We show that a central linear mapping of a projectively embedded Euclidean n -space onto a projectively embedded Euclidean m -space is decomposable into a central projection followed by a similarity if, and only if, the least singular value of a certain matrix has multiplicity $\geq 2m - n + 1$. This matrix is arising, by a simple manipulation, from a matrix describing the given mapping in terms of homogeneous Cartesian coordinates.

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1 Introduction

A linear mapping between projectively embedded Euclidean spaces is called *central*, if its exceptional subspace is not at infinity. Such a linear mapping is in general not decomposable into a central projection followed by a similarity. Necessary and sufficient conditions for the existence of such a decomposition have been given in [4] for arbitrary finite dimensions; cf. also [1], [2], [3]. However, those results do not seem to be immediately applicable on a *central axonometry*, i.e., a central linear mapping given via an axonometric figure. On the other hand, in a series of recent papers [5], [6], [7] this problem of decomposition has been discussed for central axonometries of the Euclidean 3-space onto the Euclidean plane from an elementary point of view¹.

Loosely speaking, the concept of central axonometry is a geometric equivalent to the algebraic concept of a *coordinate matrix* for a linear mapping of the underlying vector spaces. However, from the results in [2] and [4] it is also not immediate whether or not a given matrix describes (in terms of homogeneous Cartesian coordinates) a mapping that permits the above-mentioned factorization. The aim of this communication is to give a criterion for this.

Let \mathbf{I} , \mathbf{J} be finite-dimensional Euclidean vector spaces. Given a linear mapping $f : \mathbf{I} \rightarrow \mathbf{J}$ denote by $f^{\text{ad}} : \mathbf{J} \rightarrow \mathbf{I}$ its adjoint mapping. Then $f^{\text{ad}} \circ f$ is self-adjoint with eigenvalues

$$v_1 \geq \dots \geq v_r > v_{r+1} = \dots = v_n = 0.$$

Here r equals the rank of f and $n = \dim \mathbf{I}$. Moreover, each eigenvalue is written down repeatedly according to its multiplicity². The positive real numbers $\sqrt{v_1}, \dots, \sqrt{v_r}$ are frequently called the *singular values* of f . The multiplicity of a singular value of f is defined via the multiplicity of the corresponding eigenvalue of $f^{\text{ad}} \circ f$. It is immediate from the singular value decomposition that f and f^{ad} share the same singular values (counted with their multiplicities). See, e.g., [8].

These results hold true, *mutatis mutandis*, when replacing f by any real matrix, say A , and f^{ad} by the transpose matrix A^T .

¹A lot of further references can be found in the quoted papers.

²For a self-adjoint mapping the algebraic and geometric multiplicities of an eigenvalue are identical. Hence we may unambiguously use the term ‘multiplicity’.

2 Decompositions

When discussing central linear mappings it will be convenient to consider Euclidean spaces embedded in projective spaces. Thus let \mathbf{V} be an $(n + 1)$ -dimensional real vector space ($3 \leq n < \infty$) and \mathbf{I} one of its hyperplanes. Assume, furthermore, that \mathbf{I} is equipped with a positive definite inner product (\cdot) so that \mathbf{I} is a Euclidean vector space. In the projective space on \mathbf{V} , denoted by $\mathcal{P}(\mathbf{V})$, we consider the projective hyperplane $\mathcal{P}(\mathbf{I})$ as the hyperplane at infinity. The absolute polarity in $\mathcal{P}(\mathbf{I})$ is determined by the inner product on \mathbf{I} . Hence $\mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\mathbf{I})$ is a projectively embedded Euclidean space³. Similarly, let $\mathcal{P}(\mathbf{W}) \setminus \mathcal{P}(\mathbf{J})$ be an m -dimensional projectively embedded Euclidean space ($2 \leq m < n < \infty$). Given a linear mapping

$$f : \mathbf{V} \rightarrow \mathbf{W} \quad (1)$$

of vector spaces then the associate (projective) linear mapping

$$\phi : \mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\ker f) \rightarrow \mathcal{P}(\mathbf{W}), \mathbb{R}\mathbf{x} \mapsto \mathbb{R}(f(\mathbf{x})) \quad (2)$$

has $\mathcal{P}(\ker f)$ as its exceptional subspace. In the sequel we shall assume that

$$\ker f \not\subset \mathbf{I} \quad \text{and} \quad f(\mathbf{V}) = \mathbf{W}, \quad (3)$$

or, in other words, that ϕ is central and surjective⁴. Obviously, (3) is equivalent to

$$f(\mathbf{I}) = \mathbf{W}. \quad (4)$$

We recall some results [2], [4]: If \mathbf{T} is any complementary subspace of $\ker f$ in \mathbf{V} , then denote by

$$\psi_{\mathbf{T}} : \mathcal{P}(\mathbf{V}) \setminus \mathcal{P}(\ker f) \rightarrow \mathcal{P}(\mathbf{T}) \quad (5)$$

the projection with the exceptional subspace $\mathcal{P}(\ker f)$ onto $\mathcal{P}(\mathbf{T})$. The restricted mapping

$$\phi_{\mathbf{T}} := \phi | \mathcal{P}(\mathbf{T}) : \mathcal{P}(\mathbf{T}) \rightarrow \mathcal{P}(\mathbf{W}) \quad (6)$$

is a collineation and

$$\phi = \phi_{\mathbf{T}} \circ \psi_{\mathbf{T}}; \quad (7)$$

every decomposition of ϕ into a projection and a collineation is of this form. In the Euclidean vector space \mathbf{I} we have the distinguished subspace

$$\mathbf{E} := f^{-1}(\mathbf{J}) \cap \mathbf{I}. \quad (8)$$

Write

$$f_{\mathbf{E}} : \mathbf{E} \rightarrow \mathbf{J}, \mathbf{x} \mapsto f(\mathbf{x}); \quad (9)$$

this $f_{\mathbf{E}}$ is well-defined and surjective, since $\mathbf{E} \subset f^{-1}(\mathbf{J})$ and $\ker f \not\subset \mathbf{E}$. The subspace \mathbf{T} can be chosen with $\phi_{\mathbf{T}}$ being a similarity if, and only if, the least singular value of $f_{\mathbf{E}}$ has multiplicity⁵ $\geq 2m - n + 1$.

Next, we assume that $\mathcal{P}(\mathbf{T}) \not\subset \mathcal{P}(\mathbf{I})$ is orthogonal to $\mathcal{P}(\ker f)$. This means that $(\mathbf{T} \cap \mathbf{I})^{\perp} \subset \ker f \cap \mathbf{I}$ or $(\mathbf{T} \cap \mathbf{I})^{\perp} \supset \ker f \cap \mathbf{I}$. Hence $\psi_{\mathbf{T}}$ is an *orthogonal central projection*⁶. It is easily seen from [2] that ϕ permits a decomposition into an orthogonal central projection followed by a similarity if, and only if, all singular values of $f_{\mathbf{E}}$ are equal.

³We do not endow this space with a unit segment.

⁴This assumption of surjectivity is made 'without loss of generality' in most papers on this subject. It will, however, be essential several times in this paper.

⁵In [2, Satz 10] this multiplicity is printed incorrectly as $2m - n - 1$.

⁶The central projections used in elementary descriptive geometry are trivial examples of orthogonal central projections.

Finally, we are going to show that the crucial properties of $f_{\mathbf{E}}$ can be read off from another mapping: Denote by

$$p : \mathbf{I} \rightarrow \mathbf{E} \quad (10)$$

the orthogonal projection with the kernel $\mathbf{E}^\perp \subset \mathbf{I}$. Then

$$(f_{\mathbf{E}} \circ p) \circ (f_{\mathbf{E}} \circ p)^{\text{ad}} = f_{\mathbf{E}} \circ p \circ p^{\text{ad}} \circ (f_{\mathbf{E}})^{\text{ad}} = f_{\mathbf{E}} \circ (f_{\mathbf{E}})^{\text{ad}}, \quad (11)$$

since p^{ad} is the natural embedding $\mathbf{E} \rightarrow \mathbf{I}$. Thus, by (11) and the results stated in Section 1, $f_{\mathbf{E}}$ and $(f_{\mathbf{E}} \circ p)^{\text{ad}}$ have the same singular values (counted with their multiplicities). Hence, by the surjectivity of $f_{\mathbf{E}}$ and (11), all singular values of $f_{\mathbf{E}}$ are equal if, and only if, there exists a real number $v > 0$ such that

$$(f_{\mathbf{E}} \circ p) \circ (f_{\mathbf{E}} \circ p)^{\text{ad}} = v \text{id}_{\mathbf{J}}. \quad (12)$$

We shall use this in the next section.

3 A matrix characterization

Introducing homogeneous Cartesian coordinates in $\mathcal{P}(\mathbf{V})$ is equivalent to choosing a basis $\{\mathbf{b}_0, \dots, \mathbf{b}_n\}$ of \mathbf{V} such that $\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbf{I}$ is an orthonormal system. The origin is given by $\mathbb{R}\mathbf{b}_0$ and the unit points are $\mathbb{R}(\mathbf{b}_0 + \mathbf{b}_1), \dots, \mathbb{R}(\mathbf{b}_0 + \mathbf{b}_n)$. In the same manner we are introducing homogeneous Cartesian coordinates in $\mathcal{P}(\mathbf{W})$ via a basis $\{\mathbf{c}_0, \dots, \mathbf{c}_m\}$.

Theorem 1 *Suppose that $f : \mathbf{V} \rightarrow \mathbf{W}$ is inducing a surjective central linear mapping ϕ according to formula (2). Let*

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & & \vdots \\ a_{m0} & \cdots & a_{mn} \end{pmatrix} \quad (13)$$

be the coordinate matrix of f with respect to bases of \mathbf{V} and \mathbf{W} that are yielding homogeneous Cartesian coordinates. Write

$$\mathbf{a}_i := (a_{i1}, \dots, a_{in}) \in \mathbb{R}^n \text{ for all } i = 0, \dots, m \quad (14)$$

and

$$\tilde{A} := \begin{pmatrix} \mathbf{a}_1 - \frac{\mathbf{a}_0 \cdot \mathbf{a}_1}{\mathbf{a}_0 \cdot \mathbf{a}_0} \mathbf{a}_0 \\ \vdots \\ \mathbf{a}_m - \frac{\mathbf{a}_0 \cdot \mathbf{a}_m}{\mathbf{a}_0 \cdot \mathbf{a}_0} \mathbf{a}_0 \end{pmatrix}. \quad (15)$$

Then the following assertions hold true:

1. ϕ is decomposable into a central projection followed by a similarity if, and only if, the least singular value of the matrix \tilde{A} has multiplicity $\geq 2m - n + 1$.
2. ϕ is decomposable into an orthogonal central projection followed by a similarity if, and only if, there exists a real number $v > 0$ such that

$$\tilde{A}\tilde{A}^T = \text{diag}(v, \dots, v). \quad (16)$$

Proof. We read off from the top row of A that

$$a_{00}x_0 + \cdots + a_{0n}x_n = 0$$

is an equation of $f^{-1}(\mathbf{J}) \neq \mathbf{I}$ so that $\mathbf{a}_0 \cdot \mathbf{a}_0 \neq 0$. Write $\tilde{f} : \mathbf{I} \rightarrow \mathbf{J}$ for the linear mapping whose coordinate matrix with respect to $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ equals \tilde{A} . A straightforward calculation shows that

$$\tilde{f}(\mathbf{x}) = f(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{E}$$

and

$$\tilde{f}(a_{01}\mathbf{b}_1 + \dots + a_{0n}\mathbf{b}_n) = 0,$$

i.e., $\mathbf{E}^\perp \subset \ker \tilde{f}$. Thus \tilde{f} equals the mapping $f_{\mathbf{E}} \circ p$ discussed above. Now the proof is completed by translating formulae (11) and (12) into the language of matrices. ■

We remark that (3) and the linear independence of $\mathbf{a}_1, \dots, \mathbf{a}_m$ are equivalent conditions.

In contrast to the results in [5], [6], [7], the ϕ -image of the origin $\mathbb{R}\mathbf{b}_0$ does not appear in our characterization. On the other hand, we have

$$f(\mathbf{E}^\perp) = \mathbb{R}((\mathbf{a}_0 \cdot \mathbf{a}_0)\mathbf{c}_0 + \dots + (\mathbf{a}_0 \cdot \mathbf{a}_m)\mathbf{c}_m).$$

In projective terms this 1-dimensional subspace of \mathbf{W} gives the *principal point* of the mapping ϕ . Exactly if the principal point of ϕ equals the origin $\mathbb{R}\mathbf{c}_0$, then \tilde{A} arises from A merely by deleting the top row and the leading column.

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