From geometry to invertibility preservers*

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Abstract

We characterize bijections on matrix spaces (operator algebras) preserving full rank (invertibility) of differences of matrix (operator) pairs in both directions.

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1 Introduction

Marcus and Purves [19] proved that every unital invertibility preserving linear map on a matrix algebra is either an inner automorphism or an inner anti-automorphism. One of the equivalent formulations of Gleason-Kahane-Żelazko theorem [6, 16, 25] states that every unital linear functional on a complex unital Banach algebra $\mathcal{A}$ sending every invertible element into a nonzero scalar is multiplicative. Equivalently, if a linear functional $f : \mathcal{A} \to \mathbb{C}$ maps every

element $a \in \mathcal{A}$ into its spectrum $\sigma(a)$, then f is multiplicative. This two results motivated Kaplansky to formulate the question under which conditions an invertibility preserving linear unital map between two algebras must be a Jordan homomorphism [17]. A lot of work has been done on this problem (see the surveys [1, 3, 22]). We will mention here only the results that are relevant for our paper. Let X be a complex Banach space and $B(X)$ the algebra of all bounded linear operators on X. In 1986 Jafarian and Sourour [15] proved that every surjective unital linear map $\phi : B(X) \to B(X)$ preserving invertibility in both directions, i.e., having the property that A is invertible if and only if $\phi(A)$ is invertible, is either of the form $\phi(A) = TAT^{-1}$, $A \in B(X)$, for some invertible $T \in B(X)$, or of the form $\phi(A) = TA'T^{-1}$, $A \in B(X)$, for some invertible bounded linear operator $T : X' \to X$. Here, A' denotes the adjoint of A and X' the dual of X. Under the additional assumption of injectivity the assumption of preserving invertibility in both directions can be relaxed to the weaker assumption of preserving invertibility in one direction only [23]. The proof of the result of Jafarian and Sourour was simplified in [21]. It is rather easy to see that a linear map $\phi : B(X) \to B(X)$ is unital and preserves invertibility in both directions if and only if ϕ preserves the spectrum, that is, $\sigma(\phi(A)) = \sigma(A)$ for every $A \in B(X)$.

An interesting extension of Gleason-Kahane-Żelazko theorem was obtained by Kowalski and Słodkowski [18]. They proved that every functional f on a complex Banach algebra $\mathcal{A}$ (they did not assume the linearity of f) satisfying $f(a) - f(b) \in \sigma(a - b)$, $a, b \in \mathcal{A}$, is linear and multiplicative up to the constant $f(0)$. Thus, they replaced the two conditions in Gleason-Kahane-Żelazko theorem, the linearity assumption and the condition $f(a) \in \sigma(a)$, $a \in \mathcal{A}$, by a single weaker assumption and got essentially the same conclusion.

In view of this result it is natural to ask if we can do the same with the above mentioned results on invertibility preserving maps on matrix and operator algebras. Can we replace the linearity assumption and the invertibility preserving assumption by a single weaker condition similar to the one in Kowalski-Słodkowski theorem? More precisely, can we characterize bijective maps on matrix algebras and operator algebras satisfying the condition that $\phi(a) - \phi(b)$ is invertible if and only if $a - b$ is invertible?

The result of Kowalski and Słodkowski depends heavily on deep results from analysis. We will answer the above question using the results from geometry. We should first mention that there is an essential difference between the finite and the infinite-dimensional case. In the finite-dimensional case our condition will imply up to a translation the semilinearity of the maps under consideration, while in the infinite-dimensional case the elementary automatic continuity methods will imply the linearity or conjugate-linearity up to a translation. Moreover, in the finite-dimensional case it makes sense to extend our result from matrix algebras of square matrices to the spaces of rectangular matrices. Then, of course, the condition of invertibility will be replaced by the condition of being of full rank.

Our strategy when considering bijective maps ϕ on matrix spaces (operator algebras) satisfying the condition that $\phi(A) - \phi(B)$ is of full rank (invertible) if and only if $A - B$ is of full rank (invertible) will be to prove first that such maps preserve adjacency in both directions. Recall that two matrices or operators A and B are adjacent if $A - B$ is of rank one. Then we will apply the so called fundamental theorem of geometry of matrices (or its analogue for operators) to

complete the proof. This connects our results with the geometry of Grassmann spaces. Let us briefly describe this connection.

Let $M_{m,n}$, $m, n \geq 2$, be the linear space of all $m \times n$ matrices over a field $\mathbb{F}$. If σ is an automorphism of the field $\mathbb{F}$ and $A = [a_{ij}] \in M_{m,n}$ then we denote by A_σ the matrix obtained from A by applying σ entrywise, $A_\sigma = [\sigma(a_{ij})]$. The fundamental theorem of geometry of matrices states that every bijective map $\phi : M_{m,n} \to M_{m,n}$ preserving adjacency in both directions is of the form $A \mapsto T A_\sigma S + R$, where T is an invertible $m \times m$ matrix, S is an invertible $n \times n$ matrix, R is an $m \times n$ matrix, and σ is an automorphism of the underlying field. If $m = n$, then we have the additional possibility that $\phi(A) = T A_\sigma^t S + R$ where T, S, R and σ are as above, and A^t denotes the transpose of A. This theorem and its analogues for hermitian matrices, symmetric matrices, and skew-symmetric matrices were proved by Hua [7]-[14] under some mild technical assumptions that were later proved to be superfluous (see [24]). Let m, n be integers ≥ 2. We will consider the Grassmann space whose "points" are vector subspaces of $\mathbb{F}^{m+n}$ of dimension m. Chow [4] studied bijective maps on the Grassmann space preserving adjacent pairs of points in both directions. Recall that m-dimensional subspaces U and V are adjacent if $\dim(U + V) = m + 1$. Now, to each m-dimensional subspace U of $\mathbb{F}^{m+n}$ we can associate an $m \times (m + n)$ matrix whose rows are coordinates of vectors that form a basis of U. Each $m \times (m + n)$ matrix will be written in the block form $[X \ Y]$, where X is an $m \times n$ matrix and Y is an $m \times m$ matrix. Two matrices $[X \ Y]$ and $[X' \ Y']$ are associated to the same subspace U (their rows represent two bases of U) if and only if $[X \ Y] = P[X' \ Y']$ for some invertible $m \times m$ matrix P. If this is the case, then Y is invertible if and only if Y' is invertible. So, we have associated to each point in a Grassmann space a (not uniquely determined) matrix $[X \ Y]$. If Y is singular, we say that the corresponding point in the Grassmann space is at infinity. Otherwise, we observe that this point can be represented also with the matrix $[Y^{-1}X \ \ I]$. The matrix $Y^{-1}X$ is uniquely determined by the point in the Grassmann space. So, if U and V are two m-dimensional subspaces that are finite points in the Grassmann space, then they can be represented with two uniquely determined $m \times n$ matrices T and S, and it is easy to see that the subspaces U and V are adjacent if and only if the matrices T and S are adjacent. Using this connection it is possible to deduce the result of Chow on bijective maps on a Grassmann space preserving adjacency in both directions from the fundamental theorem of geometry of matrices (see [24]).

If we consider the special case when $m = n$ and replace in the fundamental theorem of geometry of matrices the condition of preserving adjacent pairs of matrices by our assumption of preserving the pairs A, B with the property that $\text{rank}\,(A - B) = n$, then this corresponds to the study of bijective maps on the Grassmann space of all vector subspaces of $\mathbb{F}^{2n}$ of dimension n that preserve the complementarity of subspaces. Such maps were studied by Blunck and the first author [2]. We suspect that this result can be deduced from our result and the other way around, but we also believe that it is easier to prove each of them separately. Namely, to prove any of these two implications seems to be difficult because of the points at infinity.

Now we state our main results. In the finite-dimensional case we will consider bijective maps on $m \times n$ matrices preserving pairs of matrices whose difference has a full rank. Of course, if we have such a map ϕ then the map $\psi : M_{n,m} \to M_{n,m}$ defined by $\psi(A) = (\phi(A^t))^t$ has the same properties. Thus, when studying

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such maps there is no loss of generality in assuming that $m \geq n$. We will do this throughout the paper. A matrix $A \in M_{m,n}$ is said to be of full rank if rank $A = n$. Let $A, B \in M_{m,n}$. We write $A \triangle B$ if $A - B$ is of full rank.

Theorem 1.1 *Let $\mathbb{F}$ be a field with at least three elements and m, n integers with $m \geq n \geq 2$. Assume that $\phi : M_{m,n} \to M_{m,n}$ is a bijective map such that for every pair $A, B \in M_{m,n}$ we have $A \triangle B$ if and only if $\phi(A) \triangle \phi(B)$. Then there exist an invertible $m \times m$ matrix T, an invertible $n \times n$ matrix S, an $m \times n$ matrix R, and an automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ such that*

$$\phi(A) = TA_\sigma S + R$$

for every $A \in M_{m,n}$. If $m = n$, then we have the additional possibility that

$$\phi(A) = TA_\sigma^t S + R, \quad A \in M_{n,n},$$

where $T, S, R \in M_{n,n}$ with T and S invertible, and σ is an automorphism of $\mathbb{F}$.

Theorem 1.2 *Let H be an infinite-dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Assume that $\phi : B(H) \to B(H)$ is a bijective map such that for every pair $A, B \in B(H)$ the operator $A - B$ is invertible if and only if $\phi(A) - \phi(B)$ is invertible. Then there exist $R \in B(H)$ and invertible $T, S \in B(H)$ such that either*

$$\phi(A) = TAS + R$$

for every $A \in B(H)$, or

$$\phi(A) = TA^t S + R$$

for every $A \in B(H)$, or

$$\phi(A) = TA^* S + R$$

for every $A \in B(H)$, or

$$\phi(A) = T(A^t)^* S + R$$

for every $A \in B(H)$. Here, A^t and A^* denote the transpose with respect to an arbitrary but fixed orthonormal basis, and the usual adjoint of A in the Hilbert space sense, respectively.

The converses of both theorems obviously hold true. In the second section we will prove the finite-dimensional case and in the third one the infinite-dimensional case. These two sections can be read independently.

2 The finite-dimensional case

In this section we will consider matrices over a field $\mathbb{F}$ with at least three elements. At a certain point in the proof of our first main theorem we will identify $m \times n$ matrices with linear operators from $\mathbb{F}^n$ into $\mathbb{F}^m$. For such operators we have the following simple lemma.

Lemma 2.1 *Let $T, S : \mathbb{F}^n \to \mathbb{F}^m$ be nonzero linear operators and assume that T has at least two-dimensional image. Then we can find linearly independent vectors $x, y \in \mathbb{F}^n$ such that Tx and Sy are linearly independent.*

Proof. Take any $y \in \mathbb{F}^n$ such that $Sy \neq 0$. The set of all vectors $z \in \mathbb{F}^n$ with the property that Tz is linearly dependent of Sy is a proper subspace of $\mathbb{F}^n$, since the image of T is not contained in the span of Sy. There exist at least two linearly independent vectors of $\mathbb{F}^n$ which are not in this particular subspace. One of them is linearly independent of y and gives the required vector x.

$\square$

We have two relations on $M_{m,n}$, that is, the relation of adjacency and $\triangle$. The following result connecting these two relations is the key step in our proof. We believe it is of some independent interest.

Proposition 2.2 *Let $A, B \in M_{m,n}$ be matrices with $A \neq B$. Then the following are equivalent:*

1. *A and B are adjacent.*

2. *There exists $R \in M_{m,n}$, $R \neq A, B$, such that for every $X \in M_{m,n}$ the relation $X \triangle R$ yields $X \triangle A$ or $X \triangle B$.*

Proof. Note that none of the above conditions are effected if we replace A and B by $PAQ - C$ and $PBQ - C$, respectively, where P and Q are invertible matrices of the appropriate size and C is any $m \times n$ matrix. Thus if the rank distance between A and B equals r then we may assume with no loss of generality that $A = 0$ and

$$B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is the $r \times r$ identity matrix and the zeros stand for the zero matrices of the appropriate size.

Assume first that A and B are adjacent. So, without loss of generality, we have $A = 0$ and $B = E_{11}$. Set $R = \lambda E_{11}$, where λ is a scalar different from 0 and 1, and E_{11} denotes the matrix with the $(1,1)$-entry equal to 1 and all other entries equal to zero. Now let $X \triangle R$. That means that $X - R$ is of full rank or equivalently, the matrix $X - R$ contains at least one invertible $n \times n$ submatrix. We have to consider two possibilities. Let first assume that one of these submatrices does not contain the first row. In this case X is of full rank and thus $X \triangle A$. Otherwise any such submatrix contains the first row and we choose one of them. We will prove that at least one of the corresponding $n \times n$ submatrices of $X - A = X$ and $X - B$ is invertible. So we restrict our attention to these $n \times n$ submatrices. In other words we deal only with the square case $m = n$. Hence $X - \lambda E_{11}$ is an invertible square matrix. If the first row of E_{11}, i.e. $(1, 0, \ldots, 0)$, is in the subspace spanned by rows $2, 3, \ldots, n$ of X then $X - \lambda E_{11} - \mu E_{11}$ is invertible for all $\mu \in \mathbb{F}$, otherwise this holds for all but one $\mu \in \mathbb{F}$. Therefore $X - \lambda E_{11} - \mu E_{11}$ is invertible for at least one of the values $\mu = -\lambda$ or $\mu = -\lambda + 1$. Equivalently, at least one of $X = X - A$ or $X - E_{11} = X - B$ is invertible, as desired. This completes the proof of the first implication.

To prove the other direction we identify $m \times n$ matrices with linear operators from $\mathbb{F}^n$ into $\mathbb{F}^m$. We assume that $A = 0$ and $B : \mathbb{F}^n \to \mathbb{F}^m$ is a linear operator whose image is at least two-dimensional. Let $R : \mathbb{F}^n \to \mathbb{F}^m$ be any linear operator, $R \neq 0, B$. We have to find a linear operator $X : \mathbb{F}^n \to \mathbb{F}^m$ such that $X - R$ is injective while X and $X - B$ are not.

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The first possibility we will treat is that at least one of the operators $B - R$ and R has rank at least two. Then, by Lemma 2.1, we can find linearly independent $x, y \in \mathbb{F}^n$ such that $Bx - Rx$ and Ry are linearly independent. We first define X on the linear span of x and y. We set $Xx = Bx$ and $Xy = 0$. No matter how we will extend X to the whole space these two equations will guarantee that $X - B$ and X will not be injective. Now, $(X - R)x = Bx - Rx$ and $(X - R)y = -Ry$ are linearly independent. It is now obvious that we can extend the linear operator X to the whole space $\mathbb{F}^n$ such that the obtained $X - R$ is injective.

It remains to consider the case when both operators $B - R$ and R are of rank one. By our assumption, B is of rank two. Hence $B = R + (B - R)$ implies that the ranges of $B - R$ and R meet at 0 only. So, if we choose any $x, y \in \mathbb{F}^n$ such that $(B - R)x \neq 0$ and $Ry \neq 0$ then $(B - R)x$ and Ry will be linearly independent. Since $\mathbb{F}$ has at least three elements, we can choose these x and y to be linearly independent. Now we can proceed as above.

$\square$

It is now easy to prove Theorem 1.1. Namely, if $\phi : M_{m,n} \to M_{m,n}$ is a bijective map preserving $\triangle$ in both directions, then, by Proposition 2.2 it preserves adjacency in both directions. Thus, the result follows directly from the fundamental theorem of geometry of matrices.

Observe that in [2] there is no need to assume that $\mathbb{F}$ has at least three elements, due to the presence of points at infinity.

3   The infinite-dimensional case

Let H be an infinite-dimensional complex Hilbert space and $x, y \in H$. The inner product of x and y will be denoted by $y^* x$. If x and y are nonzero vectors then xy^* stands for the rank one bounded linear operator defined by $(xy^*)z = (y^* z)x$, $z \in H$. Note that every bounded rank one operator can be written in this form. Two operators $A, B \in B(H)$ are said to be adjacent if $A - B$ is an operator of rank one. We write $A \triangle B$ if $A - B$ is invertible. We start with an analogue of Proposition 2.2.

Proposition 3.1 *Let $A, B \in B(H)$ with $A \neq B$. Then the following statements are equivalent:*

  1. *A and B are adjacent.*

  2. *There exists $R \in B(H)$, $R \neq A, B$, such that for every $X \in B(H)$ the relation $X \triangle R$ yields $X \triangle A$ or $X \triangle B$.*

Proof. Note that none of the above conditions are effected if we replace A and B by $A - C$ and $B - C$, respectively, where C is any bounded linear operator on H. Thus we may assume with no loss of generality that $A = 0$.

Assume first that $A = 0$ and B are adjacent, that is, B is of rank one. Set $R = 2B$. Suppose that $X - 2B$ is invertible. Then

$$X - 2B - \lambda B = (X - 2B)(I - \lambda(X - 2B)^{-1}B)$$

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is invertible if and only if $I - \lambda S$ is invertible, where $S = (X - 2B)^{-1}B$ is an operator of rank one. Every operator of rank one has at most one non-zero complex number in its spectrum. Hence, $X - 2B - (-2B) = X$ is invertible or $X - 2B - (-B) = X - B$ is invertible. This completes the proof of one direction.

Assume now that $A = 0$ and B is an operator whose image is at least two-dimensional. We have to prove that for every $R \in B(H)$, $R \neq 0, B$, there exists $X \in B(H)$ such that $X - R$ is invertible and X is singular and $X - B$ is singular. So, let $R \in B(H) \setminus \{0, B\}$.

In the next step we will prove that there exist $x, z \in H$ such that x and z are linearly independent and $Bz - Rz$ and Rx are linearly independent. It is enough to show that we can find $x, z \in H$ such that $Bz - Rz$ and Rx are linearly independent. For if x and z are linearly dependent, we can choose $u \in H$ linearly independent of x. Then $z + \lambda u$ and x are linearly independent for all nonzero λ and for all λ's small enough the vectors $B(z + \lambda u) - R(z + \lambda u) = Bz - Rz + \lambda(Bu - Ru)$ and Rx are linearly independent as well.

So, let us show that such x and z exist. Assume on the contrary that $Bz - Rz$ and Rx are linearly dependent for every x and z. Then $B - R$ and R are rank one operators with the same one-dimensional image. It follows that $B = 0$ or B is of rank one, a contradiction.

Now, we define W to be the orthogonal complement of the linear span of x and z, where x and z are as in the previous paragraph, and Z to be the orthogonal complement of Rx and $Bz - Rz$. Then there exists a bounded invertible linear operator $U : W \to Z$. Define $X \in B(H)$ with

$$Xx = 0,$$

$$Xz = Bz,$$

and

$$Xu = Uu + Ru, \quad u \in W.$$

Because of the first two equations the operators X and $X - B$ are singular. Since $(X - R)x = -Rx$, $(X - R)z = Bz - Rz$, and $(X - R)u = Uu$, $u \in W$, the operator $X - R$ is invertible, as desired.

$\qquad \square$

We continue with some technical lemmas.

Lemma 3.2 *Let $B, C \in B(H)$. Assume that for every invertible $A \in B(H)$ the operator $A - B$ is invertible if and only if $A - C$ is invertible. Then $B = C$.*

Proof. Let λ be any complex number satisfying

$$|\lambda| > \|B\|, \|C\|,$$

and $x, y \in H$ any vectors such that $y^*x = 0$. Then $\lambda(I + xy^*)$ is invertible because $(I + xy^*)(I - xy^*) = I$. Hence, $\lambda I + \lambda xy^* - B$ is invertible if and only if $\lambda I + \lambda xy^* - C$ is invertible. On the other hand,

$$\lambda I + \lambda xy^* - B = (I + xy^*)(\lambda I - B + xy^*B) = (I + xy^*)(I + xy^*B(\lambda I - B)^{-1})(\lambda I - B)$$

is invertible if and only if $I + xy^*B(\lambda I - B)^{-1}$ is invertible. Thus, $I + xy^*B(\lambda I - B)^{-1}$ is invertible if and only if $I + xy^*C(\lambda I - C)^{-1}$ is invertible, or equivalently,

for every scalar λ with $|\lambda| > \|B\|, \|C\|$, and every pair of vectors $x, y \in H$ with $y^*x = 0$ we have

$$y^*B(\lambda I - B)^{-1}x = -1 \iff y^*C(\lambda I - C)^{-1}x = -1.$$

Fix λ. Then $y^*Tx = 0$ for every pair of orthogonal vectors x and y, where $T = B(\lambda I - B)^{-1} - C(\lambda I - C)^{-1}$. It follows that $T = \mu I$ for some scalar μ. Thus, for every λ with $|\lambda| > \|B\|, \|C\|$ we have

$$B(\lambda I - B)^{-1} - C(\lambda I - C)^{-1} = g(\lambda)I$$

for some $g(\lambda) \in \mathbb{C}$. Obviously, $g(\lambda)$ is holomorphic outside the circle centered at 0 with radius $\max\{\|B\|, \|C\|\}$. Expressing the above analytic functions with the series and comparing the coefficients we get

$$B = C + \mu I$$

for some complex number μ. Our assumption implies that $\sigma(B) \setminus \{0\} = \sigma(C) \setminus \{0\}$. Here $\sigma(B)$ denotes the spectrum of B. It follows that $\mu = 0$, as desired.

$\square$

Lemma 3.3 *Let $A, B \in B(H)$ be invertible operators. Assume that for every rank one operator $xy^* \in B(H)$ the operator $A - xy^*$ is invertible if and only if $B - xy^*$ is invertible. Then $A = B$.*

Proof. Our assumptions yield that for every pair of vectors x, y the operator $I - xy^*A^{-1}$ is invertible if and only if $I - xy^*B^{-1}$ is, or equivalently, $y^*A^{-1}x = 1$ if and only if $y^*B^{-1}x = 1$. By linearity we have $y^*A^{-1}x = y^*B^{-1}x$ for every pair $x, y \in H$, and therefore, $A^{-1} = B^{-1}$. It follows that $A = B$.

$\square$

Let us recall that an additive map $T : H \to H$ is called semilinear if there is an automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ such that $T(\lambda x) = \sigma(\lambda)Tx$ for every $\lambda \in \mathbb{C}$ and every $x \in H$. Now we are ready to prove Theorem 1.2.

Let $\phi : B(H) \to B(H)$ be a bijective map such that for every pair $A, B \in B(H)$ the operator $A - B$ is invertible if and only if $\phi(A) - \phi(B)$ is invertible. After replacing ϕ by $A \mapsto \phi(A) - \phi(0)$ we may assume that $\phi(0) = 0$. Then $\phi(I)$ is invertible. Replacing ϕ by $A \mapsto \phi(I)^{-1}\phi(A)$ we may further assume that $\phi(I) = I$.

According to Proposition 3.1, ϕ preserves adjacency in both directions. Every rank one operator is adjacent to zero, every rank two operator is adjacent to some rank one operator, etc. Consequently, ϕ maps the subspace $F(H) \subset B(H)$ of all finite rank operators onto itself. So, we can apply Theorem 1.5 from [20] to conclude that there exist bijective semilinear maps $T, S : H \to H$ (with the same accompanying automorphism) such that either $\phi(xy^*) = (Tx)(Sy)^*$, $x, y \in H$, or $\phi(xy^*) = (Sy)(Tx)^*$, $x, y \in H$. The second case can be reduced to the first one if we replace ϕ by $A \mapsto \phi(A)^*$, $A \in B(H)$. So, we may assume that the first possibility holds true.

Using $\phi(I) = I$ and our assumptions we conclude that $I - xy^*$ is invertible if and only if $I - (Tx)(Sy)^*$ is invertible, $x, y \in H$. Thus, $y^*x = 1$ if and only if $(Sy)^*(Tx) = 1$, and by semilinearity,

$$(Sy)^*(Tx) = 0 \iff y^*x = 0, \quad x, y \in H.$$

Thus, the semilinear maps T and S and their inverses carry closed hyperplanes (every closed hyperplane is the orthogonal complement of some nonzero vector) onto closed hyperplanes. Hence, by [5, Lemmas 2 and 3], S and T are both linear bounded or both conjugate-linear bounded. Thus, we have $\phi(xy^*) = T(xy^*)R$, where T and $R = S^*$ are bounded invertible either both linear, or both conjugate-linear operators. Assume they are both conjugate-linear. Choosing an orthonormal basis we define $K : H \to H$ to be the conjugate-linear bijection which maps each vector x into a vector whose coordinates are obtained from the coordinates of x by complex conjugation. Of course, $K^2 = I$, the product of two conjugate-linear maps is linear, and $K(xy^*)K = ((xy^*)^*)^t$, where the transpose is taken with the respect to the chosen basis. Replacing ϕ by $A \mapsto (\phi(A)^t)^*$, $A \in B(H)$, we reduce the conjugate-linear case to the linear one.

So, we may assume that we have $\phi(xy^*) = T(xy^*)R$, where T and $R = S^*$ are bounded invertible linear operators. From $(Sy)^*(Tx) = 1 \iff y^*x = 1$ and linearity we get $(Sy)^*(Tx) = y^*x$, $x, y \in H$, which further yields that T is the inverse of R. Composing ϕ by a similarity transformation we may further assume that $\phi(xy^*) = xy^*$, $x, y \in H$.

Let $A \in B(H)$ be invertible. Applying Lemma 3.3 with $B = \phi(A)$ we see that $\phi(A) = A$.

Finally, let $B \in B(H)$ be any operator and set $C = \phi(B)$. Using Lemma 3.2 we conclude that $\phi(B) = B$. This completes the proof.

$\square$

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