APPLICATIONS OF RESULTS ON GENERALIZED POLYNOMIAL IDENTITIES IN DESARGUESIAN PROJECTIVE SPACES

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ABSTRACT. By following ideas of synthetic real projective geometry rather than classical algebraic geometry, maps in a finite-dimensional desarguesian projective space are used to generate normal curves. We aim at solving the problems of classification, automorphic collineations and generating maps of arbitrary non-degenerate normal curves and degenerate normal curves in desarguesian projective planes (also called degenerate conics). Properties of normal curves are shown, on one hand, by using methods of projective geometry as well as linear algebra and, on the other hand, by applying results on the non-existence of certain types of ordinary and generalized polynomial identities with coefficients in a not necessarily commutative field.

1. INTRODUCTION

1.1. Preface

It is well known that special curves and surfaces in a real (or complex) projective space of finite dimension permit definitions in a purely geometrical way by using *generating maps* such as projectivities, polarities, etc. Some of these definitions may be taken over word for word to more general classes of projective spaces. However, most of the classical results will no longer hold in the general case.

Normal (rational) curves in an n-dimensional real (or complex) projective space allow geometrical definitions including those of a *conic* (n=2) and a *twisted cubic* (n=3) given by J. Steiner (1832) and F. Seydewitz (1847), respectively. First results for arbitrary finite dimension are due to W.K. Clifford [11] and G. Veronese [37]. Cf. also [5,270], [6,318], [10,166], [21], [34,894] for further details and historical remarks.

By transferring one of these definitions in an even generalized way, *normal curves* have been introduced in finite dimensional desarguesian projective spaces by R. Riesinger [28]. Previous publications by E. Berz [7], B. Segre [33, 325], L.A. Rosati [30], [31], R. Artzy [3], W. Krüger [23] and R. Riesinger [27] are concerned with examples of these normal curves, especially conics. Recent articles [29], [15], [16], [17], [18], all but one by the author, are also dealing with normal curves. Finally, we mention some papers which contain results on (certain) normal curves in pappian projective spaces and on *conic-like figures* in non-pappian projective planes [8,55], [9,195], [20], [25], [35], [36].

A theorem by S.A. Amitsur [1] states that a field[1] satisfies a *generalized polynomial identity* if and only if it is of finite degree over its centre. Cf. also [13;141, 162]. This result is frequently applied in Riesinger's papers, because some geometrical problems, for example the coincidence of two degenerate conics, are equivalent to the fact that a generalized polynomial in a non-commutative indeterminant x with coefficients in a field K vanishes for all values of x in K. Hence the given polynomial either is the zero-polynomial or it yields a generalized polynomial identity. By Amitsur's theorem, the latter possibility can be excluded if K is of infinite degree over its centre. In this way geometrical problems have been solved completely under the assumption that the field K has infinite degree over its centre, but not necessarily complete solutions have been given for finite degree [27], [28]. However, as will be shown in this paper by a different approach, complete solutions of these problems can be found irrespective of whether the field K has infinite degree over its centre or not.

Interpretations of generalized polynomial identities in terms of geometry are included in [2].

1.2. Basic concepts

1.2.1. Suppose that Π is an n-dimensional desarguesian projective space ($2 \leq n < \infty$) which is regarded as a set of points $\mathcal{P}$, say, and a collection of subsets of $\mathcal{P}$ which are called lines (cf. e.g. [8]). The subspaces of Π form the lattice $(\mathrm{u}\Pi, \vee, \cap)$ with $\vee$ and $\cap$ denoting the operation signs for "join" and "intersection", respectively. Any $M \in \mathrm{u}\Pi$ determines projective spaces $\Pi(M)$, Π/M, with lattices of subspaces

[1] See 1.2.2.

$$u(\Pi(M)) = u\Pi(M) = \{X \in u\Pi \mid X \subset M\},$$

$$u(\Pi/M) = u\Pi/M = \{X \in u\Pi \mid X \supset M\},$$

respectively. We shall not distinguish a point $M \in P$ from the subspace $\{M\} \in u\Pi$ and $u\Pi/M$ will be called a *bundle* (of subspaces). The same symbol will denote a *collineation* (being a point-to-point map) and the associated *isomorphism* (which maps subspaces to subspaces).

1.2.2. The term *field* will be used for a not necessarily commuatative field, but *skewfield* always means a non-commutative field. We shall assume throughout this paper that Π is a projective space $\Pi(\mathbf{V})$ on a right vector space $\mathbf{V}$ over a field K. The set of points of $\Pi(\mathbf{V})$ is the set $P(\mathbf{V})$ of all one-dimensional subspaces of $\mathbf{V}$. If $\mathbf{U} \subset \mathbf{V}$ is a subspace of $\mathbf{V}$, then $P(\mathbf{U})$ denotes the subspace of the projective space $\Pi(\mathbf{V})$ given as $\{X \in P(\mathbf{V}) \mid X = \mathbf{x}K \text{ and } \mathbf{x} \in \mathbf{U}\}$. The *centre* of K will be written as Z, and we set $L^{\times} := L \setminus \{0\}$ for any subfield L of K.

Let $\mathbf{U}$ be an $(m+1)$-dimensional subspace of $\mathbf{V}$ with $m \geq 1$. A vector $\mathbf{u} \in \mathbf{U}$ is called *central* with respect to a given basis $\{\mathbf{p}_0, \dots, \mathbf{p}_m\}$ of $\mathbf{U}$ if $\mathbf{u} = \sum_{j=0}^{m} \mathbf{p}_j z_j$ with $z_j \in Z$, and a subspace of $\mathbf{U}$ is named *central* if it can be spanned by central vectors. The central subspaces of $\mathbf{U}$ determine *central subspaces* of $\Pi(\mathbf{U})$ with respect to the frame $F = \{P_0 = \mathbf{p}_0 K, \dots, P_m = \mathbf{p}_m K, E = \mathbf{e}K\}$ of $\Pi(\mathbf{U})$ where $\mathbf{e} = \sum_{j=0}^{m} \mathbf{p}_j$. Those projective collineations of $\Pi(\mathbf{U})$ which are fixing the frame F pointwise are induced by linear automorphisms of $\mathbf{U}$ such that

$$\mathbf{p}_j \mapsto \mathbf{p}_j c \quad (j=0,\dots,m) \text{ and } c\in K^\times.$$

A subspace $\mathcal{P}(\mathbf{S})$ of $\Pi(\mathbf{U})$ is central with respect to $\mathcal{F}$ if and only if it is invariant under all projective collineations fixing the frame $\mathcal{F}$ pointwise. The "if"-part of this assertion is streightforward by using induction on the dimension of $\mathbf{S}$, the "only if"-part is trivial. Assume, finally, that $\mathcal{P}(\mathbf{U})$ is a projective line. Then the set of all points which are central with respect to the frame $\{P_0,P_1,E\}$ is called a Z-*chain* and will be denoted by $[P_0,P_1,E]_Z$. See [4,326]. In terms of cross-ratios (CR) a Z-chain $[P_0,P_1,E]_Z$ is the set of all points X in the line $P_0P_1:=P_0\vee P_1$ satisfying

$$CR(X,E,P_1,P_0) \in Z \cup \{\infty\}.$$

The dual vector space of $\mathbf{V}$ is written as $\mathbf{V}^*$ and $\langle \mathbf{h}^*,\mathbf{v}\rangle$ stands for the image of $\mathbf{v}\in\mathbf{V}$ under the linear form $\mathbf{h}^*\in\mathbf{V}^*$. In order to simplify notation, we shall frequently write Π, $u\Pi$, $\mathcal{P}$ instead of $\Pi(\mathbf{V})$, $u\Pi(\mathbf{V})$, $\mathcal{P}(\mathbf{V})$, respectively.

1.2.3. We shall need the following

LEMMA 1.1. *Let* $s_0,\dots,s_m\in K$ *be linearly independent over the centre* Z *of* K. *Given elements* $r_0,\dots,r_m\in K$ *then*

(1.1) $\quad r_0ts_0 + \dots + r_mts_m = 0$ *for all* $t\in K$

if and only if all r_i*'s vanish simultaneously.*

Proof. The dual vector space of the right vector space K^{m+1} will be identified (as usual) with the left vector space ${}^{m+1}K$. If (1.1) holds with $r_0\neq 0$, say, then $[r_0,\dots,r_m]\in$ $\in {}^{m+1}K$ is a non-trivial linear form such that

$$\langle [r_0,\ldots,r_m],(ts_0,\ldots,ts_m)\rangle = 0 \text{ for all } t\in K.$$

In terms of the projective space $\Pi(K^{m+1})$ this means that the hyperplane $\mathcal{P}(\ker[r_0,\ldots,r_m])$ contains the subspace spanned by the points $(ts_0,\ldots,ts_m)K = (ts_0t^{-1},\ldots,ts_mt^{-1})K$ with $t\in K^\times$. But this subspace is central with respect to the standard frame of $\Pi(K^{m+1})$ according to 1.2.2. This implies the existence of a non trivial linear form $[z_0,\ldots,z_m]\in{}^{m+1}Z\subset{}^{m+1}K$ with

$$\langle [z_0,\ldots,z_m],(ts_0,\ldots,ts_m)\rangle = 0 \text{ for all } t\in K$$

and yields the contradiction $z_0s_0+\ldots+z_ms_m=0$.

The converse is trivial.□

In view of this, all remarks [27;247,249], [28,445] concerning skewfields which satisfy a generalized polynomial identity (1.1) are false.

1.2.4. In $\Pi=\Pi(V)$ we choose two different points P and Q. Let

(1.2) $\quad \zeta : u\Pi/P \to u\Pi/Q$

be a *projective isomorphism*. Then we refer to

(1.3) $\quad \Gamma(\zeta) := \{X\in\mathcal{P} \mid X\in l \text{ and } X\in l^\zeta \text{ for some line } l\ni P\}$

as being the *point set generated by* ζ. We shall also say that ζ is a *generating map of* $\Gamma(\zeta)$. Obviously $P,Q\in\Gamma(\zeta)$, because $P,Q\in PQ$ and $P\in(PQ)^{\zeta^{-1}}$, $Q\in(PQ)^\zeta$. The *fundamental subspace of* ζ is defined as the intersection of all ζ-invariant subspaces and is denoted by $G(\zeta)$.

If $\zeta':u\Pi/P'\to u\Pi/Q'$ is any generating map of $\Gamma(\zeta)$, then $\{P',Q'\}$ is called a *fundamental pair of the point set* $\Gamma(\zeta)$ and we shall use the term *fundamental point of* $\Gamma(\zeta)$ for P' as well as Q'. The intersection of $\Gamma(\zeta)=\Gamma(\zeta')$ with the fundamental subspace of ζ' is called the *improper part of* $\Gamma(\zeta)$ *with respect to* ζ'. All other points of $\Gamma(\zeta)$ form the *proper part of* $\Gamma(\zeta)$ *with respect to* ζ'. These two point sets are denoted by $\Gamma^{\times}(\zeta')$ and $\Gamma^{\circ}(\zeta')$, respectively.

Suppose that H is a hyperplane of Π which contains neither P nor Q. Then, by (1.2),

(1.4) $\quad \zeta_H : X\ (\in H) \mapsto (PX)^{\zeta} \cap H$

is a projective collineation $H \to H$ which is called the *trace map of* ζ *in* H. A point is invariant under ζ_H if and only if it is an element of $\Gamma(\zeta) \cap H$.

The next result is trivial, but important.

PROPOSITION 1.1. *If* $\zeta : \mathrm{u}\Pi/P \to \mathrm{u}\Pi/Q$ is a *projective isomorphism, then any line which is different from* PQ *and passing through* P *contains at most one point of* $\Gamma(\zeta)$ *other than* P.

A simple example of a projective isomorphism (1.2) is a *perspectivity*. Here the generated point set $\Gamma(\zeta)$ is the union of the line PQ and a hyperplane H not passing through P as well as Q. The trace map ζ_H is the identity map in H. The fundamental subspace of ζ equals the line PQ.

The following defintions are subject to the assumption that ζ is no perspectivity: The point set $\Gamma(\zeta)$ given by (1.3) is called a *normal curve*. The map ζ as well as the normal curve $\Gamma(\zeta)$ are named *degenerate* if P equals the fundamental subspace $G(\zeta)$ and *non-degenerate* otherwise. If Π is a projective plane, then a normal curve is also called a *conic* and the map (1.2) is a *projectivity*.

To illustrate these definitions, we recall the situation in a 3-dimensional real projective space [24,135]:

(1) If ζ is non-degenerate, then $\Gamma(\zeta)$ is a twisted cubic.

(2) Let $G(\zeta)$ be a plane. Now $\Gamma^{\times}(\zeta)\subset G(\zeta)$ is a conic (in the usual sense) and $\Gamma^{o}(\zeta)$ equals an affine line $RS\setminus\{R\}$, where $R\in\Gamma^{\times}(\zeta)$, $R\neq P,Q$ and $S\in\mathcal{P}\setminus G(\zeta)$.

(3) Assume that $G(\zeta)=PQ$. Then, since ζ is no perspectivity, there are two ζ-invariant planes at most. In every ζ-invariant plane one line other than PQ belongs to $\Gamma(\zeta)$. Any two such lines are skew and do not pass through P or Q.

We deduce from (3) that $\Gamma(\zeta)=\Gamma(\zeta')$ does not imply $\Gamma^{\times}(\zeta)=\Gamma^{\times}(\zeta')$, as is shown by a normal curve which is the union of two different lines with a common point. Hence we are not always able to speak unambigously of the proper and the improper part of a degenerate normal curve. We shall see in 4.2.1 that there are even normal curves which are degenerate as well as non-degenerate.

The preceding definitions are very close to the ones given in [28] and they still will make sense if we drop the convention about the finite dimensionality of Π.

2. NON-DEGENERATE NORMAL CURVES

2.1. Classification

2.1.1. In order to show that non-degenerate normal curves do exist in $\Pi=\Pi(\mathbf{V})$, we have to give an example of a non-degenerate projective isomorphism [15], [23].

Let $(\mathbf{p}_0,\dots,\mathbf{p}_n)$ be an ordered basis of $\mathbf{V}$ and write[2] $\mathbf{e}=\sum_j \mathbf{p}_j$. Then

$$F = (P_0=\mathbf{p}_0K,\dots,P_n=\mathbf{p}_nK,E=\mathbf{e}K)$$

is an ordered frame of Π. For any two consecutive points $P_{j-1},P_j\in F$ $(j=1,\dots,n)$ there is a unique involutory perspective collineation $\gamma_j:\mathcal{P}\to\mathcal{P}$ whose axis is spanned by all points of F except P_{j-1},P_j with $P_{j-1}{}^{\gamma_j}=P_j$. The product map

$$\gamma := \gamma_1\cdots\gamma_n$$

is a projective collineation satisfying

$$P_0 \xrightarrow{\gamma_1} P_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} P_n,$$

[2] We shall use $\sum_j$ as a shorthand for $\sum_{j=0}^{n}$.

$$P_1 \xmapsto{\gamma_1} P_0 \xmapsto{\gamma_2} \dots \xmapsto{\gamma_n} P_0,$$

$$\dots$$

$$P_j \xmapsto{\gamma_1} \dots \xmapsto{\gamma_{j-1}} P_j \xmapsto{\gamma_j} P_{j-1} \xmapsto{\gamma_{j+1}} \dots \xmapsto{\gamma_n} P_{j-1},$$

$$\dots$$

$$P_n \xmapsto{\gamma_1} \dots \xmapsto{\gamma_{n-1}} P_n \xmapsto{\gamma_n} P_{n-1},$$

$$E \xmapsto{\gamma_1} E \xmapsto{\gamma_2} \dots \xmapsto{\gamma_n} E.$$

Any of the perspective collineations γ_j can be induced by a linear automorphism of $\mathbf{V}$, and it is easy to see that γ is induced by the automorphism $g \in GL(\mathbf{V})$ with

(2.1) $\quad \mathbf{p}_0{}^g = \mathbf{p}_n, \ \mathbf{p}_j{}^g = \mathbf{p}_{j-1} \ (j=1,\dots,n).$

We consider the restricted map

(2.2) $\quad \varphi := \gamma|(u\Pi/P_0) : u\Pi/P_0 \to u\Pi/P_n.$

(Figure 1 illustrates the case $\dim\Pi=n=2$.) Our next task is

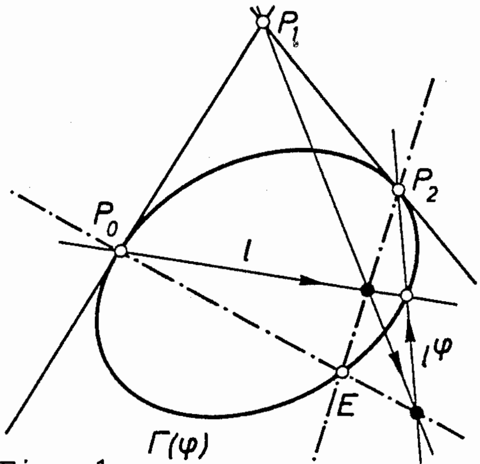

Fig. 1

to show that φ is non-degenerate. The fundamental subspace $G(\varphi)$ is invariant under φ and γ as well. But $P_n \in G(\varphi)$, and so $P_{n-j} \in G(\varphi)^{\gamma^j} = G(\varphi)$ for all $j=1,\dots,n$, whence $G(\varphi)=\mathcal{P}$. We shall refer to this non-degenerate projective isomorphism φ

as the *normal isomorphism* determined by the ordered basis $(p_0, \ldots, p_n)$ of V, and we shall also say that φ is determined by the ordered frame F of Π.

Consequently, the point set $\Gamma(\varphi)$ is a non-degenerate normal curve.

2.1.2. We need some more definitions [15]. Assume that

$$\zeta : u\Pi/P \to u\Pi/Q \quad (P \neq Q)$$

is a non-degenerate projective isomorphism. A non-empty subspace M of Π is called a *chordal subspace of* ζ if, either M is a hyperplane, or $M = L \cap L^{\zeta}$ with $L \in u\Pi/P$ and $\dim\Pi(M) = \dim\Pi(L) - 1$. Frequently we shall use the term *chord* instead of chordal subspace and a k-dimensional chord will also be called a k-*chord*.

If M is a k-chord of ζ, where $k \geq 1$ and $P, Q \notin M$, then the *trace map of* ζ *in* M is defined by formula (1.4), with H to be replaced by M, and this trace map will be denoted by ζ_M. A non-empty subspace of $\Pi(M)$ is invariant under ζ_M if and only if it is a chord of ζ.

The 0-chords of ζ are the points of the normal curve $\Gamma(\zeta)$. If we draw a line m joining two different points of $\Gamma(\zeta)$, then m is a chord in the usual sense and a 1-chord according to the definition given above. However, a 1-chord m, say, of the map ζ will not contain any point of $\Gamma(\zeta)$ at all if the trace map ζ_m has no invariant points.

The subspaces

$$(2.3) \qquad S_P^{(0)}(\zeta) := P, \; S_P^{(k)}(\zeta) := (S_P^{(k-1)}(\zeta) \vee Q)^{\zeta^{-1}}$$

$$(k = 1, \ldots, n-1)$$

are called *osculating subspaces of* ζ *in* P. The osculating subspaces of ζ in Q are given, by (2.3), as the osculating subspaces of ζ^{-1} in Q. We shall use the term *osculating* k-*subspace* for any k-dimensional osculating subspace. In addition the words *tangent* and *osculating hyperplane* will be used if k=1,k=n-1, respectively.

Clearly, we would prefer to speak of osculating subspaces of a non-degenerate normal curve rather than of osculating subspaces of a non-degenerate projective isomorphism. However, as will be shown in 2.4.2, this is not always possible if we want osculating subspaces to be *uniquely* determined by a non-degenerate normal curve. Recall that, by (1.3), a normal curve is merely a set of points and not, for example, a pair formed by the map ζ and the set $\Gamma(\zeta)$.

In the special case of a conic $\Gamma(\zeta)$, say, the tangents of ζ in P and Q can as well be defined in terms of the set $\Gamma(\zeta)$ as follows from

PROPOSITION 2.1. *If* $\zeta:u\Pi/P \to u\Pi/Q$ *is a non-degenerate projective isomorphism, then* $S_P^{(k)}(\zeta)$ *is the only* k-*chord of* ζ *which passes through* $S_P^{(k-1)}(\zeta)$ *(*k=1,...,n-1*) and meets the non-degenerate normal curve* $\Gamma(\zeta)$ *in* P *only.*

Proof. Clearly, we have $S_P^{(1)}(\zeta)\cap\Gamma(\zeta)=P$. If Π is a projective plane (n=2), then every line m, say, passing through P is a 1-chord of ζ, and $m\neq S_P^{(1)}(\zeta)$ implies $P\neq m\cap m\in\Gamma(\zeta)$.

If $n\geq 3$, then

$$S_P^{(1)}(\zeta) = (PQ)^{\zeta^{-1}} = ((PQ)^{\zeta^{-1}}\vee Q)\cap((PQ)^{\zeta^{-1}}\vee Q)^{\zeta^{-1}} = $$
$$= S_P^{(2)}(\zeta)^{\zeta^{-1}}\cap S_P^{(2)}(\zeta),$$

from which it follows that $S_P^{(1)}(\zeta)$ is a 1-chord of ζ. On the other hand, assume that $m=L\cap L^{\zeta}$ is any 1-chord of ζ, where L denotes a plane passing through P. Then $m,m^{\zeta}\subset L$ implies, firstly, $m\cap m^{\zeta}\in\Gamma(\zeta)$ and, secondly, $m\cap m^{\zeta}\neq P$ if and only if $m\neq$ $\neq S_P^{(1)}(\zeta)$.

Still assuming $n \geq 3$, the restricted map

(2.4) $\quad \bar{\zeta} := \zeta \,|\, (u\Pi / S_P^{(1)}(\zeta)) \; : \; u\Pi / S_P^{(1)}(\zeta) \to u\Pi/PQ$

is a non-degenerate projective isomorphism in the $(n-1)$-dimensional quotient space Π/P. The set of k-chords of $\bar{\zeta}$ ($k=0,\dots,n-2$ as viewed from Π/P) coincides with the set of $(k+1)$-chords of ζ which pass through P, and so the osculating k-subspace of $\bar{\zeta}$ in $S_P^{(1)}(\zeta)$ is identical with $S_P^{(k+1)}(\zeta)$. The proof is completed by induction on n.□

We remark that a normal isomorphism φ, as given by formula (2.2), has osculating subspaces ($k=0,\dots,n-1$)

(2.5) $\quad S_{P_0}^{(k)}(\varphi) = P_0 \vee \dots \vee P_k,$

$\quad\quad S_{P_n}^{(k)}(\varphi) = P_n \vee \dots \vee P_{n-k}.$

For $n \geq 3$ the restricted map $\varphi\,|\,(u\Pi/P_0P_1)$ (cf. (2.4)) is a normal isomorphism in Π/P determined by $(P_0P_1,\dots,P_0P_n,P_0E)$.

<u>2.1.3.</u> We are now in a position to solve the classification problem [15], [23], [28].

THEOREM 2.1. *Every non-degenerate projective isomorphism is normal.*

Proof. (1) If an ordered frame $(P_0,\dots,P_n,E)$ is to determine a given non-degenerate projective isomorphism $\zeta : u\Pi/P \to u\Pi/Q$, then necessarily $P_0=P$ and $P_n=Q$. We shall use induction on $\dim\Pi = n$ to prove the assertion.

(2) Suppose $n=2$. Now ζ is a non-degenerate projectivity. We shall adopt the notations $A \overset{c}{\overline{\wedge}} B$, $a \overset{C}{\overline{\wedge}} b$, say, for perspectivities $u\Pi/A \to u\Pi/B$ with axis c ($A,B \notin c$), $u\Pi(a) \to u\Pi(b)$ with centre

C ($C \notin a,b$), respectively, where A,B,C are points and a,b,c are lines. By a well known result on the decomposition of projectivities into a product of perspectivities [8,31], we have a factorization

$$\zeta = P \overset{q'}{\barwedge} P_1'' \overset{p'}{\barwedge} Q$$

with $P_1'' \notin PQ$, because ζ is non-degenerate (Fig. 2). If $E :=$

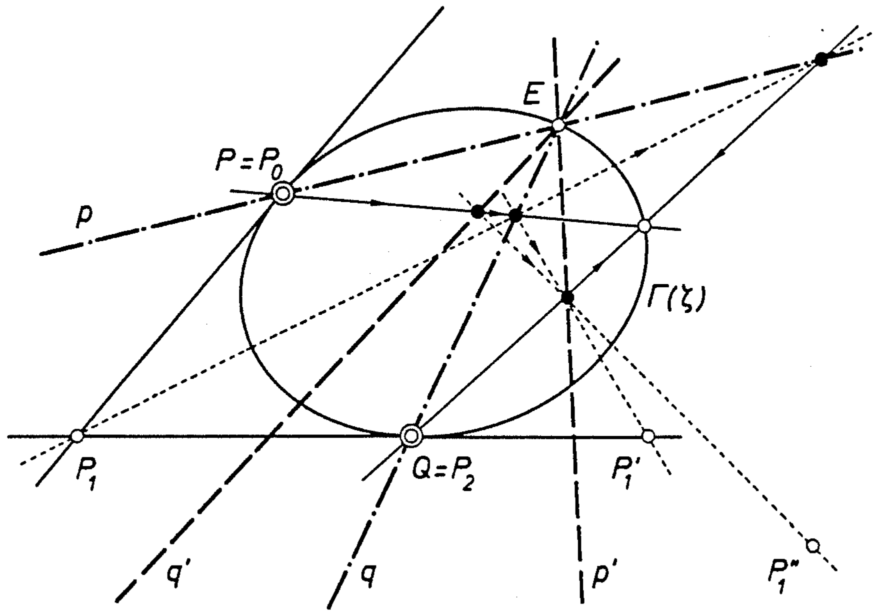

Fig. 2

$= p' \cap q'$ and $q := EQ$, then $q \overset{P}{\barwedge} q' \overset{P_1''}{\barwedge} p'$ is a perspectivity $q \overset{P_1'}{\barwedge} p'$, say, since q, p', q' are concurrent [8,31]. Hence

$$P \overset{q'}{\barwedge} P_1'' \overset{p'}{\barwedge} P_1' = P \overset{q}{\barwedge} P_1'$$

and

$$\zeta = P \overset{q'}{\barwedge} P_1'' \overset{p'}{\barwedge} Q = P \overset{q'}{\barwedge} P_1'' \overset{p'}{\barwedge} P_1' \overset{p'}{\barwedge} Q = P \overset{q}{\barwedge} P_1' \overset{p'}{\barwedge} Q.$$

Similarly, we can replace p' by the line $p := EP$ and P_1' by P_1. Write $P_0 = P$, $P_2 = Q$; then (P_0, P_1, P_2, E) determines ζ.

(3) For $n \geq 3$ we may assume, by induction hypothesis, that the restricted map $\bar{\zeta}$, as given by formula (2.4), is a

normal isomorphism in Π/P determined by an ordered frame $(PP_1,\ldots,PP_n,P\bar{E})$, say, with $P_1,\ldots,P_{n-1},P_n=Q\in S_Q^{(n-1)}(\zeta)$. Since $P\bar{E}$ is a 1-chord of ζ, there exists $(P\bar{E})\cap(P\bar{E})^{\zeta}=:E\in\Gamma(\zeta)$, and $(P=:P_0,\ldots,P_n,E)$ is an ordered frame of Π, because $E\notin$ $\notin S_Q^{(n-1)}(\zeta)$. Let φ denote the normal isomorphism determined by this ordered frame. The restrictions of ζ and φ, respectively, on $u\Pi/S_P^{(1)}(\zeta)$ coincide. Furthermore $(PE)^{\zeta}=(PE)^{\varphi}$ and

$$\begin{aligned}(PQ)^{\zeta} &= ((S_P^{(1)}(\zeta)\vee Q)\cap(S_Q^{(n-1)}(\zeta))^{\zeta^{-1}})^{\zeta} = \\ &= (S_P^{(1)}(\zeta)\vee Q)^{\varphi}\cap S_Q^{(n-1)}(\zeta) = \\ &= (P_0\vee P_n\vee P_{n-1})\cap S_Q^{(n-1)}(\zeta) = P_nP_{n-1} = (PQ)^{\varphi}\end{aligned}$$

which is sufficient for $\zeta=\varphi$ [8,126].□

THEOREM 2.2. *There is a unique non-degenerate projective isomorphism and a unique non-degenerate normal curve to within transformation under projective collineations of* Π.

Proof. By Theorem 2.1, we may restrict ourselves to normal isomorphisms. The group of projective collineations of Π, i.e. $PGL(\Pi)$, operates transitively on the set of ordered frames of Π. This completes the proof.□

2.2. Conjugate points

2.2.1 It follows from Theorems 2.1 and 2.2 that we can restrict ourselves to non-degenerate normal curves generated by normal isomorphisms. We shall investigate a normal isomorphism φ as defined by formula (2.2) and we shall use all the other notations introduced in 2.1.1.

For any point $X\in\Gamma:=\Gamma(\varphi)$ we set

$$(2.6) \qquad H_X := P_0 \vee \dots \vee P_{n-2} \vee X \text{ if } X \neq P_0,$$
$$H_X := P_0 \vee \dots \vee P_{n-1} \qquad \text{if } X = P_0.$$

The subspace H_X is a hyperplane by Proposition 2.1 and formula (2.5). Conversely, let $H \supset P_0 \vee \dots \vee P_{n-2}$ be a hyperplane. Since $\{P_i\}$ is a basis of Π, there is a point X such that

$$(2.7) \qquad X = H \cap H^{\gamma} \cap \dots \cap H^{\gamma^{n-1}}.$$

If $H=P_0\vee\dots\vee P_{n-1}$, then $X=P_0\in\Gamma$. Otherwise $X\neq P_0$, $(P_0X)=H\cap\dots\cap$ $\cap H^{\gamma^{n-2}}$ and $(P_nX)=H^{\gamma}\cap\dots\cap H^{\gamma^{n-1}}$ force $X\in\Gamma$. Now, suppose $H=$ $=P(\ker(t\mathbf{p}^*_{n-1}-\mathbf{p}^*_n))$ with $t\in K$, where $\{\mathbf{p}^*_i\}\subset \mathbf{V}^*$ denotes the dual basis of $\{\mathbf{p}_i\}$. Then X equals the one-dimensional subspace $\mathbf{x}_tK$ of $\mathbf{V}$ with[3] $\mathbf{x}_t=\sum_j \mathbf{p}_j t^j$, because $\langle t\mathbf{p}^*_j-\mathbf{p}^*_{j+1},\mathbf{x}_t\rangle=0$ for $j=0,\dots,n-1$. As t varies in K, we obtain all points of $\Gamma\setminus\{P_n\}$, hence

$$(2.8) \qquad \Gamma(\varphi) = \{\mathbf{x}_tK \mid \mathbf{x}_t=\sum_j \mathbf{p}_j t^j,\ t\in K\} \cup \{P_n\}.$$

The map $X(\in\Gamma)\to H_X(\in u\Pi/(P_0\vee\dots\vee P_{n-2}))$ is a bijection. Its inverse map may be regarded as being a *Veronese map* $P(K^2)\to P(\mathbf{V})$, where we have to identify the projective line $P(K^2)$ with a pencil of hyperplanes. We remark that (2.7) yields another way of generating a non-degenerate normal curve: it involves n projectively related pencils of hyperplanes. See [5,275] or [6,323] and the introduction of [28].

2.2.2. Let $M=P(\mathbf{M})$ be a k-chord of φ $(k\geq 1)$ with $P_0,P_n\notin M$ and write $p_{\mathbf{M}}: \mathbf{p}_nK\oplus\mathbf{M}\to\mathbf{M}$ for the projection with kernel $\mathbf{p}_nK$. It follows immediately that $g_{\mathbf{M}}:=(g|\mathbf{M})p_{\mathbf{M}}:\mathbf{M}\to\mathbf{M}$ (c.f.(2.1)) induces the trace map φ_M.

Clearly, $\mathbf{x}_t\in\Gamma\cap M$ if and only if $\mathbf{x}_t$ is an eigenvector of $g_{\mathbf{M}}$. Since

[3] Upper indices are always in brackets in order to distinguish them from exponents.

$$(2.9)\qquad x_t^{g_M} = (\sum_j p_j t^j)^{gp_M} = (\sum_{j=0} p_j t^{j-1} + p_n)^{p_M} = x_t t,$$

the vector x_t is belonging to the eigenvalue[4] $t \neq 0$ of g_M. Thus the problems to determine the eigenvectors of g_M, the eigenvalues of g_M, the intersection $\Gamma \cap M$, respectively, are equivalent. In the special case of a hyperplane $M = P(H) = P(\ker \sum_j h_j p_j^*)$, say, $x_t K \in \Gamma \cap M$ if and only if

$$<\sum_j h_j p_j^*, x_t> = h_n t^n + \ldots + h_0 = 0,$$

where $h_0 h_n \neq 0$. Cf [17], [29].

A theorem by G. Gordon and T.S. Motzkin [14,220] tells us that at most n conjugacy classes of K contain zeros of the equation $h_n t^n + \ldots + h_0 = 0$. On the other hand, we might as well infer from results by P.M. Cohn [13,207] that the spectrum of the linear automorphism g_H consists of n conjugacy classes of K at most. It is our task to give a translation into the language of geometry.

Two points $U, V \in \Gamma$ are called *conjugate* if, either $U=V$, or $U \neq V$ and the line UV meets Γ in at least three different points. A point of Γ is named *regular* if no other point of Γ is conjugate to it [7], [17].

The definition of regularity in [28]is different, but equivalent. In [33,346] the term "points of first kind" has been introduced for regular points of a non-degenerate conic.

By Proposition 1.1. points P_0, P_n are regular. Suppose that $U = x_u K$ and $V = x_v K$ are distinct. The line UV carries any point $W = (x_u + x_v w) K \in \Gamma$ $(w \neq 0)$ if and only if W $(\neq U, V)$ is invariant under the trace map φ_{UV}. This is equivalent to $x_u + x_v w$, $x_u u + x_v v w$ being linearly dependent by (2.9) or, in other

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[4] Using matrices instead of linear maps would force to speak of *right eigenvalues*; cf. [13,205].

words

(2.10) $u = w^{-1}vw$.

Hence U and V are conjugate points of Γ if and only if u and v are conjugate elements of the ground field K. If U and V are conjugate and distinct, then, by (2.10),

(2.11) $\{(x_u + x_v vws)K\} \cup \{x_v K\}$,

where s is in the centralizer of u, equals the set of invariant points of φ_{UV}. This set (2.11) is always infinite [14,221], [22,409].

2.2.3. Given a point $X \in \Gamma$ $(=\Gamma(\varphi))$ all points conjugate to X form the *conjugacy class of* X *in* Γ. By definition, this conjugacy class does not depend on the normal isomorphism φ which has been used as generating map of Γ.

PROPOSITION 2.2. *Let* M *be a* k-*dimensional subspace of* Π *(*$1 \leq k \leq n-1$*) and let* $\{Q_0, \ldots, Q_k\}$ *be a basis of* $\Pi(M)$ *which is contained in the non-degnerate normal curve* $\Gamma = \Gamma(\varphi)$ *as given in 2.1.1. The points* Q_i *are pairwise conjugate if and only if there is an additional point* $Q_{k+1} \in \Gamma$ *such that* $\{Q_0, \ldots, Q_k, Q_{k+1}\}$ *is a frame of* $\Pi(M)$.

Proof. (1) Suppose $P_0 \in M$. Then every point of $\Gamma \cap M$, other than P_0, has to be an element of $M^{\varphi} \neq M$ as well. Hence $P_0 = Q_0$, say, and none of the assertions of the criterion holds in M. Substituting φ by φ^{-1} shows $P_n \in M$ to be impossible in any event.

(2) Assume that neither P_0 nor P_n is in M. The result is

established, by definition, if M is a line. We use induction on k: If $Q_0,\ldots,Q_k$ are pairwise conjugate ($2\leq k\leq n-1$), then there is a frame $\{Q_0,\ldots,Q_{k-1},Q_k'\}$ of $\Pi(Q_0\vee\ldots\vee Q_{k-1})$ which is contained in Γ. The line joining the conjugate points Q_k and Q_k' carries the point Q_{k+1}, as required.

Conversely, if $\{Q_0,\ldots,Q_{k+1}\}\subset\Gamma$ is a frame of $\Pi(M)$, then M is a chord of φ. Let $Q_k':=Q_kQ_{k+1}\cap(Q_0\vee\ldots\vee Q_{k-1})$; this Q_k' is φ_M-invariant, hence it is in the normal curve Γ and $Q_0,\ldots,$ Q_{k-1},Q_k,Q_k' are pairwise conjugate.□

By Proposition 2.2, any $m\leq n+1$ pairwise inconjugate points of Γ are independent and $m\leq n$ implies that the proper subspace spanned by them meets Γ in no other points. Consequently, any k-dimensional subspace of Π ($1\leq k\leq n-1$) has non-empty intersection with at most k+1 conjugacy classes of a non-degenerate normal curve in Π. (Cf. the Gordon-Motzkin theorem.)

Regular points are important when transferring classical theorems on conics (e.g. Pascal's theorem [7,75]) to the general case.

2.3. Generating maps

2.3.1. We are still persuing the discussion of the normal isomorphism φ given by an ordered basis $(\mathbf{p}_0,\ldots,\mathbf{p}_n)$ of $\mathbf{V}$ according to 2.1.1, and we shall use all the other notations introduced there.

PROPOSITION 2.3. *Two ordered bases* $(\mathbf{p}_j)$, $(\mathbf{p}_j')$ *of* $\mathbf{V}$ *determine the same normal isomorphism if and only if*

(2.12) $\quad \mathbf{p}_j' = \mathbf{p}az^j$ *(j=0,...,n),*

where $a\in K^\times$ *and* $z\in Z^\times$.

Proof. If $\varphi=\varphi'$, then $\mathbf{p}_j'=\mathbf{p}_jc_j$ with $c_j\in K^\times$, as follows from (2.5). The bundle $u\Pi/P_0$ is fixed elementwise under the projective automorphism $\gamma\gamma'^{-1}$, hence there is an element $z\in$ $\in Z^{\times}$ such that $\mathbf{p}_j{}^{gg'^{-1}}=\mathbf{p}_j c_j c_{j-1}^{-1}=\mathbf{p}_j z$ $(j=1,\dots,n)$. Now $c_0=:a$ yields $\mathbf{p}'_j=\mathbf{p}_j a z^j$ by induction on $j=0,\dots,n$. Reversing the above arguments completes the proof.□

Observe that $E'=(\sum_j \mathbf{p}'_j)K$ is a regular point of $\Gamma(\varphi)$.

2.3.2. Now, in several steps, all normal isomorphisms yielding the same non-degenerate normal curve will be determined [17].

THEOREM 2.3. *Assume that* $\varphi:\mathrm{u}\Pi/P_0\to\mathrm{u}\Pi/P_n$ *is a normal isomorphism. There exists a normal isomorphism* $\varphi':\mathrm{u}\Pi/P_0\to\mathrm{u}\Pi/P_n$ *different from* φ*, but also generating the non-degenerate normal curve* $\Gamma(\varphi)$*, if and only if* $\Gamma(\varphi)\setminus\{P_0,P_n\}$ *is contained in a hyperplane and* $\dim\Pi=n\geq 3$.

Proof. (1) If $\Gamma(\varphi)\setminus\{P_0,P_n\}$ is in a hyperplane and $n\geq 3$, then write N for the subspace spanned by $\Gamma(\varphi)\setminus\{P_0,P_n\}$. By Proposition 2.2, neither P_0 nor P_n is an element of N, and we choose a hyperplane $H\supset N$ that meets P_0P_n in a point other than P_0,P_n. Let σ denote the involutory perspective collineation[5] which has axis H and maps P_0 to P_n. Then $\Gamma(\varphi)^{\sigma}=\Gamma(\varphi)$; under σ the tangent of φ in P_n is not mapped to the tangent of φ in P_0, because these two lines are skew by formula (2.5). Now $\varphi':=\sigma\varphi^{-1}\sigma^{-1}$ generates $\Gamma(\varphi)$ too, however the tangent of φ' in P_0 is not the tangent of φ in P_0, whence $\varphi\neq\varphi'$.

(2) It follows from $n=2$ that every line through P_0 has

[5]This collineation σ is not given correctly in my paper [17].

non-empty intersection with its image under φ, whence $\varphi=\varphi'$.

(3) Suppose $\{Q_0,\dots,Q_n\}\subset(\Gamma(\varphi)\setminus\{P_0,P_n\})$ to be a basis of Π. Without loss of generality let $Q_0=E$. By Proposition 2.2, the points P_0,P_n are not lying in any face of the basis $\{Q_i\}$. Write $M:=Q_0\vee\dots\vee Q_{n-1}=:P(\mathbf{M})$, $Q_{n0}:=P_0Q_n\cap M$ and $Q_{nn}:=P_nQ_n\cap$ $\cap M$.

Any normal isomorphism $\varphi':u\Pi/P_0\to u\Pi/P_n$ that generates $\Gamma(\varphi)$ has a trace map φ'_M in M that takes the ordered frame $(Q_0,\dots,Q_{n-1},Q_{n0})$ to the ordered frame $(Q_0,\dots,Q_{n-1},Q_{nn})$. In terms of the notations introduced in 2.2.2, the trace map φ'_M is induced by $g'_{\mathbf{M}}$ and formula (2.9),together with the regularity of $Q_0=E=\mathbf{e}K$, implies that $\mathbf{e}$ belongs to a central eigenvalue of $g'_{\mathbf{M}}$. Hence comparing φ and φ' yields that $g_{\mathbf{M}}g'^{-1}_{\mathbf{M}}$ has $\mathbf{e}$ as an eigenvector belonging to a central eigenvalue. Furthermore, $\{Q_0,\dots,Q_{n-1},Q_{n0}\}$ is elementwise invariant under $\varphi_M\varphi'^{-1}_M$. Thus $g_{\mathbf{M}}g'^{-1}_{\mathbf{M}}$ is a central dilatation on $\mathbf{M}$ and $\varphi_M=\varphi'_M$. This in turn is equivalent to $\varphi=\varphi'$, as required.□

There is a close connection between the decomposition of a projective collineation κ in a product of perspective collineations and the existence of κ-invariant points $\mathbf{f}K$ such that $\mathbf{f}$ is belonging to a central eigenvalue of any inducing linear automorphism for κ. See [32] with remarks given in [17].

The next theorem links geometry with algebra.

THEOREM 2.4. *If* $\varphi:u\Pi/P_0\to u\Pi/P_n$ *is a normal isomorphism, then* $\Gamma(\varphi)\setminus\{P_0,P_n\}$ *is contained in a hyperplane if and only if the ground field* K *has at most* $\dim\Pi+1=n+1$ *elements.*

Proof. (1) If the centre Z of K has cardinality $\geq n+2$, then $\Gamma(\varphi)\setminus\{P_0,P_n\}$ includes at least n+1 regular points which span all of $\mathcal{P}$ by Proposition 2.2.

(2) Let K be a skewfield with finite centre. The "only-if" result will be established if we are able to show the existence of an element $u\in K$ which is transcendental over Z [22].

Suppose that all elements of K are algebraic over Z. Take $a\in K\setminus Z$, whence $Z(a)$, the commutative subfield of K spanned by Z and a, is a Galois-field. There is a non-trivial automorphism of $Z(a)$ that fixes Z elementwise and takes a to $a^q\neq a$, say, where q is a power of a prime number. By the Skolem-Noether theorem (cf. e.g. [13,46] or [26,45] for an elementary proof), this automorphism can be extended to an inner automorphism of K. Hence there is some $u\in K$ such that $u^{-1}yu=y^q$ for all $y\in Z(a)$. According to our assumption this u is algebraic over Z. Therefore the minimal polynomials of u, a, respectively, and $uy^q=yu$ imply that the subfield generated by Z,a,u has finite degree over Z, so that it is a finite field. But, by Wedderburn's theorem, any finite field is commutative and this contradicts $ua\neq au$.

It follows from the existence of a hyperplane which contains $\Gamma(\varphi)\setminus\{P_0,P_n\}$ that there is a central hyperplane H (with respect to $\{P_0,\ldots,P_n,E\}$) having the same property, because the subspace spanned by $\Gamma(\varphi)\setminus\{P_0,P_n\}$ is central by 1.2.2. Let $H=\mathcal{P}(\ker \sum_j z_j p_j^*)$, $z_j\in Z$, $z_0z_n\neq 0$, then

(2.13) $z_nt^n + \ldots + z_0 = 0$ for all $t\in K$

which is not possible, because of $u \in K$ being transcendental over Z.

(3) The proof is completed by the trivial remark that $|\Gamma(\varphi) \setminus \{P_0, P_n\}| \leq n$ follows from $|K| \leq n+1$. □

There is no unique way from $\Gamma(\varphi)$ back to φ if and only if K satisfies a polynomial identity (2.13) (cf.[13,162]) and $n \geq 3$ which in turn is equivalent to $|K| \leq n+1$ and $n \geq 3$. We add, for the sake of completeness, that only if this "case of small ground field" is excluded "Lemma 6" and "Folgerung 2" in [28] are correct results.

As will be shown in 2.4.2, the group of automorphic collineations of a non-degenerate normal curve operates 3-fold transitively on the set of its regular points. Since a non-regular point cannot be fundamental by Proposition 1, we have the following.

THEOREM 2.5. *Two different points of a non-degenerate normal curve form a fundamental pair if and only if they are regular.*

2.4. Automorphic collineations

2.4.1. Every automorphic collineation of $\Gamma(\varphi)$ (cf.2.1.1) takes (non-) regular points of $\Gamma = \Gamma(\varphi)$ to (non-) regular points of Γ. Those linear automorphisms of $\mathbf{V}$ such that

(2.14) $\quad \mathbf{p}_j \mapsto \mathbf{p}_j z^j,$

(2.15) $\quad \mathbf{p}_j \mapsto \mathbf{p}_{n-j},$ $\qquad$ with $j=0,\ldots,n,\ z \in Z^\times$

(2.16) $\quad \mathbf{p}_j \mapsto \sum_{k=0}^{n-j} \mathbf{p}_{j+k} \binom{j+k}{k} z^k$

induce automorphic collineations of Γ: this follows immediately from formula (2.12), $\Gamma(\varphi) = \Gamma(\varphi^{-1})$ and, by the binomial theorem,

$$x_t \mapsto \sum_{j=0}^{n} \sum_{k=0}^{n-j} p_{j+k} \binom{j+k}{k} z^k t^j = x_{t+z},$$

respectively.

It is easily seen that these automorphic collineations generate a group which, regarded as transformation group on the set of regular points of Γ, is isomorphic to the group of projectivities of the projective line $P(Z^2)$ over the centre Z of K. Thus this group of automorphic collineations of Γ is sharply 3-fold transitive on the set of regular points of Γ. In view of this, we have to discuss only the stabilizer of any three regular points of Γ within the group G of all automorphic collineations of Γ.

If $\sigma \in G$ fixes P_0, P_n, E, then $\sigma^{-1}\varphi\sigma$ generates Γ. There are two cases:

(1) Let $n=2$ or $|K| \geq n+2$. Then, by Theorem 2.3 and Theorem 2.4, $\sigma^{-1}\varphi\sigma=\varphi$, and therefore the frame $\{P_0,\ldots,P_n,E\}$ is fixed elementwise under σ. Conversely, every collineation which fixes this frame elementwise is in the group G.

(2) Suppose $n \geq 3$ and $|K| \leq n+1$. The non-degenerate normal curve Γ has $|K|+1$ distinct points and either is a frame of Π or is a basis of the subspace spanned by Γ ($|\Gamma| \leq n+1$). Hence every permutation of Γ which fixes P_0, P_n, E can be extended to at least one automorphic projective collineation of Γ. See [18], [20].

To sum up, we have shown:

THEOREM 2.5. *The group G of automorphic collineations of a* *non-degenerate normal curve* Γ *has a subgroup of projective collineations which is sharply three-fold transitive on the set of regular points of* Γ. *If* $\dim\Pi=n=2$ *or* $|\Gamma|\geqq n+3$, *then the stabilizer of any three different regular points of* Γ *coincides with a group of collineations fixing a frame of* Π *pointwise. If* $n\geqq 3$ *and* $|\Gamma|\leqq n+2$, *then every permutation of* Γ *is the restriction of at least one projective collineation of* Π.

A different way to determine the group G of all automorphic collineations of Γ, working only for $[K:Z]=\infty$, can be found in [28] and involves Amitsur's theorem on generalized polynomial identities. However, as pointed out in [17], the original proof [28,440] is correct only under certain additional assumptions.

2.4.2. We finish this chapter with remarks on osculating subspaces. It is possible to associate with every point $X\in\Gamma(\varphi)$ a flag $(X=S_X^{(0)}(\varphi), S_X^{(1)}(\varphi), \ldots, S_X^{(n-1)}(\varphi))$ the elements of which are called *osculating subspaces of* φ *in* X. See [7] ($n=2$) and [15] ($n\geqq 3$) for details.

The definition of tangent of a conic used in [3] is different from the one in [7]. The definition of osculating subspaces in [28] does not make sense for $|\Gamma|\leqq n+2$ and $n\geqq 3$, because they are not determined uniquely, and fails to work in non-regular points if the characteristic of the ground field is $\neq 0$ and $<n$.

We see that a normal isomorphism φ does generate not only the non-degenerate normal curve $\Gamma(\varphi)$ but also the set of flags

$$\Gamma^{(n-1)}(\varphi) := \{(X, S_X^{(1)}(\varphi), \ldots, S_X^{(n-1)}(\varphi)) \mid X\in\Gamma(\varphi)\}.$$

The linear automorphisms of $\mathbf{V}$ given by formulae (2.14), (2.15), (2.16) induce collineations of the group $G^{(n-1)}$, i.e. the group of automorphic collineations of $\Gamma^{(n-1)}$. Clearly, demanding $\Gamma(\varphi)=\Gamma(\varphi')$ is a coarser relation than demanding $\Gamma^{(n-1)}(\varphi)=\Gamma^{(n-1)}(\varphi')$, where φ,φ' are normal isomorphisms. Hence $G^{(n-1)}$ is a subgroup of G. These two groups coincide if and only if $\Gamma(\varphi)=\Gamma(\varphi')$ does always imply $\Gamma^{(n-1)}(\varphi)=\Gamma^{(n-1)}(\varphi')$. This in turn is equivalent to $|\Gamma|\geq n+3$ or n=2. In the latter cases the term *osculating subspaces of a non-degenerate normal curve* does make sense [7], [17].

Most properties of osculating subspaces depend on the characteristic of the ground field K irrespective of whether K is commutative or not. We mention, without proof, one result: If the characteristic of K is a prime number which divides the dimension n of Π, then all osculating hyperplanes of a normal isomorphism belong to a pencil of hyperplanes. See [18], [20], [35], [36].

3. DEGENERATE CONICS

3.1. Degenerate projectivities

<u>3.1.1.</u> Let $\Pi=\Pi(\mathbf{V})$ be a projective plane. The existence of a degenerate projectivity implies that Π is non-pappian or, equivalently, that K is a skewfield. We assume (in this chapter only) that K is a skewfield.

An ordered basis $(\mathbf{p},\mathbf{q};\mathbf{a})$ of $\mathbf{V}$ and a non-central element $a\in K\setminus Z$ give rise to linear automorphisms $g_0,g_1\in GL(\mathbf{V})$ such that

$$(3.1)\qquad \mathbf{p}^{g_0}=\mathbf{p},\quad \mathbf{q}^{g_0}=\mathbf{q},\quad \mathbf{a}^{g_0}=a\mathbf{a},$$

$$(3.2)\qquad \mathbf{p}^{g_1}=\mathbf{q},\quad \mathbf{q}^{g_1}=\mathbf{p},\quad \mathbf{a}^{g_1}=\mathbf{a}.$$

The collineation induced by g_0 is a homology γ_0, say, with centre $A:=aK$, axis PQ ($P:=pK$, $Q:=qK$) and characteristic cross-ratio $CR(X^{\gamma_0},X,XA\cap PQ,A)=\hat{a}$, where $\hat{a}\subset K$ denotes the conjugacy class of a and X is any point of $\mathcal{P}\setminus(\{A\}\cup PQ)$. The map g_1 induces an involutory perspective collination denoted by γ_1 with axis AU ($U:=uK$, $\mathbf{u}=\mathbf{p}+\mathbf{q}a$) and $P^{\gamma_1}=Q$. Put $g=:g_0g_1$ and $\gamma:=\gamma_0\gamma_1$. This γ is a projective collineation which interchanges P with Q and fixes A. Hence the restricted map

$$(3.3) \qquad \zeta := \gamma|(u\Pi/P) : u\Pi/P \to u\Pi/Q$$

is a projectivity which has PQ as invariant line. Setting $\mathbf{M}:=\mathbf{a}K\oplus\mathbf{u}K$ the trace map ζ_{AU} in $AU=\mathcal{P}(\mathbf{M})$ is induced by the linear automorphism $g_{\mathbf{M}}:=(g|\mathbf{M})p_{\mathbf{M}}$, where $p_{\mathbf{M}}:\mathbf{V}\to\mathbf{M}$ is the projection with kernel $\mathbf{q}K$. Consequently, $\mathbf{a}^{g_{\mathbf{M}}}=\mathbf{a}a$, $\mathbf{u}^{g_{\mathbf{M}}}=\mathbf{u}a$ and

$$(3.4) \qquad \{(\mathbf{a}+\mathbf{u}s)K\} \cup \{\mathbf{u}K\},$$

where s is in the centralizer of a, is the set of invariant points of ζ_{AU}. The trace map ζ_{AU} is non-identical, because a is a non-central element of K. Thus, firstly, ζ_{AU} has infinitely many fixed points (cf. 2.2.2), secondly, ζ is a degenerate projectivity and, thirdly, $\Gamma(\zeta)$ is a degenerate conic. We shall say that ζ (as well as $\Gamma(\zeta)$) is determined by the ordered basis $(\mathbf{p},\mathbf{q};\mathbf{a})$ of $\mathbf{V}$ and the element $a\in K\setminus Z$.

Letting $a\in Z^\times$ the above construction yields a perspectivity ζ and $\Gamma(\zeta)=PQ\cup AU$ is no conic in the sense of our definition. However, sometimes it would be more convenaiant to use the term "degenerate conic" in this case as well (cf. 4.1.2); but we shall stick to our previous definitions as given in 1.2.4.

3.1.2. The following proposition illustrates that those examples of degenerate projectivities as introduced in 3.1.1

are, in fact, all degenerate projectivities.

PROPOSITION 3.1. *Any degenerate projectivity* $\zeta : u\Pi/P \to u\Pi/Q$ *is determined by an ordered basis of* $\mathbf{V}$ *and an element of* $K \setminus Z$.

Proof. Let $l, \tilde{l} \neq PQ$ be two different lines passing through P. Then $A := l \cap l^{\zeta}$, $\tilde{A} := \tilde{l} \cap \tilde{l}^{\zeta}$ are different points of the degenerate conic $\Gamma(\zeta)$. By Proposition 1.1, the line $A\tilde{A}$ meets PQ in a point U other than P,Q. The trace map ζ_{AU} is fixing $A, \tilde{A}, U$ but is not an identity-map, because ζ is no perspectivity. Suppose $A = \mathbf{a}K$, $U = \mathbf{u}K$, $\tilde{A} = (\mathbf{a}+\mathbf{u})K$ and $\mathbf{M} = \mathbf{a}K \oplus \mathbf{u}K$. If $g_{\mathbf{M}} \in GL(\mathbf{M})$ induces the trace map ζ_{AU}, then $\mathbf{a}$ and $\mathbf{u}$ are eigenvectors of $g_{\mathbf{M}}$ belonging to the same eigenvalue $a \in K \setminus Z$, say. Choose $\mathbf{p}, \mathbf{q} \in \mathbf{V}$ such that $P = \mathbf{p}K$, $Q = \mathbf{q}K$, $U = (\mathbf{p}+\mathbf{q}a)K$. Then $(\mathbf{p}, \mathbf{q}; \mathbf{a})$ and a determine the degenerate projectivity ζ.□

PROPOSITION 3.2. *Two ordered bases* $(\mathbf{p}, \mathbf{q}; \mathbf{a})$, $(\mathbf{p}', \mathbf{q}'; \mathbf{a}')$ *of* $\mathbf{V}$ *and elements* $a, a' \in K \setminus Z$, *respectively, such that*

$$\begin{aligned} \mathbf{p}' &= \mathbf{p}c_{00} \\ \mathbf{q}' &= \qquad\quad \mathbf{q}c_{11} \qquad\qquad\qquad (c_{jk} \in K) \\ \mathbf{a}' &= \mathbf{p}c_{02} + \mathbf{q}c_{12} + \mathbf{a}c_{22} \end{aligned}$$

determine the same degenerate projectivity if and only if there exists $z \in Z^{\times}$ *with*

$$(3.4) \qquad c_{00} = c_{11}z, \; c_{12} = c_{02}c_{22}{}^{-1}ac_{22}, \; a' = c_{22}{}^{-1}ac_{22}z.$$

Proof. In terms of the notation of 3.1.1, we have $\zeta = \zeta'$ if and only if the trace maps ζ_{AU} and ζ'_{AU} coincide or, equivalently,

$$\begin{aligned} \mathbf{a}^{g'_{\mathbf{M}}} = {} & \mathbf{p}(c_{02}a'c_{22}{}^{-1} - c_{00}c_{11}{}^{-1}c_{12}c_{22}{}^{-1}) + \mathbf{q}(*) + \\ & + \mathbf{a}c_{22}a'c_{22}{}^{-1} = \end{aligned}$$

$$= a^{g_{M_z}} = aaz$$

and

$$u^{g'_M} = pc_{00}c_{11}{}^{-1}a + q(**) = u^{g_{M_z}} = uaz$$

with $z \in Z^\times$. □

Our way of dealing with degenerate projectivities is different from the one used in [27]. We have been aiming at finding natural extensions of a given degenerate projectivity to a collineation of Π. As is shown by Proposition 3.2, our method yields no unique extending collineation, but all such collineations are still closely related.

Degenerate conics are also called *C-configurations* and can be obtained as certain planar sections of a regulus in a three dimensional non-pappian projective space [33,325]. Examples of degenerate conics in translation planes are given in [30].

3.1.3. Using the notation introduced in 3.1.1, we look at the degenerate conic $\Gamma(\zeta)=:\Gamma$. The improper part of Γ with respect to ζ is the line $\Gamma^\times(\zeta)=PQ$. This is the *only* line contained in Γ, because ζ is no perspectivity. Furthermore, $\Gamma^\times(\zeta)=G(\zeta)$ is the fundamental line of ζ. So it makes sense to call $\Gamma^\times(\zeta)$, $\Gamma^o(\zeta)$, $G(\zeta)$ the *improper part*, *proper part*, *fundamental line*, respectively, *of the degenerate conic* Γ.

We deduce the parametric representation

$$(3.5)\qquad \Gamma^o(\zeta) = \{\mathbf{y}_t K \mid \mathbf{y}_t = \mathbf{p}t + \mathbf{q}ta + \mathbf{a},\ t \in K\},$$

since $\Gamma^o(\zeta)=:\Gamma^o$ equals the set of all points $\mathbf{a}'K$ with $\mathbf{a}'$ satisfying the conditions in Proposition 3.2.

Consider the projection $\pi: P \to PQ$ through any point $\mathbf{y}_u K$, say. We obtain

$$(3.6)\qquad n := (\Gamma^o \setminus \{\mathbf{y}_u K\})^\pi = \{(\mathbf{p} + \mathbf{q}(t-u)a(t-u)^{-1})K \mid t \in K \setminus \{u\}\} =$$

$$= \{X \in PQ \mid CR(X,E,P,Q) = \hat{a}\},$$

where $E=(\mathbf{p}+\mathbf{q})K$. This set $\mathcal{H}$ is infinite [19]. By the proof of Proposition 3.1, every point of $\mathcal{H}$ is the image of infinitely many points of Γ^o and, by the last equality in (3.6), $\mathcal{H}$ does not depend on the choice of $u \in K$. Therefore every line joining two different points of Γ^o meets PQ in a point belonging to $\mathcal{H}$ and, conversely, any line which joins a point of $\mathcal{H}$ and a point of Γ^o can be spanned by two distinct point of Γ^o as well. Clearly, a point of $\mathcal{H}$ never is a fundamental point of Γ.

Next we take any line $l=P(\ker(h_0\mathbf{p}^*+h_1\mathbf{q}^*+h_2\mathbf{a}^*))$ different from PQ, i.e. $(h_0,h_1)\neq(0,0)$, where $\{\mathbf{p}^*,\mathbf{q}^*,\mathbf{a}^*\}$ is the dual basis of $\{\mathbf{p},\mathbf{q},\mathbf{a}\}$. The intersection $l\cap\Gamma^o$ is the set of all points $\mathbf{y}_tK\in\Gamma^o$ with $t\in K$ and

(3.7) $\quad h_0t + h_1ta + h_2 = 0.$

For $h_0=0$ or $h_1=0$ a unique point $\mathbf{y}_tK$ exists in accordance with Proposition 1.1. If $h_0h_1\neq 0$, then we may assume $h_1=1$, hence

(3.8) $\quad h_0t + ta + h_2 = 0.$

See [13,222] for results on the solutions of equation (3.8). Cf. also [33,331].

3.2. Generating maps

3.2.1. The crucial result on degenerate conics is

THEOREM 3.1. *Any two degenerate projectivities* ζ,ζ' *yield the same degenerate conic if and only if there are ordered bases* $(\mathbf{p},\mathbf{q};\mathbf{a})$, $(\mathbf{p}',\mathbf{q}';\mathbf{a})$ *of* $\mathbf{V}$ *and elements* $a, a'\in K\setminus Z$, *re-*

spectively, which determine ζ, ζ' *such that, either*

(3.10) $[a:Z] \neq 2,\ a' = (z_0+z_1a)(z_2+z_3a)^{-1}$

$$\mathbf{p}' = \mathbf{p}z_1w + \mathbf{q}z_0w,$$

$$\mathbf{q}' = \mathbf{p}z_3w + \mathbf{q}z_2w,$$

with $z_i \in Z,\ z_1z_2-z_0z_3 \neq 0,\ w \in K^\times$, *or*

(3.11) $[a:Z] = 2,\ a' = z_0 + z_1a,$

$$\mathbf{p}' = \mathbf{p}(-vz_0+(w-m_1v)z_1) + \mathbf{q}(-wz_0+m_0vz_1),$$

$$\mathbf{q}' = \mathbf{p}v \qquad\qquad\qquad + \mathbf{q}w,$$

with $a^2=m_0+m_1a,\ z_i, m_i \in Z,\ v,w \in K,\ (v,w) \neq (0,0),\ v^{-1}w \notin \hat{a}$ *(when* $v \neq 0$*), where* $\hat{a} \subset K$ *denotes the conjugacy class of* a.

Proof. (1) Suppose $\Gamma(\zeta)=\Gamma(\zeta')$. We read off from Proposition 3.1 that ζ,ζ' are determined, respectively, by ordered bases $(\mathbf{p},\mathbf{q};\mathbf{a})$, $(\mathbf{p}',\mathbf{q}';\mathbf{a})$ of $\mathbf{V}$ and elements $a,a' \in K\setminus Z$, say. Since $\mathbf{p}K \vee \mathbf{q}K = \mathbf{p}'K \vee \mathbf{q}'K$, we obtain

(3.12) $\mathbf{p}' = \mathbf{p}c_{00} + \mathbf{q}c_{10}$

$$\mathbf{q}' = \mathbf{p}c_{01} + \mathbf{q}c_{11},$$

where (c_{jk}) is an invertible matrix with entries in K. From formulae (3.5) and (3.12) we deduce

$$\Gamma^0(\zeta') = \{\mathbf{y}'_tK \mid \mathbf{y}'_t = \mathbf{p}(c_{00}t+c_{01}ta')+\mathbf{q}(c_{10}t+c_{11}ta')+\mathbf{a},\ t \in K\}.$$

Hence, by $\Gamma(\zeta)=\Gamma(\zeta')$,

$$(c_{00}t+c_{01}ta')a = (c_{10}t+c_{11}ta')$$

for all $t \in K$ or, equivalently,

(3.13) $c_{10}t - c_{00}ta + c_{11}ta' - c_{01}ta'a = 0$ for all $t \in K$.

The left-side coefficients of this identity (3.13) cannot vanish simultaneously, because (c_{jk}) is a regular matrix.

Then, by Lemma 1.1, there exists a non-trivial linear combination $z_0+z_1a-z_2a'-z_3a'a=0$ $(z_i\in Z)$. Thus

(3.14) $a' = (z_0+z_1a)(z_2+z_3a)^{-1}$

is an element of the commutative subfield $Z(a)\subset K$ and substitution in (3.13) implies

(3.15) $$(c_{10}z_2+c_{11}z_0)t + (c_{10}z_3-c_{00}z_2+c_{11}z_1-c_{01}z_0)ta + + (-c_{00}z_3-c_{01}z_1)ta^2 = 0 \text{ for all } t\in K.$$

If $1,a,a^2$ are linearly independent over Z, i.e. $[a:Z]>2$, then all left-side coefficients in (3.15) have to vanish simultaneously by Lemma 1.1. Therefore

(3.16) $$(c_{jk}) = \begin{pmatrix} z_1 & -z_3 \\ -z_0 & z_2 \end{pmatrix} \cdot \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$$

with $w\in K^\times$ and $z_1z_2-z_0z_3\neq 0$.

If $[a:Z]=2$, then $a^2=m_0+m_1a$ where $m_0,m_1\in Z$, $m_0\neq 0$. We may assume $z_2=1$, $z_3=0$ in (3.14). Substituting in (3.15) we have

(3.17) $$(c_{10}+c_{11}z_0-c_{01}z_1m_0)t + (-c_{00}+c_{11}z_1-c_{01}(z_0+z_1m_1))ta = = 0 \text{ for all } t\in K.$$

The same arguments as before yield

(3.18) $$(c_{jk}) = \begin{pmatrix} w-m_1v & v \\ m_0v & w \end{pmatrix} \cdot \begin{pmatrix} z_1 & 0 \\ -z_0 & 1 \end{pmatrix}$$

with $v,w\in K$. Since (c_{jk}) is invertible, its right column rank equals 2. Thus $v=0$ forces $w\neq 0$ and $w=0$ implies $v\neq 0$. Suppose, finally, that $vw\neq 0$. Then $v^{-1}w-m_1\neq m_0w^{-1}v$ and consequently

(3.19) $$(v^{-1}w)^2 \neq m_0 + m_1(v^{-1}w).$$

By [12,302] or [13,54], the inequality (3.19) is equivalent to $(v^{-1}w)$ not conjugate to a. (Another proof of this can be given by discussing the intersection of a line and a non-degenerate conic; cf. 2.2.2.)

(2) By reversing the above arguments, it is clear that (3.15) or (3.17) implies $\Gamma(\zeta')\subset\Gamma(\zeta)$ and the reader will easily show that $\Gamma(\zeta')=\Gamma(\zeta)$, as required.□

The proof of Theorem 3.1 follows the same pattern as that of "Satz2" in [27]. In contrast to [27], where most of this chapter's material has been taken from, nearly all of our results will turn out to be immediate consequences of Theorem 3.1. Thus it is possible to omit some lengthy calculations including another application of results on generalized polynomial identities [27,248-249].

3.2.3. A subset $\mathfrak{c}$ of the fundamental line of a degenerate conic Γ is called a *fundamental chain of* Γ if $\mathfrak{c}$ is a maximal set with the following property: Any two different points $P',Q'\in\mathfrak{c}$ form a fundamental pair of Γ.

COROLLARY 3.1. *Let* $\Gamma(\zeta)$ *be a degenerate conic given by* (3.3). *If* $[a:Z]\neq 2$, *then*

(3.20) $[P,Q,E]_Z$

where $E=(\mathbf{p}+\mathbf{q})K$ *is the only fundamental chain of* $\Gamma(\zeta)$. *If* $[a:Z]=2$, *then any point* $Q'=\mathbf{q}'K\in PQ\setminus n$, $\mathbf{q}'=\mathbf{p}v+\mathbf{q}w$, *lies in a fundamental chain* $\mathfrak{c}_{Q'}$ *of* $\Gamma(\zeta)$ *given by*

(3.21) $\mathfrak{c}_{Q'} = [P',Q',E']_Z$

where $P'=\mathbf{p}'K$, $\mathbf{p}'=\mathbf{p}(w-m_1v)+\mathbf{q}m_0v$, $E'=(\mathbf{p}'+\mathbf{q}')K$, *and any two different fundamental chains of* $\Gamma(\zeta)$ *have empty intersection.*

Proof. Theorem 3.1 immediately establishes the result, since $\{X\in PQ\,|\,\{X,Q'\}$ is a fundamental pair of $\Gamma(\zeta)\}$ is a is a Z-chain (cf. 1.2.2) as well as a fundamental chain of $\Gamma(\zeta)$.□

We remark that for $[a:Z]=2$ the line PQ is covered (disjointly) by the fundamental chains of $\Gamma(\zeta)$ and the set n. The concept of "ordinary fundamental point" ("gewöhnlicher Grundpunkt"), as has been introduced in [27], will not be used in this article, because "fundamental chain" seems more appropriate.

The following result may be regarded as "Pascal's theorem" for degenerate conics [27].

PROPOSITION 3.3. *Let* Γ *be a degenerate conic. Given three pairwise different points* P_1, P_2, P_3 *in the same fundamental chain of* Γ *and three pairwise different points* A_1, A_2, A_3 *in the proper part of* Γ, *then*

$$P_1A_2 \cap P_2A_1,\quad P_1A_3 \cap P_3A_1,\quad P_2A_3 \cap P_3A_2$$

are three collinear points.

Proof. By calculation.□

3.4. Automorphic collineations and classification

3.4.1. Suppose that, as before, a degenerate conic $\Gamma=\Gamma(\zeta)$ is given by formula (3.3). We shall frequently adopt notions of affine geometry by regarding PQ as line at infinity. The term *$\dot{f}$-automorphism of* $\mathbf{V}$ will be used as a shorthand for any bijective semi-linear map $\mathbf{V}\to\mathbf{V}$ with respect to an automorphism $\dot{f}$ of the skewfield K.
If a collineation $\kappa \in P\Gamma L(\Pi)$ fixes the line PQ, then κ is induced by one and only one $\dot{f}$-automorphism $f \in \Gamma L(\mathbf{V})$ such that

(3.22) $\quad \mathbf{a}^f = \mathbf{p}c_{02} + \mathbf{q}c_{12} + \mathbf{a}.$

This collineation κ is projective if and only if $\dot{f}$ is an inner automorphism of K, whereas $\dot{f}=\mathrm{id}_K$ yields the normal subgroup of those projective collineations which preserve all affine ratios in $\mathcal{P}\setminus PQ$. Since $\kappa^{-1}\zeta\kappa$ is a generating map of Γ^{κ} which is determined by $(\mathbf{a}^f,\mathbf{p}^f;\mathbf{q}^f)$ and $a^{\dot{f}}$, Proposition 3.1 and Theorem 3.1 give necessary and sufficient conditions for f to induce an automorphic collineation of Γ:

THEOREM 3.2. *Let* $\Gamma(\zeta)$ *be determined by an ordered basis* $(\mathbf{p},\mathbf{q};\mathbf{a})$ *of* $\mathbf{V}$ *and an element* $a\in K\setminus Z$. *Any* $\dot{f}$*-automorphism* $f\in\Gamma L(\mathbf{V})$ *satisfying* (3.22) *induces an automorphic collineation of* $\Gamma(\zeta)$ *if and only if, firstly, the conditions stated in Theorem 3.1. for* $\mathbf{a}',\mathbf{p}',\mathbf{q}'$ *hold when substituting* $\mathbf{a}^f,\mathbf{p}^f,\mathbf{q}^f$, *respectively, and, secondly, there exists* $u\in K$ *such that*

$$\mathbf{a}^f = \mathbf{p}u + \mathbf{q}ua + \mathbf{a}.$$

Those linear maps $f\in GL(\mathbf{V})$ whose matrices with respect to $(\mathbf{p},\mathbf{q};\mathbf{a})$ equal

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & ua \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } u\in K$$

induce a normal subgroup of automorphic translations of $\Gamma= =\Gamma(\zeta)$ which operates regularly on the proper part of Γ. The stabilizer of $A=\mathbf{a}K$ within the group of all automorphic collineations of Γ is induced by $\dot{f}$-automorphisms with matrices (written as a product)

$$\begin{pmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_1 & -z_3 & 0 \\ -z_0 & z_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

for $[a:Z]\neq 2$, $a^{f}=(z_0+z_1a)(z_2+z_3a)^{-1}$ and

$$\begin{pmatrix} w-m_1v & v & 0 \\ m_0v & w & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_1 & 0 & 0 \\ -z_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $[a:Z]=2$, $a^{\dot{f}}= z_0+z_1a$. Here u,v,w are subject to the conditions stated in Theorem 3.1. If $[a:Z]\neq 2$, then the orbit of $Q=\mathbf{q}K$ under this stabilizer is a subset q of the only fundamental chain of Γ. The "size" of this subset q depends on the existence of "suitable" automorphisms of K. If $[a:K]=$ $=2$, then the orbit of Q under this stabilizer equals $PQ\setminus n$; the orbit of Q under the stabilizer of A within the subgroup of automorphic collineations of Γ which preserve all affine ratios ($\dot{f}=\mathrm{id}_K$) is also $PQ\setminus n$ for $[a:Z]=2$.

The matrices

$$\begin{pmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with } w\in Z^{\times}$$

yield linear maps inducing a subgroup of automorphic homologies of Γ whose common centre is A.

3.4.2. We turn to the problem of classification [27]. If we are given two degenerate conics, then (possibly after applying a projective collineation on one of them) we may assume that these conics are determined by the same ordered basis of $\mathbf{V}$ and two elements of $K\setminus Z$. From Theorem 3.2 we deduce immediately

THEOREM 3.3. *Given two degenerate conics* $\Gamma(\zeta), \Gamma(\zeta')$ *which are determined by the same ordered basis of* **V** *and elements* $a, a' \in K \setminus Z$, *respectively, there is a (projective) collineation mapping* $\Gamma(\zeta')$ *onto* $\Gamma(\zeta)$ *if and only if there is an (inner) automorphism* $\dot{f}$ *of* K *such that, either*

(3.23) $[a:Z] \neq 2$, $a'^{\dot{f}} = (z_0+z_1a)(z_2+z_3a)^{-1}$

with $z_1z_2-z_0z_3 \neq 0$, $z_i \in Z$, *or*

(3.24) $[a:Z] = 2$, $a^{\dot{f}} \in Z(a)$.

This brings to an end our discussion of degenerate conics.

4. DEGENERATE NORMAL CURVES

4.1. A few results

4.1.1. Suppose that $\Pi(\mathbf{V})$ is an n-dimensional projective space. Let

$$\zeta : u\Pi/P \to u\Pi/Q \quad (P \neq Q)$$

be a degenerate projective isomorphism whose fundamental subspace $G(\zeta)$ is k-dimensional $(1 \leq k \leq n-1)$.

If k=1, then the improper part $\Gamma^{\times}(\zeta)$ of the normal curve $\Gamma(\zeta)$ with respect to ζ is the line PQ, whereas $k \geq 2$ implies that $\Gamma^{\times}(\zeta)$ is a non-degenerate normal curve in the projective space $\Pi(G)$ $(G := G(\zeta))$, because the restricted map $\zeta|(u\Pi(G))/P$ is non-degenerate.

Next let G be no hyperplane. Then $\zeta|u\Pi/G$ is a projective automorphism of $u\Pi/G$. If $X \in \Gamma^{\circ}(\zeta)$, i.e. the proper part of $\Gamma(\zeta)$ with respect to ζ, then $(X \vee G)^{\zeta} = X \vee G$ and $\Gamma(\zeta) \cap (X \vee G)$

is the point set generated by $\zeta|(u\Pi(X\vee G))/P$. This makes clear that $\Gamma^{o}(\zeta)$ is empty if ζ has no invariant (k+1)-dimensional subspaces.

4.1.2. The preceding discussion tells us that at first projective isomorphisms ζ with a fundamental hyperplane have to be studied. Here the generated point set $\Gamma(\zeta)$ is either the union of two distinct lines (n=2; cf. 3.1.1) or a degenerate normal curve ($n\geq 3$).

According to [28,445] there exists always a basis $\{\mathbf{p}_0,\ldots,\mathbf{p}_{n-1},\mathbf{a}\}$ of $\mathbf{V}$ such that

(4.1) $\quad \Gamma^{\times}(\zeta) = \{\mathbf{x}_tK \mid \mathbf{x}_t=\sum_{j=0}^{n-1}\mathbf{p}_j t^j,\ t\in K\} \cup \{\mathbf{p}_{n-1}K\}$

and

(4.2) $\quad \Gamma^{o}(\zeta) = \{\mathbf{y}_tK \mid \mathbf{y}_t=\sum_{j=0}^{n-1}\mathbf{p}_j ta^j+\mathbf{a},\ t\in K\}$

with $a\in K^{\times}$. A hyperplane $H=P(\ker\sum_{j=0}^{n-1}h_j\mathbf{p}_j^*+h\mathbf{a}^*)$, where $\{\mathbf{p}_0^*,\ldots,\mathbf{p}_{n-1}^*,\mathbf{a}^*\}$ is the dual basis of $\{\mathbf{p}_0,\ldots,\mathbf{p}_{n-1},\mathbf{a}\}$, contains $\Gamma^{o}(\zeta)$ if and only if

(4.3) $\quad h_0t + h_1ta + \ldots + h_{n-1}ta^{n-1} + h = 0$ for all $t\in K$

or, equivalently, h=0 and $[a:Z]\leq n-1$ by Lemma 1.1. On the other hand, it is easily seen that the dimension of the subspace spanned by $\Gamma^{o}(\zeta)$ equals $\min\{[a:Z],n\}$. (Use "Satz 1" in [17] and "9.3" in [28].) Hence $a\in Z^{\times}$ if and only if $\Gamma^{o}(\zeta)$ is an affine line. Some examples of automorphic collineations of $\Gamma(\zeta)$ are included in [28].

4.1.3. Returning to the general case and assuming $k\leq n-2$ it follows that every ζ-invariant (k+1)-dimensional subspace has non-empty intersection with $\Gamma^{\cup}(\zeta)$. Results concerning the case k=n-2 can be found in [28].

4.2. Final remarks

<u>4.2.1.</u> Let $\Pi(V)$ be a 4-dimensional projective space and let K be the Galois-field of order 2. A non-degenerate normal curve in Π is just a triangle by (2.8). On the other hand, it is easy to see that there exists a degenerate projective isomorphism ζ, say, the generated point set of which is a triangle as well: We have to ensure only that the fundamental subspace $G(\zeta)$ is a plane and that none of the seven hyperplanes passing through $G(\zeta)$ is ζ-invariant. This forces $\Gamma^{\times}(\zeta)$ to be a triangle in $G(\zeta)$ and $\Gamma^{o}(\zeta)=\emptyset$, as required.

This example shows that a normal curve may be degenerate as well as non-degenerate, but all normal curves with this property are not known to the author. Cf. however [28,436].

<u>4.2.2.</u> In general, the problems of classification, automorphic collineations and generating maps seem to be unsolved for degenerate normal curves. It should also be interesting to discuss, for example, the group of collineations fixing the point set $\Gamma^{o}(\zeta)$ given by (4.2) irrespective of what happens to $\Gamma^{\times}(\zeta)$ given by (4.1).

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