Radical parallelism on projective lines and non-linear models of affine spaces

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Abstract

We introduce and investigate an equivalence relation called *radical parallelism* on the projective line over a ring. It is closely related with the Jacobson radical of the underlying ring. As an application, we present a rather general approach to non-linear models of affine spaces and discuss some particular examples.

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1 Introduction

If two points of the projective line over a ring are non-distant then they are also said to be *parallel*. This terminology goes back to the projective line over the real dual numbers, where parallel points represent parallel spears of the Euclidean plane [1, 2.4]. In general, this parallelism of points is not an equivalence relation. In the present article we shall introduce another concept of "parallelism" on the projective line over a ring. In order to avoid ambiguity we call this the *radical parallelism*, since it reflects the *Jacobson radical* of a ring R in terms of the projective line over R. The two kinds of parallelism coincide exactly for local rings.

The radical parallelism is defined and discussed in Section 2. There are several results on the projective line over a local ring which can be generalized to an arbitrary ring R as follows: Consider radically parallel points instead of parallel points and the Jacobson radical of R instead of the only maximal ideal of a local ring. For example, the radical parallelism is always an equivalence relation. It is the equality relation if, and only if, $\operatorname{rad} R = \{0\}$. Next, in Section 3, we consider a K-algebra R and the associated affine chain geometry. Its automorphism group contains bijective transformations $R \to R$ (without "exceptional points") which are not affine transformations, provided that $\operatorname{rad} R \neq \{0\}$ and $K \neq \mathrm{GF}(2)$. In particular, when $\dim_K R$ is finite, then these mappings are birational, i.e., they are Cremona transformations.

We may regard R as an affine space over K and fix one of the non-affine transformations from the above. Then R together with the images of the lines under this transformation yields a non-linear model of the affine space R over K. Two particular cases of such models are investigated in detail. The first example arises from the ring of dual numbers over K. It yields, for $K = \mathbb{R}$, the well-known *parabola model* of the real affine plane; cf. [12, p. 67]. For an arbitrary ground field K, a similar parabola model is described. However, some properties of the parabola model of the real affine plane do not hold any more if K has characteristic 2. This is due to the fact that in this case all tangent lines of a parabola are parallel. The second example is based upon the three-dimensional K-algebra of upper 2×2-matrices over K. As before, we obtain a kind of "parabola model" which can be easily described in terms of the associated chain geometry.

Throughout this paper we shall only consider associative rings with a unit element 1, which is inherited by subrings and acts unitally on modules. The trivial case $1 = 0$ is excluded. The group of invertible elements of a ring R will be denoted by R^*.

Let us recall the definition of the projective line over a ring R: Consider the free left R-module R^2. Its automorphism group is the group $\mathrm{GL}_2(R)$ of invertible 2×2-matrices with entries in R. A pair $(a, b) \in R^2$ is called *admissible*, if there exists a matrix in $\mathrm{GL}_2(R)$ with (a, b) being its first row. Following [8, p. 785], the *projective line over* R is the orbit of the free cyclic submodule $R(1, 0)$ under the action of $\mathrm{GL}_2(R)$. So

$$\mathbb{P}(R) := R(1, 0)^{\mathrm{GL}_2(R)}$$

or, in other words, $\mathbb{P}(R)$ is the set of all $p \subset R^2$ such that $p = R(a, b)$ for an admissible pair $(a, b) \in R^2$. Two such pairs represent the same point exactly if they are left-proportional by a unit in R. We adopt the convention that points are represented by admissible pairs only. (Cf. [5, Proposition 2.1] for the possibility to represent points also by non-admissible pairs.)

The point set $\mathbb{P}(R)$ is endowed with the symmetric and anti-reflexive relation *distant* ($\triangle$) defined by

$$\triangle := (R(1, 0), R(0, 1))^{\mathrm{GL}_2(R)}.$$

Letting $p = R(a, b)$ and $q = R(c, d)$ gives then

$$p \triangle q \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

The vertices of the *distant graph* on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, two vertices of this graph are joined by an edge if, and only if, they are distant. Given a point $p \in \mathbb{P}(R)$ let

$$\mathbb{P}(R)_p := \{x \in \mathbb{P}(R) \mid x \triangle p\}$$

be the neighbourhood of p in the distant graph.

We shall no longer use the term "parallel points" in the present paper, but we speak instead of "non-distant points" ($\not\triangle$). The sign $\parallel$ will be used for the radical parallelism which is defined below.

The *Jacobson radical* of a ring R, denoted by $\operatorname{rad} R$, is the intersection of all the maximal left (or right) ideals of R. It is a two sided ideal of R and its elements can be characterized as follows:

$$b \in \operatorname{rad} R \Leftrightarrow 1 - ab \in R^* \text{ for all } a \in R \Leftrightarrow 1 - ba \in R^* \text{ for all } a \in R; \tag{1}$$

see [10, pp. 53–54].

2 Radical parallelism

A point $p \in \mathbb{P}(R)$ is called *radically parallel* to a point $q \in \mathbb{P}(R)$ if

$$x \vartriangle p \;\Rightarrow\; x \vartriangle q \tag{2}$$

holds for all $x \in \mathbb{P}(R)$. In this case we write $p \parallel q$. Clearly, the relation $\parallel$ is reflexive and transitive; we shall see in due course that $\parallel$ is in fact an equivalence relation.

Each matrix $\gamma \in \mathrm{GL}_2(R)$ determines an automorphism $\mathbb{P}(R) \to \mathbb{P}(R) : p \mapsto p^\gamma$ of the distant graph. Hence, by definition,

$$p \parallel q \;\Leftrightarrow\; p^\gamma \parallel q^\gamma \tag{3}$$

holds for all $p, q \in \mathbb{P}(R)$ and all $\gamma \in \mathrm{GL}_2(R)$.

The connection between the radical parallelism on $\mathbb{P}(R)$ and the Jacobson radical of R is as follows:

Theorem 2.1 *The point $R(1,0)$ is radically parallel to $q \in \mathbb{P}(R)$ exactly if there is an element b in the Jacobson radical $\operatorname{rad} R$ such that $q = R(1,b)$.*

Proof: We start with a characterization of $\operatorname{rad} R$ in terms of matrices. For all $a, b \in R$ we have

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-ba & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}. \tag{4}$$

So, by (1), we get

$$b \in \operatorname{rad} R \;\Leftrightarrow\; \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \text{ for all } a \in R. \tag{5}$$

Clearly, we have

$$\mathbb{P}(R)_{R(1,0)} = \{x \in \mathbb{P}(R) \mid x \vartriangle R(1,0)\} = \{R(a,1) \mid a \in R\}. \tag{6}$$

So (5) shows immediately that $R(1,0)$ is radically parallel to every point $q = R(1,b)$ with $b \in \operatorname{rad} R$.

Conversely, suppose that $R(1,0) \parallel q$. Then $R(0,1) \vartriangle R(1,0)$ implies $R(0,1) \vartriangle q$. So we may set $q = R(1,b)$ with $b \in R$. Now (6) and $R(1,0) \parallel q$ imply that the right hand side of (5) is fulfilled, whence $b \in \operatorname{rad} R$. $\square$

In order to obtain an alternative description of the radical parallelism we consider the factor ring $R/\operatorname{rad} R =: \overline{R}$ and the canonical epimorphism $R \to \overline{R} : a \mapsto a + \operatorname{rad} R =: \overline{a}$. It has the crucial property

$$a \in R^* \;\Leftrightarrow\; \overline{a} \in \overline{R}^* \tag{7}$$

for all $a \in R$; cf. [10, Proposition 4.8].

Now we turn to the corresponding projective lines. The mapping

$$\mathbb{P}(R) \to \mathbb{P}(\overline{R}) : p = R(a,b) \mapsto \overline{R}(\overline{a}, \overline{b}) =: \overline{p} \tag{8}$$

is well defined and surjective [5, Proposition 3.5]. Furthermore, as a geometric counterpart of (7) we have

$$p \vartriangle q \Leftrightarrow \overline{p} \vartriangle \overline{q} \tag{9}$$

for all $p, q \in \mathbb{P}(R)$, where we use the same symbol to denote the distant relations on $\mathbb{P}(R)$ and on $\mathbb{P}(\overline{R})$, respectively. See [5, Propositions 3.1, 3.2].

Theorem 2.2 *The mapping given by (8) has the property*

$$p \parallel q \Leftrightarrow \overline{p} = \overline{q} \tag{10}$$

for all $p, q \in \mathbb{P}(R)$.

Proof: Let $q = R(c, d) \in \mathbb{P}(R)$. Then (7) shows that $\overline{R}(\overline{1}, \overline{0}) = \overline{q}$ if, and only if, $c \in R^*$ and $d \in \operatorname{rad} R$ or, equivalently, $q = R(1, b)$ with $b := c^{-1}d \in \operatorname{rad} R$. So from Theorem 2.1 we get

$$R(1, 0) \parallel q \Leftrightarrow \overline{R}(\overline{1}, \overline{0}) = \overline{q} \tag{11}$$

for all $q \in \mathbb{P}(R)$. Now consider arbitrary points $p, q \in \mathbb{P}(R)$. There is a matrix $\gamma \in \mathrm{GL}_2(R)$ with $p^\gamma = R(1, 0)$. We have

$$\overline{r^\gamma} = \overline{r}^{\,\overline{\gamma}} \tag{12}$$

for all $r \in \mathbb{P}(R)$, where $\overline{\gamma} \in \mathrm{GL}_2(\overline{R})$ is obtained by applying the canonical epimorphism to the entries of the matrix $\gamma \in \mathrm{GL}_2(R)$; cf. [5, Proposition 3.1]. With (3), (11), and (12) we conclude

$$p \parallel q \Leftrightarrow p^\gamma = R(1, 0) \parallel q^\gamma \Leftrightarrow \overline{p^\gamma} = \overline{R}(\overline{1}, \overline{0}) = \overline{q^\gamma} \Leftrightarrow \overline{p}^{\overline{\gamma}} = \overline{q}^{\overline{\gamma}} \Leftrightarrow \overline{p} = \overline{q}. \tag{13}$$

This completes the proof. $\square$

As an immediate consequence of Theorem 2.2 we obtain:

Corollary 2.3 *The radical parallelism $\parallel$ on the projective line over a ring is an equivalence relation.*

In particular, $\parallel$ is a symmetric relation despite its (seemingly asymmetric) definition in formula (2). Since $p \parallel q$ means that the neighbourhood of p in the distant graph is a subset of the neighbourhood of q, we get, by virtue of this symmetry:

Corollary 2.4 *The neighbourhood of a point $p \in \mathbb{P}(R)$ in the distant graph cannot be a proper subset of the neighbourhood of a point $q \in \mathbb{P}(R)$.*

Furthermore, we have

$$\#\{x \in \mathbb{P}(R) \mid x \parallel p\} = \#\operatorname{rad} R \tag{14}$$

for all $p \in \mathbb{P}(R)$; in fact, Theorem 2.1 implies that (14) holds for $p = R(1, 0)$, whence the assertion follows from the transitive action of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$ and (3). Thus the "size" of $\operatorname{rad} R$ can be recovered from the distant graph on $\mathbb{P}(R)$ as the cardinality of an (arbitrarily chosen) class of radically parallel points. In particular, $\parallel$ is the equality relation if, and only if, $\operatorname{rad} R = \{0\}$.

Another easy consequence of (9) and Theorem 2.2 is

$$p \parallel q \Leftrightarrow \overline{p} = \overline{q} \Rightarrow \overline{p} \not\triangle \overline{q} \Leftrightarrow p \not\triangle q \tag{15}$$

for all $p, q \in \mathbb{P}(R)$. Note that here our assumption $1 \neq 0$ is essential, since it guarantees that $\triangle$ is an antireflexive relation. (The only point of the projective line over the zero-ring is distant to itself.) In general, however, the converse of (15) is not true:

Theorem 2.5 *The relations "radically parallel" and "non-distant" on $\mathbb{P}(R)$ coincide if, and only if, R is a local ring.*

Proof: Since $\mathrm{GL}_2(R)$ acts transitively on $\mathbb{P}(R)$ and leaves $\triangle$ and $\parallel$ invariant, it suffices to characterize those rings where

$$\{x \in \mathbb{P}(R) \mid R(1,0) \not\triangle x\} = \{x \in \mathbb{P}(R) \mid R(1,0) \parallel x\}. \tag{16}$$

Furthermore, we recall the following property: If a pair $(a, b) \in R^2$ is admissible and so the first row of an invertible matrix, then the first column of the inverse matrix shows that (a, b) is unimodular, i.e., there are $a', b' \in R$ such that $aa' + bb' = 1$.
Now let R be local. Then $R \setminus R^* = \operatorname{rad} R$, since this is the only maximal left ideal in R; see [10, Theorem 19.1]. The previous remark on unimodularity, applied to the local ring R, shows that at least one entry of each admissible pair $(a, b) \in R^2$ is a unit. From this we get that a point $x \in \mathbb{P}(R)$ satisfies $R(1,0) \not\triangle x$ if, and only if, $x = R(1, b)$ with $b \in \operatorname{rad} R$. But this is equivalent to $R(1,0) \parallel x$ by Theorem 2.1.
Conversely, suppose that (16) holds. Choose any non-unit $b \in R$. Then $R(1, b)$ is a point and $R(1,0) \not\triangle R(1,b)$ implies $b \in \operatorname{rad} R$ by Theorem 2.1. Hence $R \setminus R^* \subset \operatorname{rad} R$ and, since $\operatorname{rad} R \subset R \setminus R^*$ holds trivially, $R \setminus R^* = \operatorname{rad} R$ is a two-sided ideal. But this means that R is a local ring. □

Corollary 2.3, Corollary 2.4, and (14) generalize well-known results on the projective line over a local ring. Cf. [8, Proposition 2.4.1].

3 Non-linear models of affine spaces

In this section R is a ring and $K \neq R$ is a field contained in the centre of R. So R is an algebra over K with finite or infinite dimension. The point set of the *chain geometry* $\Sigma(K, R)$ is the projective line over R, the *chains* are the K-sublines of $\mathbb{P}(R)$; cf. [8, p. 790].
We fix the point $R(1,0) =: \infty$. By (6), the mapping

$$\iota : R \to \mathbb{P}(R)_\infty : z \mapsto R(z, 1) \tag{17}$$

is a bijection. We consider R as an affine space over K.
For each subset $\mathcal{S} \subset \mathbb{P}(R)$ let $(\mathcal{S} \cap \mathbb{P}(R)_\infty)^{\iota^{-1}}$ be the *affine trace* of $\mathcal{S}$. The affine traces of the chains through ∞ are precisely the so-called *regular lines* $Ku + v$ ($u \in R^*$, $v \in R$); cf. [8, Proposition 3.5.3]. By reversing the order of the coordinates in Theorem 2.1, we obtain

$$(\operatorname{rad} R)^\iota = \{x \in \mathbb{P}(R) \mid x \parallel R(0,1)\},$$

whence $\operatorname{rad} R$ is the affine trace of $\{x \in \mathbb{P}(R) \mid x \parallel R(0,1)\}$. One can easily check that the affine trace of $\{x \in \mathbb{P}(R) \mid x \not\triangle R(0,1)\}$ equals $R \setminus R^*$. In general, however, $(R \setminus R^*)^\iota$ is not equal to $\{x \in \mathbb{P}(R) \mid x \not\triangle R(0,1)\}$. Let, for example, $R = \mathbb{R} + \mathbb{R}j$ be the ring of real anormal-complex numbers, where $j^2 = 1$ and $j \notin \mathbb{R}$; cf. [1, p. 44]. Then $R(1-j, 1+j) \not\triangle R(0,1)$, but $R(1-j, 1+j) \notin R^\iota$, because $1+j$ is not invertible.

Each matrix $\gamma \in \mathrm{GL}_2(R)$ defines an automorphism of the chain geometry $\Sigma(K, R)$. Let us write D_γ for the set of all points $z \in R$ such that $z^{\iota\gamma} \in \mathbb{P}(R)_\infty$. Then the mapping

$$\gamma' : D_\gamma \to R : z \mapsto z^{\iota\gamma\iota^{-1}} \tag{18}$$

is injective, but in general the domain and the image of γ' will be proper subsets of R.

Theorem 3.1 *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2-matrix over R. Then the following hold:*

(a) *The matrix γ is invertible and the corresponding mapping γ', given by (18), is defined for all points of R if, and only if,*

$$a, d \in R^*, \text{ and } b \in \operatorname{rad} R. \tag{19}$$

(b) *If (19) is satisfied then the corresponding mapping $\gamma' : R \to R$ is an affine transformation for $b = 0$, and a non-affine bijective transformation for $b \in \operatorname{rad} R \setminus \{0\}$ and $K \neq \mathrm{GF}(2)$.*

Proof: (a) Let $\gamma \in \mathrm{GL}_2(R)$ and suppose that γ' is defined for all points of R. So we obtain

$$(\mathbb{P}(R)_\infty)^\gamma \subset \mathbb{P}(R)_\infty. \tag{20}$$

By definition, the distant relation $\triangle$ is invariant under $\mathrm{GL}_2(R)$. Therefore $(\mathbb{P}(R)_\infty)^\gamma = \mathbb{P}(R)_{\infty^\gamma}$, so that (20) is equivalent to $R(a,b) = \infty^\gamma \parallel \infty$. Thus $a \in R^*$ and $b \in \operatorname{rad} R$ by Theorem 2.2. Furthermore, $R(c,d) \triangle R(a,b) \parallel \infty$ yields $R(c,d) \triangle \infty$, so that $d \in R^*$.

Conversely, we infer from (19) that $a - bd^{-1}c \in R^*$. Hence

$$\begin{pmatrix} -1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a + bd^{-1}c & 0 \\ d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{21}$$

shows that $\gamma \in \mathrm{GL}_2(R)$. It follows from $d \in R^*$, $b \in \operatorname{rad} R$, and (1) that $zb + d \in R^*$ for all $z \in R$. Therefore γ yields the mapping

$$\gamma' : R \to R : z \mapsto (zb + d)^{-1}(za + c) \tag{22}$$

with domain $D_\gamma = R$.

(b) We deduce from (a) that (20) is satisfied. By Corollary 2.4, applied to the points ∞ and ∞^γ, it follows that $(\mathbb{P}(R)_\infty)^\gamma = \mathbb{P}(R)_{\infty^\gamma} = \mathbb{P}(R)_\infty$, whence the injective mapping γ' is bijective. There are two cases:

If $\infty = \infty^\gamma$ then $b = 0$. This implies that $\gamma' : R \to R : z \mapsto d^{-1}(za + c)$ is an affine transformation; see also [8, Lemma 3.5.7].

If $\infty \neq \infty^\gamma$ then $b \in \operatorname{rad} R \setminus \{0\}$. We observe, as above, that the first and the third matrix on the left hand side of (21) both yield affine transformations. Hence we may confine our attention to the transformation

$$\beta' : R \to R : z \mapsto (1 - zb)^{-1} z \tag{23}$$

arising from the second matrix in (21). Let now $K \neq \mathrm{GF}(2)$. Then there is an element $k \in K \setminus \{0, 1\}$. The image of the line K under β' carries the points $0^{\beta'} = 0$, $1^{\beta'} = (1 - b)^{-1}$, and $k^{\beta'} = (1 - kb)^{-1} k$. These points are non-collinear, since $b \in \operatorname{rad} R \setminus \{0\}$ implies that $b \notin K$, whence $1 - b$ and $1 - kb$ are linearly independent over K. Thus β' cannot be an affine transformation. □

The following example shows that we cannot drop the assumption $K \neq \mathrm{GF}(2)$ in Theorem 3.1 (b). Let $R = \mathrm{GF}(2) + \mathrm{GF}(2)\varepsilon$ be the ring of dual numbers over $\mathrm{GF}(2)$, where $\varepsilon^2 = 0$ and $\varepsilon \notin \mathrm{GF}(2)$. The invertible matrix $\delta := \begin{pmatrix} 1 & \varepsilon \\ 0 & 1+\varepsilon \end{pmatrix}$ yields a transformation on $\mathbb{P}(R)$ that interchanges ∞ with $R(1, \varepsilon)$, but fixes the remaining four points of $\mathbb{P}(R)$. Hence $\delta' = \mathrm{id}_R$ is an affine transformation, even though $\infty^\delta \neq \infty$.

Suppose that γ' is a non-affine bijection according to Theorem 3.1 (b). We obtain a *non-linear model* of the affine space on the K-vector space R by applying the bijection γ' to the points and lines of this affine space. So we get a "new" space which has the same point set, but the β'-images of the "old" lines will be the lines in the "new" sense. In view of Theorem 3.1 (b), such non-linear models of affine spaces are possible whenever the radical of R is non-zero and $K \neq \mathrm{GF}(2)$. It would be interesting to describe explicitly the "new lines" in a purely geometric way. However, this is beyond the scope of this article. Below we just give two examples, one of it generalizes the well-known *parabola model* of the real affine plane [12, p. 67].

We have several distinguished subgroups of $\mathrm{GL}_2(R)$ which, by Theorem 3.1, fix $\mathbb{P}(R)_\infty$ as a set. The commutative group

$$B := \left\{ \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \mid b \in \operatorname{rad} R \right\} \tag{24}$$

acts regularly on the set of points that are radically parallel to ∞; cf. Theorem 2.1. For each $\beta \in B$ the induced mapping $\beta' : R \to R$ is given by (23). Next, there is the commutative group

$$T := \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in R \right\}. \tag{25}$$

Each $\tau \in T$ fixes ∞ and, by Theorem 3.1, it yields the translation $\tau' : R \to R : z \mapsto z + c$. Every translation of R arises in this way. A transformation $\tau \in T$ need not fix every point $p \parallel \infty$. In fact, if τ is the matrix in formula (25) then $p = R(1, b)$, with $b \in \operatorname{rad} R$, remains fixed if, and only if,

$$bcb = 0. \tag{26}$$

For a subset $S \subset R$ let $\operatorname{ann}(S) := \{a \in R \mid aS = Sa = 0\}$ denote the *annihilator* of S in R. So, for example, $c \in \operatorname{ann}(\operatorname{rad} R)$ implies that (26) is fulfilled for all $b \in \operatorname{rad} R$. Finally, a straightforward calculation shows that

$$N := \left\{ \begin{pmatrix} 1+n_1 & 0 \\ n_2 & 1 \end{pmatrix} \mid n_1, n_2 \in \operatorname{ann}(\operatorname{rad} R) \cap \operatorname{rad} R \right\} \tag{27}$$

is a commutative subgroup of $\mathrm{GL}_2(R)$. (Observe that $(1+n_1)(1-n_1)=1$.) Each $\nu \in N$ stabilizes ∞ and, by Theorem 3.1, it determines an affinity $\nu' : R \to R : z \mapsto z(1+n_1)+n_2$. The groups N and B have the property that

$$\nu\beta = \beta\nu \text{ for all } \nu \in N \text{ and all } \beta \in B. \tag{28}$$

Every point $p \parallel \infty$ remains fixed under every transformation $\nu \in N$. For, clearly, $\infty^\nu = \infty$ and, since p can be written as ∞^β with $\beta \in B$, we obtain $p^\nu = \infty^{\beta\nu} = \infty^{\nu\beta} = p$ from (28).

We adopt the notation $B' := \{\beta' \mid \beta \in B\}$; T' and N' are defined similarly.

In the remainder of this section, we suppose that $m := \dim_K R$ is finite. Then the so-called *cone of singularity* $R \setminus R^*$ is an algebraic set; see [8, Remark 3.5.4]. Also, the affine trace of a chain is an *affine normal rational curve* of degree $\leq m$, provided that it has at least two points in common with $\mathbb{P}(R)_\infty$; see [8, Theorem 3.6.5]. According to [8, p. 804], the mappings given in (18) are *Cremona transformations*; cf. also [2] and [3, p. 129]. In particular, the mappings described in Theorem 3.1 (b) are bijective Cremona transformations $R \to R$.

Let s be the dimension of the Jacobson radical of R. Then $1 \notin \operatorname{rad} R$ implies $s \leq m-1$. All elements of $\operatorname{rad} R$ are nilpotent; see [10, Proposition 4.18]. Thus $y^{s+1}=0$ for all $y \in \operatorname{rad} R$. So, for each $\beta \in B$, formula (23) can be written in polynomial form as

$$\beta' : R \to R : z \mapsto \left(1 + zb + \cdots + (zb)^s\right) z. \tag{29}$$

The final part of this section is devoted to the investigation of two particular examples, where we are able to describe explicitly the images of the lines under a fixed non-identical transformation $\beta' \in B'$. It will be easy to show that non-regular lines go over to non-regular lines and that the images of the regular lines are "certain" parabolas. Our main objective is to make more precise this last statement. We rule out, however, the field with two elements from our discussion, because in an affine space over $\mathrm{GF}(2)$ a parabola has only two points, and it would take rather complicated formulations to include this case.

Example 3.2 Let $R = K + K\varepsilon$ be the ring of dual numbers over K, where $K \neq \mathrm{GF}(2)$. This is a local commutative ring, and its radical is $K\varepsilon$. The lines parallel to $K\varepsilon$ are called *vertical*. In formula (29) we may put $z = z_1 + z_2\varepsilon$ and $b = t\varepsilon$ with $z_1, z_2, t \in K$. Thus we get

$$\beta' : R \to R : (z_1 + z_2\varepsilon) \mapsto z_1 + (tz_1^2 + z_2)\varepsilon. \tag{30}$$

We assume that $t \neq 0$. All non-regular lines or, said differently, all vertical lines are invariant under β'. In order to describe the images of the regular lines, we consider the group N'. Its transformations are obtained from (27) by substituting $n_1 = l_1\varepsilon$ and $n_2 = l_2\varepsilon$, where $l_1, l_2 \in K$, and this gives

$$\nu' : R \to R : z_1 + z_2\varepsilon \mapsto z_1 + (z_1 l_1 + z_2 + l_2)\varepsilon. \tag{31}$$

Then, either $l_1 \neq 0$, whence ν' is a non-trivial shear with the vertical axis $z_1 = -l_2/l_1$, or $l_1 = 0$, whence ν' is a *vertical translation*, i.e. a translation along the vertical line $K\varepsilon$. Altogether, since l_1 and l_2 can be chosen arbitrarily in K, the transformations in N' are all the shears with a vertical axis and all the vertical translations. From a projective point of view this is the group of all elations whose centre is the point at infinity of all the vertical lines.

If L is a regular line then there is a $\nu' \in N'$ such that $L^{\nu'} = K$; for if L and K are parallel then ν' can be chosen as a vertical translation, and otherwise as a non-trivial shear whose vertical axis contains the point $K \cap L$. Hence the group N' acts transitively and, by the commutativity of N', even regularly on the set of regular lines.

It is clear that the image under β' of the regular line K is a parabola C, say, with an equation $z_2 = tz_1^2$. By the transitivity of N' on the set of regular lines and by (28), the set of β'-images of the regular lines is the orbit of the parabola C under the action of the group N', i.e. the set of all parabolas with an equation

$$z_2 = tz_1^2 + l_1 z_1 + l_2 \text{ with } l_1, l_2 \in K. \tag{32}$$

In projective terms this is a net of conics mutually osculating at the point at infinity of all vertical lines.

For each translation $\tau' : z \mapsto z + c$, $c \in R$, the point $\infty^\beta = R(1, t\varepsilon)$ is fixed under τ, because $(t\varepsilon)c(t\varepsilon) = 0$; cf. (26). But C is the affine trace of a chain through ∞^β; so $C^{\tau'}$ is the affine trace of a chain through $\infty^{\beta\tau} = \infty^\beta$, whence, by the above, $C^{\tau'} \in C^{N'}$. Therefore $C^{T'} \subset C^{N'}$. There are two cases:

If $\operatorname{char} K = 2$ then $C^{T'} \neq C^{N'}$, since in this case for every parabola in $C^{T'}$ all its tangent lines are parallel to the line K, whereas there is a non-trivial shear $\nu' \in N'$ which maps C to a parabola whose mutually parallel tangent lines are not parallel to K.

Suppose now that $\operatorname{char} K \neq 2$. Then equation (32) can be written in the form

$$z_2 + \frac{l_1^2}{4t} - l_2 = t\left(z_1 + \frac{l_1}{2t}\right)^2.$$

Hence for each $\nu' \in N$ there exists a translation $\tau' \in T'$ with $C^{\nu'} = C^{\tau'}$. This is illustrated (with $\nu' \neq \tau'$) in Figure 1. (We just proved an affine version of a theorem on osculating conics; see [6, 2.5.4] or [11, Satz 2].) Thus $C^{T'} = C^{N'}$. So the mapping (30) leads us in a natural way to the aforementioned *parabola model* of the affine plane over K, $\operatorname{char} K \neq 2$. The point set of this model is the ring R; its line set consists of all vertical lines together with all translates of the parabola C.

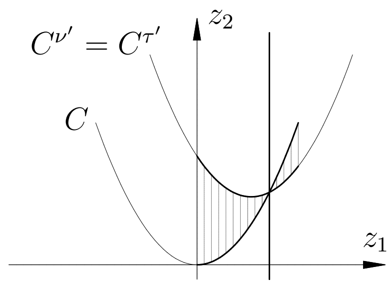

Figure 1.

Observe that there is also a parabola model for $\operatorname{char} K = 2$. However, since $C^{T'} \neq C^{N'}$, we have to use all the vertical lines and the orbit of C under the group N' (rather than the translation group) in order to obtain its line set.

The paper [16] gives, for the real dual numbers, an explicit description and some applications of the transformations described in Theorem 3.1 (b).

Example 3.3 Let R be the ring of upper triangular 2×2-matrices over K, where $K \neq \mathrm{GF}(2)$. So, R has a K-basis

$$j_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ j_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \varepsilon := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There are two maximal ideals in R, namely $Kj_1 + K\varepsilon$ and $Kj_2 + K\varepsilon$, and their union is the cone of singularity. The radical is $K\varepsilon$. A line or plane is said to be *vertical* if it is parallel to $K\varepsilon$. In formula (29) we may put $z = z_1 j_1 + z_2 j_2 + z_3\varepsilon$ and $b = t\varepsilon$ with $z_1, z_2, z_3, t \in K$. Thus we get

$$\beta' : R \to R : z_1 j_1 + z_2 j_2 + z_3 \varepsilon \mapsto z_1 j_1 + z_2 j_2 + (z_3 + t z_1 z_2)\varepsilon. \tag{33}$$

We assume that $t \neq 0$. All vertical lines are invariant under β'. Each point on the cone of singularity remains fixed. Consider a plane which is parallel to one of the planes of the cone of singularity. The restriction of β' to such a plane is a planar shear, fixing the intersection of the plane with the cone of singularity. Hence all non-regular lines go over to non-regular lines.

The group N' is obtained from (27) by putting $n_1 = l_1\varepsilon$ and $n_2 = l_2\varepsilon$, where $l_1, l_2 \in K$. So we get

$$\nu' : R \to R : z_1 j_1 + z_2 j_2 + z_3 \varepsilon \mapsto z_1 j_1 + z_2 j_2 + (z_1 l_1 + z_3 + l_2)\varepsilon. \tag{34}$$

If $l_1 \neq 0$ then ν' is a non-trivial *admissible shear*, i.e. a shear in the direction of $K\varepsilon$ with an axis parallel to the plane $z_1 = 0$. In fact, the axis of ν' is the vertical plane $z_1 = -l_2/l_1$. If $l_1 = 0$ then ν' is a *vertical translation*, i.e. a translation along the vertical line $K\varepsilon$. Altogether, the transformations in N' are all the admissible shears and all the vertical translations. In projective terms this is the group of all elations whose centre is the point at infinity of all the vertical lines and whose axis is a plane through the line at infinity of all the planes $z_1 = \text{const}$.

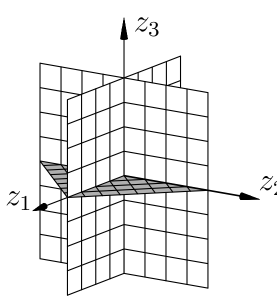

Figure 2.

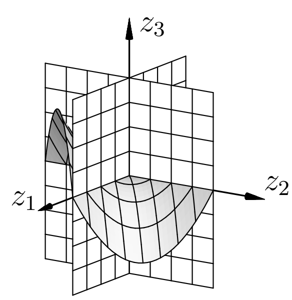

Figure 3.

Let P be the plane with equation $z_3 = 0$. It is clear that $P^{\beta'}$ is a hyperbolic paraboloid H with equation $z_3 = t z_1 z_2$, and that the set of images of the regular lines in P is the set $\mathcal{H}$ of all the parabolas contained in H. (Figure 2 shows, for $K = \mathbb{R}$, the cone of singularity and the plane P. In Figure 3 their images under β' are displayed.) As in Example 3.2, the group N' operates regularly on the set of non-vertical lines in a vertical plane which is non-parallel to $z_1 = 0$: If L is a regular line then there is a unique vertical plane V_L through L, and this plane is not parallel to $z_1 = 0$. Thus there is a unique mapping $\nu \in N$ such that $L^{\nu'} = V_L \cap P$. So, by this action of N' and by (28), the set of β'-images of the regular lines is the union of all orbits $C^{N'}$ with $C \in \mathcal{H}$.

An alternative description is possible using the translation group T'. (The straightforward calculations leading to the following results are left to the reader.) Fix a parabola $C \in \mathcal{H}$ lying in the vertical plane V, say. Then $\mathcal{V} := \{V^{\tau'} \cap H \mid \tau' \in T'\}$ is a set of parabolas. It follows that each parabola in $\mathcal{V}$ is a translate of a parabola in $C^{N'}$ and vice versa. There are two cases. If $\operatorname{char} K = 2$ then no parabola in $\mathcal{V} \setminus \{C\}$ is a translate of C. If $\operatorname{char} K \neq 2$ then all parabolas in $\mathcal{V}$ are translates of C. (This is well known for $K = \mathbb{R}$.)

Irrespective of $\operatorname{char} K$, the β'-images of the regular lines are—up to translations—precisely the parabolas in $\mathcal{H}$. Furthermore, if $\operatorname{char} K \neq 2$ then this result remains true if $\mathcal{H}$ is replaced by $\mathcal{H}_0 := \{C \in \mathcal{H} \mid 0 \in C\}$. Also we obtain the following *parabola model* of the affine 3-space over K. The point set of this model is the ring R; its line set consists of all non-regular lines together with all translates of the parabolas in $\mathcal{H}$ (for arbitrary characteristic of K) or in $\mathcal{H}_0$ (for $\operatorname{char} K \neq 2$ only).

If $K = \mathbb{R}$ then R is isomorphic to the ring of *real ternions*. A detailed investigation of the chain geometry over the real ternions can be found in [2].

The mappings discussed in Example 3.2 are closely related with the geometry of the *isotropic* (or: *Galilean*) *plane*. Likewise, Example 3.3 leads to a three-dimensional Cayley-Klein geometry, namely the geometry of the *pseudo-isotropic space*. We refer, among others, to [13], [17], [7, p. 136], and [14, p. 24].

The parabola model of the real affine plane is the starting point of the theory of *shift planes*. See [15, p. 420]. Such a plane arises, for example, from the real affine plane if the vertical lines and the translates of a curve which is in a certain sense "close to a parabola" are defined to be the "new lines". Similarly, it seems plausible that "in the neighbourhood" of our parabola model of the real affine 3-space there could exist so called $\mathbb{R}^3$-*spaces* (in the sense of [4]) other than the real affine 3-space. The reader should consult [9] for results and a lot of references on this interesting class of topological geometries.

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