Projective Representations
I. Projective lines over rings

Andrea Blunck[*]   Hans Havlicek

Dedicated to Armin Herzer on the occasion of his 70th birthday.

Abstract

We discuss representations of the projective line over a ring R with 1 in a projective space over some (not necessarily commutative) field K. Such a representation is based upon a (K, R)-bimodule U. The points of the projective line over R are represented by certain subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one of their complements. In particular, distant points go over to complementary subspaces, but in certain cases, also non-distant points may have complementary images.
Mathematics Subject Classification (1991): 51C05, 51A45, 51B05.

1 Introduction

Each ring R with 1, containing in its centre a (necessarily commutative) field F with $1 \in F$, gives rise to a *chain geometry* $\Sigma(F, R)$. For a survey, see [11]. In [4] we introduced the concept of a *generalized chain geometry* $\Sigma(F, R)$; now R is a ring with 1 containing a (not necessarily commutative) field F subject to $1 \in F$. In both cases the point set of $\Sigma(F, R)$ is $\mathbb{P}(R)$, i.e., the *projective line* over R, and the chains are the F-sublines.

In the present paper we introduce representations of the projective line over an arbitrary ring R in a projective space over some field K. In a second publication our results will be applied to obtain representations of generalized chain geometries.

The starting point of our investigation is A. Herzer's approach [11] to obtain a model of a chain geometry $\Sigma(F, R)$ for a finite-dimensional F-algebra R by means of a faithful right R-module U with finite F-dimension, say r. Here the points of the projective line $\mathbb{P}(R)$ are in one-one correspondence with certain $(r-1)$-dimensional subspaces of the $(2r-1)$-dimensional projective space $\mathbb{P}(F, U \times U)$. In our more general setting we use a (K, R)-bimodule U; so U is a left K-vector space and at the same time a right R-module. We neither do assume that K is a subset of R, nor that U is a faithful R-module, nor that the K-dimension of U is finite. A *projective representation* obtained in this way maps the points of $\mathbb{P}(R)$ into the set of those subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one

[*]Supported by a Lise Meitner Research Fellowship of the Austrian Science Fund (FWF), project M529-MAT.

of their complements. This mapping is injective if, and only if, U is a faithful R-module. In this case we speak of a *projective model* of $\mathbb{P}(R)$.

If U' is a sub-bimodule of U then there are representations of $\mathbb{P}(R)$ that stem from the action of R on U' and U/U'. In general, these *induced representations* are not injective, even if U is faithful. This is one of the reasons why we also discuss non-injective representations. The examples at the end of the paper illustrate how these induced representations can sometimes be used in order to describe models of $\mathbb{P}(R)$ in terms of $\mathbb{P}(K, U \times U)$.

2 The projective line over a ring

Let R be a ring. Throughout this paper we shall only consider rings with 1 (where the trivial case $1 = 0$ is not excluded). The group of invertible elements of the ring R will be denoted by R^*. Consider the free left R-module R^2. Its automorphism group is the group $\mathrm{GL}_2(R)$ of invertible 2×2-matrices with entries in R. According to [4], [11], the *projective line over* R is the orbit

$$\mathbb{P}(R) := R(1,0)^{\mathrm{GL}_2(R)}$$

of the free cyclic submodule $R(1,0)$ under the action of $\mathrm{GL}_2(R)$. Since $R^2 = R(1,0) \oplus R(0,1)$, the elements (the *points*) of $\mathbb{P}(R)$ are exactly those free cyclic submodules of R^2 that have a free cyclic complement.

A pair $(a, b) \in R^2$ is called *admissible*, if there exist $c, d \in R$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$. So $\mathbb{P}(R) = \{R(a,b) \subset R^2 \mid (a,b) \text{ admissible}\}$. However, in certain cases the points of $\mathbb{P}(R)$ may also be represented by non-admissible pairs, as we will see below.

We recall that a pair $(a, b) \in R^2$ is *unimodular*, if there exist $x, y \in R$ such that $ax + by = 1$, i.e., if there is an R-linear form $R^2 \to R$ mapping (a, b) to 1. This is equivalent to saying that the right ideal generated by a and b is the whole ring R.

Obviously, each admissible pair (a, b) is unimodular. If R is commutative, then admissibility and unimodularity are equivalent. W. Benz in [1] considers only commutative rings and defines the projective line using unimodular pairs.

Proposition 2.1 *Let $(a, b) \in R^2$ be admissible, and let $s \in R$. Put $(a', b') := s(a, b)$. Then*

(1) *s is left invertible $\Longleftrightarrow R(a,b) = R(a',b')$.*

(2) *s is right invertible $\Longleftrightarrow (a',b')$ is admissible.*

Proof: (1): If there is an $l \in R$ with $ls = 1$, then $(a,b) = l(a',b')$. So $R(a,b) = R(a',b')$.
If $R(a,b) = R(a',b')$, then there is an $l \in R$ such that $(a,b) = l(a',b')$. Since (a,b) is admissible, it is also unimodular, and so there are $x, y \in R$ with $1 = ax + by = lsax + lsby = ls$. Hence s is left invertible.

(2): If s is right invertible, then $sr = 1$ for some $r \in R$. An easy calculation shows that

$$\gamma = \begin{pmatrix} s & 0 \\ 1 - rs & r \end{pmatrix} \in \mathrm{GL}_2(R), \text{ with } \gamma^{-1} = \begin{pmatrix} r & 1 - rs \\ 0 & s \end{pmatrix}. \tag{1}$$

There is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, whence $\left(\begin{smallmatrix} a' & b' \\ * & * \end{smallmatrix}\right) = \gamma\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, as required.
If (a', b') is admissible, then there are $x', y' \in R$ with $a'x' + b'y' = 1$. So $s(ax' + by') = 1$, i.e., s has a right inverse. $\square$

Note that the statement of Proposition 2.1 remains true if one substitutes "admissible" by "unimodular", however, the proof of (2)"$\Rightarrow$" then has to be modified.
Rings with the property that $ab = 1$ implies $ba = 1$ are called *Dedekind-finite* (see e.g. [13]). From Proposition 2.1 we obtain

Proposition 2.2 *Let R be a ring. Then the following are equivalent:*

(1) *R is Dedekind-finite.*

(2) *If $R(a, b) \in \mathbb{P}(R)$, then (a, b) is admissible.*

(3) *No point of $\mathbb{P}(R)$ is properly contained in another point of $\mathbb{P}(R)$.*

Remark 2.3 If R is not Dedekind-finite, then each point $p \in \mathbb{P}(R)$ belongs to an infinite sequence of points

$$\ldots \subsetneq p_{-2} \subsetneq p_{-1} \subsetneq p_0 = p \subsetneq p_1 \subsetneq p_2 \subsetneq \ldots$$

Namely, let γ be the matrix of formula (1), where $sr = 1 \neq rs$. Then Proposition 2.1 shows that the points $p_i := p^{\gamma^i}$ are as desired.

Recall that according to F.D. Veldkamp [15], [16] the ring R has *stable rank* 2, if for each unimodular pair $(a, b) \in R^2$ there is a $c \in R$ such that $a + bc$ is right invertible. The following results on rings of stable rank 2 can be found in [15] (results 2.10 and 2.11):

Remark 2.4 Let R be of stable rank 2. Then R is Dedekind-finite and each unimodular $(a, b) \in R^2$ is admissible.

Note that Herzer's definition of stable rank 2 in [11] seems to be stronger but actually coincides with Veldkamp's because of 2.4. Moreover, it is not necessary to distinguish between left and right stable rank 2 because by [15], 2.2, the opposite ring (with reversed multiplication) of a ring of stable rank 2 also has stable rank 2.
Using results of [13], § 20, one obtains that each left (or right) artinian ring has stable rank 2 (called "left stable range 1" in [13]). We shall need the following special case:

Remark 2.5 Assume that R contains a subfield K such that R is a *finite-dimensional* left (or right) vector space over K. Then R is of stable rank 2. In particular, R is Dedekind-finite.

Here by a *subfield* we mean a not necessarily commutative field $K \subset R$ with $1 \in K$.
We turn back to the projective line over an arbitrary ring. The point set $\mathbb{P}(R)$ is endowed with the symmetric relation $\triangle$ (*"distant"*) defined by

$$\triangle := \{R(1, 0), R(0, 1)\}^{\mathrm{GL}_2(R)}$$

i.e., two points $p, q \in \mathbb{P}(R)$ are distant exactly if there is a $\gamma \in \mathrm{GL}_2(R)$ mapping $R(1, 0)$ to p and $R(0, 1)$ to q. Distance can also be expressed in terms of coordinates:

Remark 2.6 Let $p = R(a,b)$, $q = R(c,d) \in \mathbb{P}(R)$ with admissible (a,b), (c,d). Then

$$p \triangle q \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

Note that this is independent of the choice of the *admissible* representatives (a,b), (c,d). In addition, $\triangle$ is anti-reflexive exactly if $1 \neq 0$; compare [11].

By definition, two points of $\mathbb{P}(R)$ are distant if, and only if, they are complementary submodules of R^2. There are several possibilities for points being non-distant, which all can occur as the following examples show:

Examples 2.7 (1) Let R be a ring that is not Dedekind-finite. Let $\gamma \in \mathrm{GL}_2(R)$ be defined as in Remark 2.3. Then $p = R(1,0)^\gamma = R(s,0)$ and $q = R(0,1)$ are non-distant: They have a trivial intersection but they do not span R^2.
Now consider $p' = R(1,0)^{\gamma^{-1}} = R(r, 1-rs)$ (see formula (1)). Then p' and q are non-distant: They span R^2, but $(1-rs)(r,1-rs) = (0,1-rs) \neq (0,0)$ lies in their intersection.

(2) Let R contain a subfield K such that R, considered as left vector space over K, has finite dimension n. Then all points of $\mathbb{P}(R)$ are n-dimensional subspaces of the left vector space R^2. In particular, two points have a trivial intersection exactly if they span R^2.

In Example 4.7 below we will see an example of a commutative (and hence Dedekind-finite) ring where non-distant points intersect trivially.

3 Homomorphisms

Now we want to study mappings between projective lines over rings that are induced by ring homomorphisms.

From now on, we will follow the convention that whenever a point of $\mathbb{P}(R)$ is given in the form $R(a,b)$, we always assume that the pair $(a,b) \in R^2$ is admissible.

Let R, S be rings. The distance relations on $\mathbb{P}(R)$ and $\mathbb{P}(S)$ are denoted by $\triangle_R$ and $\triangle_S$, respectively. Consider a ring homomorphism $\varphi : R \to S$, where we always suppose that 1_R is mapped to 1_S. Associated to φ is a homomorphism $\mathrm{M}(2 \times 2, R) \to \mathrm{M}(2 \times 2, S)$, mapping $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a^\varphi & b^\varphi \\ c^\varphi & d^\varphi \end{smallmatrix}\right)$, which will also be denoted by φ. Its restriction to $\mathrm{GL}_2(R)$ is a group homomorphism into $\mathrm{GL}_2(S)$. This implies that if $(a,b) \in R^2$ is admissible, so is $(a^\varphi, b^\varphi) \in S^2$, and we can introduce the mapping

$$\bar{\varphi} : \mathbb{P}(R) \to \mathbb{P}(S) : R(a,b) \mapsto S(a^\varphi, b^\varphi).$$

Proposition 3.1 *Let $\varphi : R \to S$ be a ring homomorphism. Then for $\bar{\varphi} : \mathbb{P}(R) \to \mathbb{P}(S)$ the following statements hold:*

(1) $\bar{\varphi}$ preserves distance, *i.e.*, $\forall p, q \in \mathbb{P}(R) : \ p \triangle_R q \Rightarrow p^{\bar{\varphi}} \triangle_S q^{\bar{\varphi}}$.

(2) $\bar\varphi$ *is* compatible with the action of $\mathrm{GL}_2(R)$*, i.e.* $\forall p \in \mathbb{P}(R)\ \forall \gamma \in \mathrm{GL}_2(R):\ p^{\gamma\bar\varphi} = p^{\bar\varphi\gamma^\varphi}$.

(3) $\bar\varphi$ *is injective if, and only if,* φ *is.*

Proof: Only (3) deserves our attention. Let φ be injective. Assume that $R(a,b)^{\bar\varphi} = R(c,d)^{\bar\varphi}$ holds for $R(a,b)$, $R(c,d) \in \mathbb{P}(R)$. Then there is an $s \in S^*$ with $(a^\varphi, b^\varphi) = s(c^\varphi, d^\varphi)$. Since $(c,d) \in R^2$ is unimodular, there are $x, y \in R$ with $s = s1 = s1^\varphi = s(cx+dy)^\varphi = a^\varphi x^\varphi + b^\varphi y^\varphi \in R^\varphi$. Analogously, one sees that $s^{-1} \in R^\varphi$. Hence $s \in (R^\varphi)^*$, which equals $(R^*)^\varphi$, since φ is injective. So $R(a,b) = R(c,d)$.
Now let $\bar\varphi$ be injective, and assume $a^\varphi = b^\varphi$ for $a, b \in R$. Then $R(1,a)^{\bar\varphi} = S(1,a^\varphi) = S(1,b^\varphi) = R(1,b)^{\bar\varphi}$, whence $R(1,a) = R(1,b)$ and so $a = b$. $\square$

We call the mapping $\bar\varphi : \mathbb{P}(R) \to \mathbb{P}(S)$ the *homomorphism of projective lines induced by* $\varphi : R \to S$. Such homomorphisms map distant points to distant points. However, they may also map non-distant points to distant points: Consider e.g. the homomorphism $\mathbb{P}(\mathbb{Z}) \to \mathbb{P}(\mathbb{Q})$ induced by the natural inclusion $\mathbb{Z} \to \mathbb{Q}$. This injective homomorphism actually is a bijection, since each element of $\mathbb{P}(\mathbb{Q})$ can be represented by a pair of relatively prime integers. The points $\mathbb{Z}(1,0)$ and $\mathbb{Z}(1,2)$ are non-distant because $\left(\begin{smallmatrix}1 & 0\\ 1 & 2\end{smallmatrix}\right)$ is not invertible over $\mathbb{Z}$. However, their image points $\mathbb{Q}(1,0)$ and $\mathbb{Q}(1,2)$ are different and hence distant in $\mathbb{P}(\mathbb{Q})$.

The following gives a characterization of the homomorphisms $\bar\varphi$ that preserve also non-distance. By $\mathrm{rad}(R)$ we denote the (Jacobson) *radical* of the ring R (cf. [13]).

Proposition 3.2 *Let* $\bar\varphi : \mathbb{P}(R) \to \mathbb{P}(S)$ *be induced by the ring homomorphism* $\varphi : R \to S$. *Then the following statements are equivalent:*

(1) $\forall p, q \in \mathbb{P}(R):\ p^{\bar\varphi} \triangle_S q^{\bar\varphi} \Rightarrow p \triangle_R q$.

(2) $\forall y \in R : y^\varphi \in S^* \Rightarrow y \in R^*$.

(3) $\ker(\varphi) \subset \mathrm{rad}(R)$ *and* $(R^\varphi)^* = S^* \cap R^\varphi$.

Proof: (1) $\Rightarrow$ (2): For $r \in R$ with $r^\varphi \in S^*$ we have $S(1,0) \triangle_S S(1,r^\varphi)$. Hence condition (1) implies $R(1,0) \triangle_R R(1,r)$ and thus $r \in R^*$.
(2) $\Rightarrow$ (1): Let $p^{\bar\varphi} \triangle_S q^{\bar\varphi}$ hold for $p, q \in \mathbb{P}(R)$. Choose $\gamma \in \mathrm{GL}_2(R)$ with $p^\gamma = R(1,0)$. Then $q^\gamma = R(x,y)$ for a certain admissible pair $(x,y) \in R^2$. By 3.1(2), we have $S(1,0) = p^{\gamma\bar\varphi} = p^{\bar\varphi\gamma^\varphi} \triangle_S q^{\bar\varphi\gamma^\varphi} = q^{\gamma\bar\varphi} = S(x^\varphi, y^\varphi)$, and hence $y^\varphi \in S^*$. So, by (2), $y \in R^*$. This implies $p^\gamma \triangle_R q^\gamma$ and thus also $p \triangle_R q$.
(2) $\Leftrightarrow$ (3): See [8], Lemma 1.5. $\square$

As the example $\mathbb{P}(\mathbb{Z}) \to \mathbb{P}(\mathbb{Q})$ above shows, the ring homomorphism φ need not be surjective if $\bar\varphi$ is.

We now consider the case where $\varphi : R \to S$ is a surjective homomorphism of rings. It is not clear whether in general $\bar\varphi$ also is surjective. We study special cases.

According to J.R. Silvester [14] we introduce the following notions for a ring R:

The *elementary linear group* $\mathrm{E}_2(R)$ is the subgroup of $\mathrm{GL}_2(R)$ generated by the *elementary transvections*, i.e, by all matrices of the form $\left(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ $(x \in R)$. The group $\mathrm{GE}_2(R)$

is the subgroup of $\mathrm{GL}_2(R)$ generated by $\mathrm{E}_2(R)$ and all diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(R)$. Note that $\mathrm{E}_2(R)$ is normal in $\mathrm{GE}_2(R)$. If $\mathrm{GE}_2(R) = \mathrm{GL}_2(R)$, then R is called a GE_2*-ring*. Examples of GE_2-rings and also of rings that are not GE_2-rings can be found in [14], p.114 and p.121, respectively. Important for us is the following:

Remark 3.3 (See [9], 4.2.5.) Let R be a ring of stable rank 2. Then R is a GE_2-ring.

Lemma 3.4 *Let R be a GE_2-ring. Then $\mathbb{P}(R) = R(1,0)^{\mathrm{E}_2(R)}$.*

Proof: Let $p = R(1,0)^\gamma \in \mathbb{P}(R)$, with $\gamma \in \mathrm{GL}_2(R)$. Since $\mathrm{GL}_2(R) = \mathrm{GE}_2(R)$ and $\mathrm{E}_2(R)$ is normal in $\mathrm{GE}_2(R)$, we have $\gamma = \delta\eta$, where $\delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\eta \in \mathrm{E}_2(R)$. So $p = R(1,0)^\gamma = R(a^{-1},0)^\gamma = R(1,0)^\eta \in R(1,0)^{\mathrm{E}_2(R)}$. $\square$

Now we can state conditions that imply that with $\varphi : R \to S$ also $\bar{\varphi}$ is surjective.

Proposition 3.5 *Let $\varphi : R \to S$ be a surjective homomorphism of rings. Then also $\bar{\varphi} : \mathbb{P}(R) \to \mathbb{P}(S)$ is surjective, if one of the following conditions is satisfied:*

(1) *S is a GE_2-ring.*

(2) $\ker(\varphi) \subset \mathrm{rad}(R)$.

(3) *R is the internal direct product of $\ker(\varphi)$ and some ideal $R' \subset R$.*

Proof: (1): Consider a point $q \in \mathbb{P}(S)$. By Lemma 3.4 we have $q = S(1,0)^\eta$, where $\eta \in \mathrm{E}_2(S)$, i.e., η is a product of elementary transvections. Since $\varphi : R \to S$ is surjective, each elementary transvection has a preimage under $\varphi : \mathrm{M}(2 \times 2, R) \to \mathrm{M}(2 \times 2, S)$ that is an elementary transvection over R. Hence $\eta = \gamma^\varphi$, where $\gamma \in \mathrm{E}_2(R)$, and so by 3.1(2) we obtain $q = R(1,0)^{\bar{\varphi}\eta} = R(1,0)^{\gamma\bar{\varphi}} \in \mathbb{P}(R)^{\bar{\varphi}}$.
(2): Follows from [5], Lemma 1.14.
(3): In this case, $\mathrm{GL}_2(R)$ consists exactly of the sums $\gamma + \gamma'$, where $\gamma \in \mathrm{GL}_2(\ker(\varphi))$ and $\gamma' \in \mathrm{GL}_2(R')$. Moreover, $\varphi|_{\mathrm{GL}_2(R')} : \mathrm{GL}_2(R') \to \mathrm{GL}_2(S)$ is an isomorphism of groups. This yields the assertion. $\square$

Note that one could also use Proposition 3.2 in order to prove assertion (2) above, since the radical of $\mathrm{M}(2 \times 2, R)$ consists exactly of all matrices with entries in $\mathrm{rad}(R)$.

4 Projective representations

The projective representations we are now aiming at are based upon the following.

Remark 4.1 (see [2]). Let U be a left vector space over a field K, and let $S = \mathrm{End}_K(U)$ be its endomorphism ring. Moreover, let $\mathcal{G}$ be the set of all subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one of their complements. Then

$$\Psi : \mathbb{P}(S) \to \mathcal{G} : S(\alpha, \beta) \mapsto U^{(\alpha,\beta)} := \{(u^\alpha, u^\beta) \mid u \in U\}$$

is a well-defined bijection mapping distant points of $\mathbb{P}(S)$ to complementary subspaces in $\mathcal{G}$ and non-distant points to non-complementary subspaces. Moreover, the groups $\mathrm{GL}_2(S)$ and $\mathrm{Aut}_K(U \times U)$ are isomorphic and their actions on $\mathbb{P}(S)$ and on $\mathcal{G}$, respectively, are equivalent via Ψ. In particular, the mappings induced on $\mathcal{G}$ by $\mathrm{GL}_2(S)$ arise from *projective collineations* of the projective space $\mathbb{P}(K, U \times U)$.

Let now K be a field and let R be a ring. A left vector space U over K is called a (K, R)-*bimodule*, if U is a (unitary) right R-module such that for all $k \in K$, $u \in U$, $a \in R$ the equality $k(u \cdot a) = (ku) \cdot a$ holds. If U is a (K, R)-bimodule, then $\varphi : R \to \mathrm{End}_K(U)$ with $a^\varphi = \rho_a : u \mapsto u \cdot a$ is a ring homomorphism.

If, on the other hand, there is a homomorphism $\varphi : R \to \mathrm{End}_K(U)$, then U becomes a (K, R)-bimodule by setting $u \cdot a := u^{\rho_a}$, where $\rho_a = a^\varphi$. A homomorphism $\varphi : R \to \mathrm{End}_K(U)$ is also called a *K-linear representation* of R.

So, the concepts of a K-linear representation of R and a (K, R)-bimodule are equivalent. Whenever we consider a (K, R)-bimodule U, we denote by φ the associated linear representation, and for $a \in R$ we write ρ_a for the endomorphism $a^\varphi : u \mapsto u \cdot a$.

A (K, R)-bimodule U and the associated linear representation φ are called *faithful*, exactly if φ is an injection.

Combining 3.1 and 4.1, we obtain our main result:

Theorem 4.2 *Let U be a (K, R)-bimodule. Then the mapping*

$$\Phi := \bar{\varphi}\Psi : \mathbb{P}(R) \to \mathcal{G} : R(a, b) \mapsto U^{(\rho_a, \rho_b)}$$

maps distant points of $\mathbb{P}(R)$ to complementary subspaces in $\mathbb{P}(K, U \times U)$. The bimodule U is faithful if, and only if, Φ is injective.

Thus, to each homomorphism $\varphi : R \to \mathrm{End}_K(U)$ corresponds a mapping Φ (see above). We call Φ a *projective representation* of $\mathbb{P}(R)$, and a *faithful* projective representation if U is faithful. We are interested in the image of $\mathbb{P}(R)$ under a projective representation. If the representation is faithful, then $\Phi : \mathbb{P}(R) \to \mathbb{P}(R)^\Phi$ is a bijection, and the image $\mathbb{P}(R)^\Phi$ can be seen as a model of $\mathbb{P}(R)$ in the projective space; we then call $\mathbb{P}(R)^\Phi$ a *projective model* of $\mathbb{P}(R)$. Otherwise, one obtains a model of the projective line over another ring:

Proposition 4.3 *Let $J = \mathrm{ann}(U)$ be the* annihilator *of U, i.e., the kernel of the representation $\varphi : R \to \mathrm{End}_K(U)$. Then the following statements hold:*

(1) *The mapping $\varphi_f : R/J \to \mathrm{End}_K(U)$ with $\rho_{a+J} : u \mapsto u^{\rho_a}$ is a faithful K-linear representation of R/J. Hence $\Phi_f = \overline{\varphi_f}\Psi$ is a faithful projective representation of $\mathbb{P}(R/J)$.*

(2) *The projective model $\mathbb{P}(R/J)^{\Phi_f}$ contains $\mathbb{P}(R)^\Phi$.*

(3) *The mapping $\bar{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/J)$ induced by the canonical epimorphism $\pi : R \to R/J$ is surjective if, and only if, $\mathbb{P}(R/J)^{\Phi_f} = \mathbb{P}(R)^\Phi$.*

Recall that Proposition 3.5 gives conditions under which the assumptions of statement (3) are met.

The representation $\varphi : R \to S = \mathrm{End}_K(U)$ gives rise to a group homomorphism $\varphi : \mathrm{GL}_2(R) \to \mathrm{GL}_2(S) \cong \mathrm{Aut}_K(U \times U)$. Using 3.1(2) and 4.1 we obtain the following:

Proposition 4.4 *Let U be a (K, R)-bimodule, and let $\gamma \in \mathrm{GL}_2(R)$. Then the induced mapping*

$$\mathbb{P}(R)^{\Phi} \to \mathbb{P}(R)^{\Phi} : R(a,b)^{\Phi} \mapsto R(a,b)^{\gamma\Phi}$$

is induced by a projective collineation of $\mathbb{P}(K, U \times U)$.

Finally, Proposition 3.2 yields the following:

Proposition 4.5 *Let U be a (K, R)-bimodule. Then the corresponding projective representation Φ maps non-distant points to non-complementary subspaces exactly if for each $a \in R$ the condition $\rho_a \in \mathrm{Aut}_K(U)$ implies $a \in R^*$.*

Note that from $\rho_a \in \mathrm{Aut}_K(U)$ and $a \in R^*$ one obtains that $(\rho_a)^{-1} = \rho_{a^{-1}}$.
We mention two classes of examples where the condition of Proposition 4.5 is satisfied.

Examples 4.6 (1) Let R contain K as a subfield. Then $U = R$ is a left vector space over K, and $\varphi : R \to \mathrm{End}_K(U)$ with $\rho_a : x \mapsto xa$ is a faithful linear representation of R, called the *regular representation*. In this case Φ is the identity, where the submodule $R(a,b) \in \mathbb{P}(R)$ is considered as a projective subspace of $\mathbb{P}(K, U \times U)$. So points of $\mathbb{P}(R)$ are distant exactly if their Φ-images are complementary. This reflects the algebraic fact that the endomorphism $\rho_a : R \to R : x \mapsto xa$ is a bijection exactly if $a \in R^*$.

(2) Let U be a faithful (K, R)-bimodule. Assume moreover that R contains a subfield L such that R is a finite-dimensional left vector space over L. Then the projective representation Φ maps non-distant points to non-complementary subspaces:
In view of (1), it suffices to show that for each $a \in R$ with $\rho_a \in \mathrm{Aut}_K(U)$ the L-linear mapping $R \to R : x \mapsto xa$ is injective. Suppose $xa = 0$ for $x \in R$. Then for all $u \in U$ we have $0 = u \cdot 0 = u \cdot (xa) = (u \cdot x)^{\rho_a}$. Since ρ_a is an automorphism, this implies $u \cdot x = 0$ for all $u \in U$, and hence $x = 0$ because U is a faithful R-module.

We proceed by giving an example of a faithful projective representation where non-distant points appear as complementary subspaces:

Example 4.7 Let K be any commutative field, let R be the polynomial ring $R = K[X]$, and let $U = K(X)$ be its field of fractions. Then U contains K and R, and thus is a faithful (K, R)-bimodule in a natural way. Obviously, $\rho_X : u \mapsto uX$ is a bijection on U, but $X \notin R^*$. This means that e.g. $R(1,0)$ and $R(1,X)$ are non-distant points of $\mathbb{P}(R)$, but their images $U^{(1,0)} = U \times \{0\}$ and $U^{(1,\rho_X)} = \{(u, uX) \mid u \in U\}$ are complementary subspaces of $\mathbb{P}(K, U \times U)$. Note that $R(1,0)$ and $R(1,X)$, considered as submodules of R^2, also intersect trivially, but they do not span R^2 (compare 2.7).
Note, moreover, that we could also interpret the elements $R(a,b)^{\Phi} = U(a,b)$ as points of the projective line over the field U. Hence any two such elements must be complementary.
In a similar way one can also construct examples where R is not contained in any field: Let R and U be as above. Let $R[\varepsilon]$ be the ring of *dual numbers* over R, with ε central, $\varepsilon \notin K$,

and $\varepsilon^2 = 0$. Then ε is a zero-divisor and hence $R[\varepsilon]$ is not embeddable into any field. Now take $U[\varepsilon]$ and proceed as above.

Let U be a (K, R)-bimodule. A subset $U' \subset U$ is called a *sub-bimodule* of U, if U' is a subspace of the left vector space U over K and at the same time a submodule of the right R-module U. The linear representation of R given by the bimodule U' is $\varphi' : a \mapsto \rho_a|_{U'}$. The faithful representation $(\varphi')_f : R/\mathrm{ann}(U') \to \mathrm{End}_K(U')$ will be called the *induced faithful representation*.

The projective representation Φ' associated to φ' maps the points of $\mathbb{P}(R)$ to certain subspaces of the projective space $\mathbb{P}(K, U' \times U')$, more exactly, $\mathbb{P}(R)^{\Phi'}$ is a subset of the set $\mathcal{G}'$ of all subspaces of $\mathbb{P}(K, U' \times U')$ that are isomorphic to one of their complements.

Now $\mathbb{P}(K, U' \times U')$ is a projective subspace of $\mathbb{P}(K, U \times U)$, and we can compare the images of $\mathbb{P}(R)$ under the projective representations Φ and Φ'. One obtains the following geometric interpretation:

Proposition 4.8 *Let U' be a sub-bimodule of the (K, R)-bimodule U, and let Φ' and Φ be the associated projective representations of $\mathbb{P}(R)$. Then for each $p \in \mathbb{P}(R)$ we have*

$$p^{\Phi'} = p^{\Phi} \cap (U' \times U').$$

In particular, each p^{Φ} meets the projective subspace $\mathbb{P}(K, U' \times U')$ in an element of $\mathcal{G}'$.

Proof: First consider $p = R(1, 0)$. Then $p^{\Phi'} = U' \times \{0\} = (U \times \{0\}) \cap (U' \times U') = p^{\Phi} \cap (U' \times U')$. Now consider an arbitrary $p \in \mathbb{P}(R)$. Then $p = R(1, 0)^{\gamma}$ for some $\gamma \in \mathrm{GL}_2(R)$. The induced automorphism γ^{φ} of $U \times U$ leaves $U' \times U'$ invariant, it coincides on $U' \times U'$ with $\gamma^{\varphi'} \in \mathrm{Aut}_K(U' \times U')$. This yields the assertion. $\square$

Note that the Φ'-image of $\mathbb{P}(R)$ is contained in the image of $\mathbb{P}(R/\mathrm{ann}(U'))$ under the induced faithful representation $(\Phi')_f$. According to 4.3(3), the two sets coincide exactly if the mapping $\overline{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/\mathrm{ann}(U'))$, associated to the canonical epimorphism $\pi : R \to R/\mathrm{ann}(U')$, is surjective.

Proposition 4.9 *Let $U = U' \oplus U''$ be a (K, R)-bimodule. Let φ, φ', φ'' be the associated representations of R. Then for each $p \in \mathbb{P}(R)$ we have $p^{\Phi} = p^{\Phi'} \oplus p^{\Phi''}$.*

Proof: As in the proof of Proposition 4.8 we first verify the assertion for $p = R(1, 0)$ (with the help of 4.8) and then use the action of $\mathrm{GL}_2(R)$. $\square$

Let again U' be a sub-bimodule of the (K, R)-bimodule U. Then also $\widetilde{U} = U/U'$ is a (K, R)-bimodule, corresponding to the representation $\widetilde{\varphi} : R \to \mathrm{End}_K(\widetilde{U})$, where $\widetilde{\rho}_a : u + U' \mapsto u^{\rho_a} + U'$. The kernel of this representation is the ideal consisting of all $a \in R$ such that the image of ρ_a is contained in U'. As above, we obtain an *induced faithful representation* $(\widetilde{\varphi})_f : R/\ker(\widetilde{\varphi}) \to \mathrm{End}_K(\widetilde{U})$.

The projective representation $\widetilde{\Phi}$ maps $\mathbb{P}(R)$ into the set $\widetilde{\mathcal{G}}$ of all subspaces of $\mathbb{P}(K, \widetilde{U} \times \widetilde{U})$ that are isomorphic to one of their complements. Now the projective space $\mathbb{P}(K, \widetilde{U} \times \widetilde{U})$ is canonically isomorphic to the projective space of all subspaces of $\mathbb{P}(K, U \times U)$ containing $U' \times U'$, because $(U \times U)/(U' \times U') \cong \widetilde{U} \times \widetilde{U}$. We shall identify the elements of $\widetilde{\mathcal{G}}$ with their images under this isomorphism. So we can compare $\widetilde{\Phi}$ and Φ, and the same procedure as before yields

Proposition 4.10 *Let $\widetilde{U} = U/U'$, and let $\widetilde{\Phi}$ be the associated projective representation of $\mathbb{P}(R)$. Then for each $p \in \mathbb{P}(R)$ we have*

$$p^{\widetilde{\Phi}} = p^{\Phi} + (U' \times U').$$

In particular, each $p^{\Phi} + (U' \times U')$ is an element of $\widetilde{\mathcal{G}}$.

As before, one may also consider the induced faithful representation $(\widetilde{\Phi})_f$ of $\mathbb{P}(R/\ker(\widetilde{\varphi}))$.

5 Examples

In this section we study some examples. Note that we consider only rings R that are finite-dimensional left vector spaces over a subfield K. Then for each ideal I of R also the ring R/I is finite dimensional over K, whence R/I is of stable rank 2 and hence a GE_2-ring (compare 2.5 and 3.3). So Proposition 3.5 implies that in all our examples the mapping $\bar{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/I)$ induced by the canonical epimorphism $\pi : R \to R/I$ is surjective.

Example 5.1 Let $K = R$ be any (not necessarily commutative) field and let $U = K^2$ with componentwise action $(x_1, x_2)\cdot k = (x_1 k, x_2 k)$. Then U is the direct sum of the sub-bimodules $U_1 = K(1,0)$ and $U_2 = K(0,1)$, on which $R = K$ acts faithfully in the natural way. The representations induced in the skew lines $U_i \times U_i$ are faithful and map $\mathbb{P}(K)$ onto the set of all points of $U_i \times U_i$. Moreover, $\beta := \Phi_1^{-1}\Phi_2$ is a bijection between these two projective lines, which is linearly induced and hence a projectivity. The elements of the projective model $\mathbb{P}(K)^{\Phi}$ in $\mathbb{P}(K, U \times U)$ are exactly the lines joining a point of $U_1 \times U_1$ and its β-image in $U_2 \times U_2$. So $\mathbb{P}(K)^{\Phi}$ is a regulus in 3-space (compare [6]).
The same applies if $U = K^n$. Then one obtains a regulus in a $(2n-1)$-dimensional projective space (see [3]), i.e., a generalization to the not necessarily pappian case of a family of $(n-1)$-dimensional subspaces on a Segre manifold $S_{n-1,1}$ (compare [7]).

Example 5.2 Example 5.1 above can be modified in the following way: Let $\alpha_1, \alpha_2 : K \to K$ be field monomorphisms. Then K acts faithfully on $U = K^2$ via $(x_1, x_2)\cdot k = (x_1 k^{\alpha_1}, x_2 k^{\alpha_2})$. The induced projective models of $\mathbb{P}(K)$ in the projective lines $U_i \times U_i$ are projective sublines over the subfields K^{α_i}. In general, the bijection β between the two models is not K-semilinearly induced.
We mention one special case: If $K = \mathbb{C}$, $\alpha_1 = \mathrm{id}$, and α_2 is the complex conjugation, then the projective model of $\mathbb{P}(\mathbb{C})$ is a set of lines in the 3-space $\mathbb{P}(\mathbb{C}, U \times U)$. It can be interpreted as follows: The α_2-semilinear bijection β extends to a collineation of order two which fixes a Baer subspace (with $\mathbb{R}$ as underlying field). The lines of the projective model of $\mathbb{P}(\mathbb{C})$ meet this Baer subspace in a regular spread (elliptic linear congruence). See [10] for a generalization of this well-known classical result that the regular spreads of a real 3-space can

be characterized (in the complexified space) as those sets of lines that join complex-conjugate points of two skew complex-conjugate lines.

Example 5.3 Let K be any field. Let $U = R = K^n$, with componentwise addition and multiplication. For $i \in \{1, \dots, n\}$, let $U_i = Kb_i$, where b_i runs in the standard basis. Then U_i is a sub-bimodule of U, the induced faithful action is the ordinary action of K. Hence the projective model $\mathbb{P}(R)^\Phi = \mathbb{P}(R)$ meets the line $U_i \times U_i$ in all points. Moreover, *each* $(n-1)$-dimensional projective subspace of $\mathbb{P}(K, U \times U)$ that meets all the lines $U_i \times U_i$ belongs to $\mathbb{P}(R)$, because $\mathrm{GL}_2(R) \cong \mathrm{GL}_2(K) \times \ldots \times \mathrm{GL}_2(K)$.

If $n = 2$, the set $\mathbb{P}(R)$ is a generalization to the not necessarily pappian case of a *hyperbolic linear congruence*.

Example 5.4 Let K be any field. Let $U = R = K[\varepsilon]$, where $\varepsilon \notin K$, $\varepsilon^2 = 0$ and $\varepsilon k = k^\alpha \varepsilon$ for some fixed $\alpha \in \mathrm{Aut}(K)$. This is a ring of *twisted dual numbers* over K. It is a local ring with $I = K\varepsilon$ the maximal ideal of all non-invertible elements. So $U' = I$ is a sub-bimodule of $U = R$, with $\mathrm{ann}(U') = I$, and on U' we have the induced faithful representation $(\varphi')_f$ of $R/I \cong K$ with $k\varepsilon \cdot a = ka^\alpha \varepsilon$. So each point of $U' \times U'$ is incident with a line of our projective model $\mathbb{P}(R) = \mathbb{P}(R)^\Phi$.

Now consider the bimodule $\widetilde{U} = R/U' \cong K$. The kernel of the induced representation $\widetilde{\varphi}$ is again I. As before, it is easily seen that each plane through $U' \times U'$ contains a line of $\mathbb{P}(R)$.

The relation $\not\triangle$ is an equivalence relation on $\mathbb{P}(R)$, because R is a local ring. Easy calculations show that elements of $\mathbb{P}(R)$ belong to the same equivalence class exactly if they meet $U' \times U'$ in the same point or, equivalently, if they together with $U' \times U'$ span the same plane. So there is a bijection β between the points of $U' \times U'$ and the planes through $U' \times U'$ such that for each $p \in \mathbb{P}(R)$ we have $p \subset (p \cap (U' \times U'))^\beta$. This bijection β is given by $K(k^\alpha \varepsilon, l^\alpha \varepsilon) \mapsto K(k, l) \oplus (U' \times U')$.

Moreover, one can compute that the projective model $\mathbb{P}(R)$ consists of *all* lines in $\mathbb{P}(K, U \times U)$ that meet $U' \times U'$ in a unique point, say q, and then lie in the plane q^β.

In case $\alpha = \mathrm{id}$ the bijection β is a projectivity. So then the set $\mathbb{P}(R)$ is a generalization of a *parabolic linear congruence*. The ring R is then the ordinary ring of dual numbers over K. In the general case β is only semilinearly induced. If $K = \mathbb{C}$ and α is the complex conjugation, then R is the ring of *Study's quaternions* (see [12], p.445).

Example 5.5 Let R be the ring of upper triangular 2×2-matrices with entries in K. Then $U = K^2$ is in a natural way a faithful (K, R)-bimodule. Moreover, $U' = K(0, 1)$ is a sub-bimodule with $\mathrm{ann}(U') = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in K\}$. So $R/\mathrm{ann}(U') \cong K$, and the induced faithful representation is the ordinary action of K on U'. This means that each point of $U' \times U'$ is on a line of the projective model $\mathbb{P}(R)^\Phi$.

Now consider $\widetilde{U} = U/U'$. The kernel of the induced action is $J = \{\left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right) \mid b, c \in K\}$. So $R/J \cong K$, and also here we have the ordinary action of K on $\widetilde{U} \cong K(1, 0)$. Hence each plane through $U' \times U'$ contains a line of $\mathbb{P}(R)^\Phi$.

Up to now, we are in the same situation as in Example 5.4. An easy calculation shows that the projective model $\mathbb{P}(R)^\Phi$ consists of *all* lines that meet $U' \times U'$ in a point. This is the generalization of a *special linear complex* to the not necessarily pappian case.

Example 5.6 Let $U = R = K[\varepsilon, \delta]$ with $\varepsilon \notin K$, $\delta \notin K[\varepsilon]$, ε, δ central, and $\varepsilon^2 = \delta^2 = \varepsilon\delta = 0$. The projective model $\mathbb{P}(R)^\Phi = \mathbb{P}(R)$ is a set of planes in 5-space.

The ring R is a local ring with maximal ideal $I = K\varepsilon + K\delta = U'$. Moreover, $\operatorname{ann}(U') = I$, and $R/I \cong K$ acts on U' componentwise. So according to 5.1 the induced model of $\mathbb{P}(K)$ in the 3-space $U' \times U'$ is a regulus $\mathcal{R}$.

Now consider $\widetilde{U} = R/U'$. Then $\ker(\widetilde{\varphi}) = I$, and we have the ordinary faithful action of K on $\widetilde{U} \cong K$. So all hyperplanes (4-spaces) through $U' \times U'$ contain an element of $\mathbb{P}(R)$.

As in Example 5.4 the elements of $\mathbb{P}(R)$ fall into equivalence classes with respect to $\not\triangle$, such that equivalent elements have the same intersection and the same join with $U' \times U'$. This yields a bijection β between the regulus $\mathcal{R}$ and the set of all hyperplanes through $U' \times U'$. As in 5.4, case $\alpha = \mathrm{id}$, this bijection is a projectivity. A calculation shows that $\mathbb{P}(R)$ consists of *all* planes that meet the 3-space $U' \times U'$ in an element of $\mathcal{R}$, say X, and then lie in the hyperplane X^β.

References

[1] W. Benz. *Vorlesungen über Geometrie der Algebren.* Springer, Berlin, 1973.

[2] A. Blunck. Regular spreads and chain geometries. *Bull. Belg. Math. Soc.*, 6:589–603, 1999.

[3] A. Blunck. Reguli and chains over skew fields. *Beiträge Algebra Geom.*, 41:7–21, 2000.

[4] A. Blunck and H. Havlicek. Extending the concept of chain geometry. *Geom. Dedicata* (to appear).

[5] A. Blunck and M. Stroppel. Klingenberg chain spaces. *Abh. Math. Sem. Univ. Hamburg*, 65:225–238, 1995.

[6] H. Brauner. *Geometrie projektiver Räume II.* BI–Wissenschaftsverlag, Mannheim, 1976.

[7] W. Burau. *Mehrdimensionale projektive und höhere Geometrie.* Dt. Verlag d. Wissenschaften, Berlin, 1961.

[8] J.C. Ferrar and F.D. Veldkamp. Neighbor-preserving homomorphisms between projective ring planes. *Geom. Dedicata*, 18:11–33, 1985.

[9] A.J. Hahn and O.T. O'Meara. *The Classical Groups and K-Theory.* Springer, Berlin, 1989.

[10] H. Havlicek. Spreads of right quadratic skew field extensions. *Geom. Dedicata*, 49:239–251, 1994.

[11] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry.* Elsevier, Amsterdam, 1995.

[12] H. Karzel and G. Kist. Kinematic algebras and their geometries. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry.* D. Reidel, Dordrecht, 1985.

[13] T.Y. Lam. *A First Course in Noncommutative Rings.* Springer, New York, 1991.

[14] J.R. Silvester. *Introduction to Algebraic K-Theory.* Chapman and Hall, London, 1981.

[15] F.D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry.* D. Reidel, Dordrecht, 1985.

[16] F.D. Veldkamp. Geometry over rings. In F. Buekenhout, editor, *Handbook of Incidence Geometry*. Elsevier, Amsterdam, 1995.

Institut für Geometrie
Technische Universität
Wiedner Hauptstraße 8–10
A–1040 Wien, Austria