Spheres of Quadratic Field Extensions

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1. Introduction

The concepts of the geometry of field extensions can be found in the book of W. BENZ [2]. There it is shown that the geometry arising from the real quaternions and the complex numbers has a point model, namely the 2-spheres on a euclidian 4-sphere in a real 5-dimensional euclidian space. This representation of the projective line over the quaternions as a 4-sphere was already known to E. STUDY [19]. In part IV of that paper it is shown how to link this 4-sphere with a manifold of lines in a complex projective 3-space by means of the KLEIN mapping[1].

In the present note we take up these ideas by constructing a point model for the geometry $\Sigma(K,L)$ which belongs to a quadratic extension L/K, where L is a proper skew field and K is a commutative field.

In a first step we represent the projective line over L in terms of a spread $\mathcal{S}_{L/K}$ in a 3-dimensional projective space $\mathcal{P}$ over K. Then we transfer this spread to the KLEIN quadric representing the lines of $\mathcal{P}$. This establishes immediately a point model of $\mathcal{S}_{L/K}$, say $\mathbf{S}_{L/K}$, in a 5-dimensional projective space $\hat{\mathcal{P}}$ over K. (If K would be a skew field too then the spread $\mathcal{S}_{L/K}$, but no such representation would exist; cf. [7].)

Although our approach is aiming at a point model in a projective space over K, it also yields a point model over the centre Z of L: There is a unique 5-dimensional BAER subspace Π of $\hat{\mathcal{P}}$ which contains $\mathbf{S}_{L/K}$; the underlying field of Π is Z. The set $\mathbf{S}_{L/K}$ is an elliptic quadric of Π ("4-sphere") which is equal to the intersection of Π with the KLEIN quadric.

In addition we shall characterize the images of the chains (K-sublines). It will also be proved that the automorphism group of $\Sigma(K,L)$ is represented by a group of automorphic collineations of $\mathbf{S}_{L/K}$. Both results depend on the embedding of Π in $\hat{\mathcal{P}}$, but the point model itself also may be seen in terms of Π alone. From this point of view it is a generalization of the classical result.

The skew field L is a 4-dimensional quadratic algebra over Z. H. HOTJE has shown how to construct a quadric model for the chain geometries arising from certain quadratic algebras. See [13], [14] or the survey article [10]. It will be shown that $\mathbf{S}_{L/K}$ equals the HOTJE model belonging to $\Sigma(Z,L)$.

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[1] STUDY points out explicitly the fact that this manifold is (in today's language) both a spread and a dual spread [19,IV,177].

We frequently shall make use of PLÜCKER coordinates and some results of projective line geometry which can be found e.g. in [4], [11], [12] or [17].

2. Spherical models

2.1. Let K be a commutative field and let L be a right quadratic extension field[2] of K. The centre of L will be denoted by Z.

If L is a skew field then K is a maximal commutative subfield of L. Hence K and the centralizer of K in L coincide so that $Z \subset K$. We infer

$$|L:K|_{\text{left}} = |K:Z|_{\text{left}} = |K:Z|_{\text{right}} = |L:K|_{\text{right}} = 2$$

from Corrollary 2 in [6,49]. Hence L is a 4-dimensional quadratic Z-algebra, i.e. L is a quaternion skew field [16,169-171]; cf. also chapter 13 in [20]. Let $\{1,a\}$ be a basis of K over Z. There exist $\lambda_1,\mu_1 \in Z$, $\mu_1 \neq 0$ such that[3]

$$a^2+a\lambda_1+\mu_1 = 0. \tag{$1_{1,2}$}$$

In choosing a basis of L over K we have to distinguish three cases:

Case 1. L is a skew field and K is a GALOIS extension of Z. Hence

$$(\bar{\ }) : K \to K,\ u = \xi+a\eta \ \mapsto \ \bar{u} = \xi-(\lambda_1+a)\eta,\ (\xi,\eta \in Z) \tag{2_1}$$

is an automorphism of order 2 which fixes Z elementwise. By the Skolem-Noether theorem (cf. the Corrollary in [6,46]) this automorphism $(\bar{\ })$ extends to an inner automorphism of L. Hence we can choose $i \in L\setminus K$ such that

$$i^{-1}ui = \bar{u} \text{ for all } u \in K. \tag{3_1}$$

Case 2. L is a skew field and K/Z is not GALOIS. Hence

$$\operatorname{Char} K = 2,\ \lambda_1 = 0, \tag{2_2}$$

since a has to be inseparable over Z. We infer from

$$(l+a^{-1}la)a = la+a^{-1}l\mu_1 = la+al = a(l+a^{-1}la) \text{ for all } l \in L$$

that $l+a^{-1}la \in K$. So there exists an $i \in L\setminus K$ such that

$$a^{-1}ia = i+1. \tag{3_2}$$

Case 3. L is commutative. Choose any $i \in L\setminus K$.

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[2]The term *field* is used for a *not necessarily commutative field* whereas *skew field* always means a *non-commutative field*. We do not exclude the case that L is commutative in order to illustrate how things will alter in the non-commutative case.

[3]We refer to the three cases described below in the numbering of formulae: e.g. (4) holds in all cases, but $(1_{1,2})$ in case 1 and case 2 only.

In every case there exist $\lambda_2,\mu_2 \in Z\cap K$, $\mu_2 \neq 0$ such that

$$i^2+i\lambda_2+\mu_2 = 0. \tag{4}$$

Multiplication in L is subject to the commutation rule

$$ui = iu^S+u^D \text{ for all } u \in K, \tag{5}$$

where S is an injective endomorphism of K and D is an S-derivation of K; cf. [5], [6,56]. But L is 4-dimensional over Z and S fixes Z elementwise, whence S is Z-linear and surjective.

Case 1. We claim that

$$S = (\bar{\ }),\ D = 0,\ \lambda_2 = 0. \tag{6_1}$$

By formulae (3_1) and (5), $i^{-1}ui = u^S+i^{-1}u^D \in K$ for all $u \in K$, whence D is a zero-derivation. Thus $\mathrm{id}_K \neq S = (\bar{\ })$. We infer from $S^2 = \mathrm{id}_K$ that $i^2 \in Z$, whence $\lambda_2 = 0$.

Case 2. We claim that

$$S = \mathrm{id}_K,\ (\xi+a\eta)^D = a\eta \text{ for all } \xi,\eta \in Z,\ \lambda_2 = 1. \tag{6_2}$$

The first assertion is obvious. By (3_2), $ai = ia+a$ so that $a^D = a$. Now the second assertion follows, because D is a Z-linear mapping with kernel Z. Finally, $\lambda_2 = 1$, because of (4), (3_2) and

$$(i+1)^2+(i+1)\lambda_2+\mu_2 = i^2+1+i\lambda_2+\lambda_2+\mu_2 = 0.$$

Case 3. Obviously

$$S = \mathrm{id}_K,\ D = 0. \tag{6_3}$$

2.2. Let $\mathcal{P}$ be the projective space on L^2 over K. The points of $\mathcal{P}$ are the 1-dimensional K-subspaces of L^2. The set of lines of $\mathcal{P}$ is denoted by $\mathcal{L}$. An ordered basis of L^2 over K is given by

$$\mathbf{b}_0 := (1,0),\ \mathbf{b}_1 := (i,0),\ \mathbf{b}_2 := (0,1),\ \mathbf{b}_3 := (0,i). \tag{7}$$

Write $\hat{\mathcal{P}}$ for the projective space on the vector space $L^2\wedge L^2$ over K. The family

$$(\mathbf{b}_0\wedge\mathbf{b}_1,\mathbf{b}_0\wedge\mathbf{b}_2,\mathbf{b}_0\wedge\mathbf{b}_3,\mathbf{b}_2\wedge\mathbf{b}_3,\mathbf{b}_1\wedge\mathbf{b}_3,\mathbf{b}_1\wedge\mathbf{b}_2) \tag{8}$$

is an ordered basis of $L^2\wedge L^2$. Coordinates always are understood with respect to bases (7) or (8). The KLEIN mapping

$$\sigma : \mathcal{L} \to \hat{\mathcal{P}},\ (\mathbf{c}K)\vee(\mathbf{d}K) \mapsto (\mathbf{c}\wedge\mathbf{d})K$$

is injective. Its image set is the KLEIN quadric, say Q, with equation

$$x_{01}x_{23}-x_{02}x_{13}+x_{03}x_{12} = 0.$$

The non-degenerate bilinear form

$$\Big(\sum_{i<j} x_{ij}(\mathbf{b}_i\wedge\mathbf{b}_j),\ \sum_{k<l} y_{kl}(\mathbf{b}_k\wedge\mathbf{b}_l)\Big)\ \mapsto\ x_{01}y_{23}+x_{23}y_{01}-x_{02}y_{13}-x_{13}y_{02}+x_{03}y_{12}+x_{12}y_{03}$$

describes the projective polarity $\perp$ which is associated to the KLEIN quadric. Recall that two lines of $\mathcal{L}$ have a common point if, and only if, their σ-images are conjugate with respect to $\perp$.

2.3. A partition of the group $(L^2,+)$ is given by

$$\{(1,l)L \mid l \in L\} \cup \{(0,1)L\}. \tag{9}$$

If L^2 is regarded as a right vector space over L then the projective line over L is obtained from (9); cf. [2,320]. If, however, L^2 is regarded as right vector space over K then (9) yields a spread of lines, say $\mathcal{S}_{L/K}$, in the projective space $\mathcal{P}$ defined in 2.2. We shall adopt this second point of view. Cf. also [8], [9].

Every 2-dimensional subspace $(1,l)L \subset L^2$ is spanned by vectors $(1,l)$, (i,li). Putting $l =: u+iv$ $(u,v \in K)$ gives

$$\begin{aligned}(1,l) &= \mathbf{b}_0+\mathbf{b}_2u+\mathbf{b}_3v,\\ (i,li) = \big(i,iu^S+u^D+i(iv^S+v^D)\big) &= \mathbf{b}_1+\mathbf{b}_2(u^D-\mu_2v^S)+\mathbf{b}_3(u^S-\lambda_2v^S+v^D).\end{aligned}$$

Write ω for the line arising from $(0,1)L$; the point ω^σ equals $(\mathbf{b}_2\wedge\mathbf{b}_3)K$. We obtain the following injective parametric representation of $(\mathcal{S}_{L/K}\setminus\{\omega\})^\sigma$:

$$(u,v)\ \mapsto\ \Big(\sum_{i<j} p_{ij}(\mathbf{b}_i\wedge\mathbf{b}_j)\Big)K$$

where the p_{ij}'s, as functions of $(u,v) \in K^2$, are given by

$$\begin{aligned}p_{01} &= 1, & p_{23} &= uu^S-\lambda_2uv^S+uv^D-u^Dv+\mu_2vv^S,\\ p_{02} &= u^D-\mu_2v^S, & p_{13} &= -v,\\ p_{03} &= u^S-\lambda_2v^S+v^D, & p_{12} &= -u.\end{aligned} \tag{10}$$

2.4. If L is commutative (case 3) then formula (10) simplifies to

$$\begin{aligned}p_{01} &= 1, & p_{23} &= u^2-\lambda_2uv+\mu_2v^2,\\ p_{02} &= -\mu_2v, & p_{13} &= -v,\\ p_{03} &= u-\lambda_2v, & p_{12} &= -u.\end{aligned} \tag{10_3}$$

It is well known that now $\mathcal{S}_{L/K}$ is an elliptic linear congruence of lines (regular spread) and that $\mathcal{S}_{L/K}{}^\sigma$ is an elliptic quadric which equals the intersection of the KLEIN quadric with a 3-dimensional subspace, say $\mathcal{T}$, of $\hat{\mathcal{P}}$. Cf. e.g. chapter 18.6.2 in [10]. By abuse of language we shall refer to $\mathbf{S}_{L/K} := \mathcal{T}\cap Q$ as a 2-*sphere*. The subspace $\mathcal{T}$ is given by equations

$$x_{03}-\lambda_2x_{13}+x_{12} = 0,\ x_{02}-\mu_2x_{13} = 0$$

and the line $\mathcal{T}^{\perp}$ is spanned by points with coordinates

$$\big(0,\lambda_2,1,0,0,1\big)^{\mathsf{T}},\ \big(0,\mu_2,0,0,-1,0\big)^{\mathsf{T}}.$$

Intersection of $\mathcal{T}^{\perp}$ with the KLEIN quadric yields an equation

$$X^2+\lambda_2 XY+\mu_2 Y^2 = 0 \tag{11_3}$$

and putting $Y = 1$ in (11_3) brings back the minimal equation (4) of i.

If the ground field of $\mathcal{P}$ and $\hat{\mathcal{P}}$ is extended from K to L then we get projective spaces $\mathcal{P}_{(L)}$ and $\hat{\mathcal{P}}_{(L)}$ on the vector spaces $L^2 \otimes_K L$ over L and $(L^2 \wedge L^2) \otimes_K L$ over L, respectively. But $(L^2 \wedge L^2) \otimes_K L$ is easily seen to be canonically isomorphic to $(L^2 \otimes_K L) \wedge (L^2 \otimes_K L)$ so that the extended KLEIN quadric $Q_{(L)}$ may be identified with the KLEIN quadric representing the lines of $\mathcal{P}_{(L)}$. The solutions of equation (11_3) over L yield the common points of the extended line $(\mathcal{T}^{\perp})_{(L)}$ with $Q_{(L)}$. These points correspond to those lines in $\mathcal{P}_{(L)}$ which intersect all extended lines of the spread $\mathcal{S}_{L/K}$. Hence, by (11_3) and (4), the number of such lines is two or one if L/K is GALOIS or not, respectively. Cf. [3], [10] for different proofs of this. But it is also easy to see if L/K is GALOIS or not without extending the ground field:

Lemma 1. *Let L/K be a quadratic field extension, where both L and K are commutative, and let $\mathbf{S}_{L/K}$ be the image of the spread $\mathcal{S}_{L/K}$ under the Klein mapping. The following assertions are equivalent:*
(a) L/K is Galois.
(b) The intersection over all tangent planes of the 2-sphere $\mathbf{S}_{L/K}$ is empty.

Proof. The intersection over all tangent planes of $\mathbf{S}_{L/K}$ equals $\mathcal{T}\cap\mathcal{T}^{\perp}$, since $\mathbf{S}_{L/K}$ is spanning $\mathcal{T}$. Calculating $\mathcal{T}\cap\mathcal{T}^{\perp}$ is equivalent to solving the linear homogeneous system

$$2X+\lambda_2 Y = 0,\ \lambda_2 X+2\mu_2 Y = 0 \tag{12_3}$$

with determinant $\Delta = 4\mu_2-\lambda_2{}^2$. If Char$K \neq 2$, then both (a) and (b) hold true. However, when Char$K = 2$ then (12_3) has only the trivial solution if, and only if, $\lambda_2 \neq 0$, i.e. L/K is GALOIS. ■

If L/K is not GALOIS then $\mathcal{T}^{\perp} \subset \mathcal{T}$, whence the knot of the sphere $\mathbf{S}_{L/K}$ is a line. Cf. chapter 23.2.C in [18].

2.5. Now cases 1 and 2 will be discussed. We shall use the following concept: Let $\mathbf{V}$ be a vector space over K and let $\mathbf{a}_0,\dots,\mathbf{a}_r \in \mathbf{V}$ be linearly independent vectors. Then an *r-dimensional Z-subspace* of the projective space on $\mathbf{V}$ is given as the set of all points $\mathbf{p}K$, where $\mathbf{p} \in \mathbf{V}\setminus\{\mathbf{o}\}$ is a linear combination of $\mathbf{a}_0,\dots,\mathbf{a}_r$ with coefficients in Z. Such an r-dimensional Z-subspace is uniquely determined by $r+2$ of its points which are in general position.

Theorem 1. *Let L/K be a quadratic field extension, where L is non-commutative and K is commutative. The image of the spread $\mathcal{S}_{L/K}$ under the Klein map $\sigma : \mathcal{L} \to \hat{\mathcal{P}}$ equals the intersection of the Klein quadric with a 5-dimensional Z-subspace of $\hat{\mathcal{P}}$, say Π. This intersection is an elliptic quadric with respect to Π.*

Proof. We substitute in formula (10) by putting

$$u =: \alpha+a\beta, \; v =: \gamma+a\delta \text{ with } (\alpha,\beta,\gamma,\delta) \in Z^4.$$

Case 1. By $a^S = \overline{a} = -(\lambda_1+a)$ and $a^2 = -(a\lambda_1+\mu_1)$ formula (10) becomes

$$\begin{aligned} p_{01} &= 1, & p_{23} &= \alpha^2-\lambda_1\alpha\beta+\mu_1\beta^2+\mu_2(\gamma^2-\lambda_1\gamma\delta+\mu_1\delta^2), \\ p_{02} &= -\mu_2(\gamma+\overline{a}\delta), & p_{13} &= -\gamma-a\delta, \\ p_{03} &= \alpha+\overline{a}\beta, & p_{12} &= -\alpha-a\beta. \end{aligned} \tag{10_1}$$

We apply the collineation $\kappa : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ given by the regular matrix

$$(a-\overline{a})^{-1}\begin{pmatrix} a-\overline{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & \overline{a} \\ 0 & -a\mu_2^{-1} & 0 & 0 & \overline{a} & 0 \\ 0 & 0 & 0 & a-\overline{a} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & \mu_2^{-1} & 0 & 0 & -1 & 0 \end{pmatrix}$$

and obtain as parametric representation of $(\mathcal{S}_{L/K}\setminus\{\omega\})^{\sigma\kappa}$

$$\begin{aligned} p_{01} &= 1, & p_{23} &= \alpha^2-\lambda_1\alpha\beta+\mu_1\beta^2+\mu_2(\gamma^2-\lambda_1\gamma\delta+\mu_1\delta^2), \\ p_{02} &= \alpha, & p_{13} &= \beta, \\ p_{03} &= \gamma, & p_{12} &= \delta. \end{aligned} \tag{13_1}$$

Case 2. Here $a^2 = \mu_1$ turns (10) into

$$\begin{aligned} p_{01} &= 1, & p_{23} &= \alpha^2+\alpha\gamma+\mu_1\beta^2+\mu_1\beta\delta+\mu_2(\gamma^2+\mu_1\delta^2), \\ p_{02} &= \mu_2\gamma+a(\beta+\mu_2\delta), & p_{13} &= \gamma+a\delta, \\ p_{03} &= \alpha+\gamma+a\beta, & p_{12} &= \alpha+a\beta \end{aligned} \tag{10_2}$$

and the collineation $\kappa : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ given by the regular matrix

$$a^{-1}\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & a\mu_2 & a \\ 0 & 0 & a & 0 & 0 & a \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

yields a parametric representation of $(\mathcal{S}_{L/K}\setminus\{\omega\})^{\sigma\kappa}$

$$\begin{aligned} p_{01} &= 1, & p_{23} &= \alpha^2+\alpha\gamma+\mu_1\beta^2+\mu_1\beta\delta+\mu_2(\gamma^2+\mu_1\delta^2), \\ p_{02} &= \alpha, & p_{13} &= \beta, \\ p_{03} &= \gamma, & p_{12} &= \delta. \end{aligned} \tag{13_2}$$

There exists a unique 5-dimensional Z-subspace Π of $\hat{\mathcal{P}}$ such that Π^κ is determined by the basis (8). It is immediate from the stereographic parametrization (13_1) or (13_2) that $(\mathcal{S}_{L/K}\setminus\{\omega\})^{\sigma\kappa}$ is a paraboloid of Π^κ when $p_{01} = 0$ is regarded as equation of an hyperplane at infinity. This paraboloid is elliptic, since $\mathcal{S}_{L/K}$ is a spread. By means of (13_1) and (13_2) we extend this paraboloid to a regular projective quadric of Π^κ with equation

$$x_{02}{}^2 - x_{02}x_{13}\lambda_1 + x_{13}{}^2\mu_1 + \mu_2(x_{12}{}^2\mu_1 - x_{12}x_{03}\lambda_1 + x_{03}{}^2) - x_{01}x_{23} = 0 \qquad (14_1)$$

and

$$x_{02}{}^2 + x_{02}x_{03} + x_{13}{}^2\mu_1 + x_{13}x_{12}\mu_1 + x_{12}{}^2\mu_1\mu_2 + x_{03}{}^2\mu_2 + x_{01}x_{23} = 0, \qquad (14_2)$$

respectively; the image of this projective quadric under κ^{-1} will be written as $\mathbf{S}_{L/K}$. The quadric $\mathbf{S}_{L/K}{}^\kappa \subset \Pi^\kappa$ is elliptic, its affine points are given by (13_1) and (13_2), respectively, and $\omega^{\sigma\kappa}$ is its only point at infinity. Therefore $\mathbf{S}_{L/K} = \mathcal{S}_{L/K}{}^\sigma$. Obviously $\mathbf{S}_{L/K} \subset Q\cap\Pi$. On the other hand equations (14_1) and (14_2) describe, respectively, a quadric of $\hat{\mathcal{P}}$. A lengthy, but straightforward calculation shows that this is equal to Q^κ. Thus $\mathbf{S}_{L/K} = Q\cap\Pi$, as required. ∎

The centre of the skew field L is infinite, since $|L:Z| = 4$. Therefore $\mathbf{S}_{L/K}$ contains 7 points in general position and Π, being a BAER subspace of $\hat{\mathcal{P}}$, is uniquely determined. By abuse of language we shall refer to $\mathbf{S}_{L/K}$ as a 4-*sphere* of Π. The polar system which is determined by the 4-sphere $\mathbf{S}_{L/K}$ is always regular, since it is the restriction of $\perp$ to the lattice of subspaces of Π. This is a major difference to the case when L is commutative; cf. Lemma 1.

3. Chains and Traces

3.1. The field extension L/K gives rise to an incidence structure $\Sigma(K,L)$ which is formed by the points of the projective line on L^2 (over L) and whose blocks, called *chains*, are the K-sublines. Moreover we shall be concerned with *traces*, i.e. Z-sublines, if L is a skew field. See [2,326]. However, in contrast to [2], L^2 will again be regarded as vector space over K, so that the spread $\mathcal{S}_{L/K}$ is endowed with subsets called chains and traces, respectively.

3.2. If L is commutative then the chains are exactly those reguli which are contained in $\mathcal{S}_{L/K}$. Hence the σ-images of the chains are precisely the non-degenerate conics (*circles*, 1-*spheres*) contained in $\mathbf{S}_{L/K}$. This is well known; see e.g. [10].

3.3. For every chain $\mathcal{C}$ in $\mathcal{S}_{L/K}$ there is at least one *transversal line* t of $\mathcal{C}$ even if L is a skew field. This means that the mapping

$$\ell\ (\in \mathcal{C})\ \mapsto \ell\cap t\ (\in t)$$

is a bijection of $\mathcal{C}$ onto t; cf. [8]. On the other hand, given a line $t \notin \mathcal{S}_{L/K}$ then the set of all lines of $\mathcal{S}_{L/K}$ which have a point in common with t is a chain with t being one of its transversal lines. All chains are in one orbit with respect to the group of automorphic projective collineations of the spread $\mathcal{S}_{L/K}$. So it is sufficient to discuss the *standard chain* $\mathcal{K}$, say, which is given by the transversal line $\mathbf{b}_0K\vee\mathbf{b}_2K$.

Theorem 2. *Let L/K be a quadratic field extension, where L is non-commutative and K is commutative, and let $\mathcal{C}$ be a chain of $\mathcal{S}_{L/K}$. There exists a unique linear congruence of lines, say $\mathcal{M}$, such that $\mathcal{C} = \mathcal{S}_{L/K}\cap\mathcal{M}$. This congruence of lines is either hyperbolic or parabolic. The field K is a Galois extension of the centre of L if, and only if, $\mathcal{M}$ is hyperbolic.*

Proof. Let $\mathcal{C} = \mathcal{K}$ be the standard chain. We obtain a parametric representation of $(\mathcal{K}\setminus\{\omega\})^\sigma$ by putting $\gamma = \delta = 0$ in formulae (10_1) and (10_2), respectively.

Case 1. We read off from formula (10_1) that $\mathcal{K}^\sigma$ is contained in the 3-dimensional subspace $\mathcal{X} \subset \hat{\mathcal{P}}$ given by the system

$$x_{02} = 0,\ x_{13} = 0$$

and we infer from formula (13_1) that $\mathcal{K}^{\sigma\kappa}$ is spanning a three dimensional subspace of $\hat{\mathcal{P}}$. Hence there is a unique linear congruence of lines, say $\mathcal{M}$, given by $\mathcal{M}^\sigma = \mathcal{X}\cap Q$, such that $\mathcal{C} \subset \mathcal{M}$. The congruence $\mathcal{M}$ is hyperbolic, since $\mathcal{X}^\perp$ is a secant of Q which is spanned by points with coordinates

$$\big(0,0,0,0,1,0\big)^{\mathsf{T}} \text{ and } \big(0,1,0,0,0,0\big)^{\mathsf{T}},$$

respectively. These two points are the σ-images of the axes of the congruence $\mathcal{M}$. These axes coincide with the only two transversal lines of $\mathcal{K}$ (cf. [8]). Note that $\mathcal{X}\cap Q$ is a hyperbolic (doubly ruled) quadric of $\mathcal{X}$.

Case 2. Repeat the argumentation of case 1: However now the subspace $\mathcal{X}$ is given by the system

$$x_{03}+x_{12} = 0,\ x_{13} = 0$$

and the congruence $\mathcal{M}$ is parabolic, since $\mathcal{X}^\perp$ is a tangent line of Q which is spanned by points with coordinates

$$\big(0,0,1,0,0,1\big)^{\mathsf{T}} \text{ and } \big(0,1,0,0,0,0\big)^{\mathsf{T}},$$

respectively. The second point is the σ-image of the axis of the congruence $\mathcal{M}$. This axis is the only transversal line of $\mathcal{K}$ (cf. [8]). Note that $\mathcal{X}^\perp \subset \mathcal{X}$ and that $\mathcal{X}\cap Q$ is a quadratic cone of $\mathcal{X}$ whose tangent planes belong to the pencil of planes in $\mathcal{X}$ with axis $\mathcal{X}^\perp$. ∎

Theorem 3. *Let L/K be a quadratic field extension, where L is non-commutative and K is commutative, and let $\mathbf{S}_{L/K}$ be the image of the spread $\mathcal{S}_{L/K}$ under the Klein mapping σ. A subset* $\mathbf{C}$ of $\mathbf{S}_{L/K}$ *is the σ-image of a chain $\mathcal{C} \subset \mathcal{S}_{L/K}$ if, and only if, there exists a 3-dimensional subspace $\mathcal{X}$ of $\hat{\mathcal{P}}$ with the following properties:*

1. *$\mathcal{X}\cap\Pi$ is a 3-dimensional subspace of Π and $\mathbf{C} = \mathcal{X}\cap\mathbf{S}_{L/K}$ is an elliptic quadric of the subspace $\mathcal{X}\cap\Pi$ (over Z).*
2. *$\mathcal{X}\cap Q$ contains a line of $\hat{\mathcal{P}}$ (over K).*

Proof. (a) Let $\mathcal{K}$ be the standard chain and define $\mathcal{X}$ as in Theorem 2. Then, by (10_1) and (10_2), putting $\gamma = \delta = 0$, and application of κ^{-1}

$$\mathcal{K}^\sigma = (\mathcal{X}\cap\Pi)\cap\mathbf{S}_{L/K} = \mathcal{X}\cap\mathbf{S}_{L/K} = (\mathcal{X}\cap\Pi)\cap Q$$

is an elliptic quadric of a 3-dimensional subspace of the Z-subspace Π. It was shown in Theorem 2 that $\mathcal{X}\cap Q$ is either a hyperbolic quadric (case 1) or a quadratic cone (case 2), whence $\mathcal{X}\cap Q$ contains a line of $\hat{\mathcal{P}}$.

(b) Suppose that $\mathcal{X}$ is satisfying conditions 1 and 2. Then define

$$\mathcal{C} \subset \mathcal{S}_{L/K} \text{ by } \mathcal{C}^\sigma = \mathcal{X}\cap\mathbf{S}_{L/K} = \mathbf{C} \text{ and } \mathcal{M} \subset \mathcal{L} \text{ by } \mathcal{M}^\sigma = \mathcal{X}\cap Q.$$

Hence $\mathcal{X}\cap Q$ is a ruled quadric in $\mathcal{X}$. However $\mathcal{X}\cap\mathbf{S}_{L/K}$ contains at least 5 points in general position with respect to $\Pi\cap\mathcal{X}$ and these points are also in general position with respect to $\mathcal{X}$. Thus $\mathcal{X}\cap Q$ is either a hyperbolic quadric or a quadratic cone and $\mathcal{M}$ is either a hyperbolic or parabolic linear congruence of lines, respectively. Denote by $t \in \mathcal{L}$ an axis of this congruence, whence

$$T := t^\sigma \in \mathcal{X}^\perp\cap Q.$$

Moreover $t \notin \mathcal{S}_{L/K}$, since no line of $\mathcal{C}$ is skew to t, but all lines of $\mathcal{S}_{L/K}$ are mutually skew. We deduce from $T \notin \mathbf{S}_{L/K} = Q\cap\Pi$ and $T \in Q$ that $T \notin \Pi$.

The set of all lines of the spread $\mathcal{S}_{L/K}$ which are intersecting t is a chain $\mathcal{C}'$ and

$$\mathcal{C}'^\sigma = T^\perp\cap\mathbf{S}_{L/K} = T^\perp\cap\Pi\cap Q.$$

Clearly $T^\perp\cap\Pi$ is a subspace of Π. If $T^\perp\cap\Pi$ would be a hyperplane relative to Π then T would also be in Π, since the polarity determined by $\mathbf{S}_{L/K}$ is induced by $\perp$ and it would take $T^\perp\cap\Pi$ back to $T^{\perp\perp} = T \in \Pi$, an absurdity. On the other hand $T^\perp\cap\Pi$ contains $\mathbf{C}$ so that $\mathcal{X}\cap\Pi = T^\perp\cap\Pi$. To sum up we have shown that

$$\mathcal{C}^\sigma = \mathbf{C} = \mathcal{X}\cap\Pi\cap Q = T^\perp\cap\Pi\cap Q = \mathcal{C}'^\sigma$$

which establishes that $\mathcal{C}$ is a chain. ■

There may be 3-dimensional subspaces $\mathcal{Y} \subset \hat{\mathcal{P}}$ such that both $\mathcal{Y}\cap\mathbf{S}_{L/K}$ and $\mathcal{Y}\cap Q$ are elliptic quadrics[4].

3.4. Let L be a skew field. The incidence structure which is formed by a fixed chain $\mathcal{C}$ and the set of all traces which are a part of $\mathcal{C}$ is isomorphic to the chain geometry $\Sigma(Z,K)$. So it is natural to ask if there is a 3-dimensional Z-subspace of $\mathcal{P}$ such that the chain $\mathcal{C}$ and the traces within $\mathcal{C}$, when restricted to this Z-subspace, represent - up to a collineation - the spread $\mathcal{S}_{K/Z}$ and its chains (Z-sublines), respectively. The answer is that such a Z-subspace exists, but it is by no means uniquely determined.

We illustrate this for the standard chain $\mathcal{K} \subset \mathcal{S}_{L/K}$. Denote by Ψ the Z-subspace of $\mathcal{P}$ determined by the basis (7). We shall show that there exists a collineation $\rho \in \mathrm{PGL}(\mathcal{P})$ such that $\mathcal{K}$ and $\Psi^{\rho^{-1}}$ have the required property[5]: Define ρ as the projective collineation of $\mathcal{P}$ given in cases 1 and 2 by the matrices

$$\begin{pmatrix} -\bar{a} & -a & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -\bar{a} & -a \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} a & a & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a & a \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. Then the corresponding automorphic collineation of the Klein quadric is described by one of the 6×6 matrices written down below and $\mathcal{K}^{\rho\sigma}\setminus\{\omega^\sigma\}$ has the parametric representations

$$\begin{pmatrix} a-\bar{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a}^2 & a\bar{a} & 0 & a^2 & a\bar{a} \\ 0 & -\bar{a} & -\bar{a} & 0 & -a & -a \\ 0 & 0 & 0 & a-\bar{a} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -\bar{a} & -a & 0 & -a & -\bar{a} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \alpha-\lambda_1\beta-a\beta \\ \alpha^2-\lambda_1\alpha\beta+\mu_1\beta^2 \\ 0 \\ -\alpha-a\beta \end{pmatrix} = (a-\bar{a}) \begin{pmatrix} 1 \\ -\mu_1\beta \\ \alpha-\lambda_1\beta \\ \alpha^2-\lambda_1\alpha\beta+\mu_1\beta^2 \\ -\beta \\ -\alpha \end{pmatrix} \qquad (15_1)$$

and

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & \mu_1 & 0 & \mu_1 & \mu_1 \\ 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & a & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a\beta \\ \alpha+a\beta \\ \alpha^2+\mu_1\beta^2 \\ 0 \\ a\beta+\alpha \end{pmatrix} = a \begin{pmatrix} 1 \\ \mu_1\beta \\ \alpha \\ \alpha^2+\mu_1\beta^2 \\ \beta \\ \alpha \end{pmatrix}, \qquad (15_2)$$

respectively, with $\alpha,\beta \in Z$. Both (15_1) and (15_2) are in accordance with (10_3) when replacing $\alpha,\beta,\lambda_1,\mu_1$ by u,v,λ_2,μ_2 and ruling out the non-zero factors $(a-\bar{a})$ and a, respectively. Applying ρ^{-1} establishes the result.

In order to see that $\Psi^{\rho^{-1}}$ is not uniquely determined by $\mathcal{K}$, we may transform the standard chain under any of the collineations κ_c ($c \in K\setminus\{0\}$) which, de-

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[4]There need not be a chain $\Sigma(K,L)$ which contains a given trace and a given point off that trace; cf. however condition (∗) in [2,334].

[5]$\mathcal{K}$ yields a spread of Ψ which belongs to $\Sigma(Z,Z(i))$ rather than $\Sigma(Z,K)$.

pending on the two cases, are given by the matrices

$$\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix},$$

respectively. It is easily seen that $\mathcal{K}$, but not $\Psi^{\rho^{-1}}$, is left invariant under all collineations κ_c, since K is infinite.

Now it is immediate that the σ-images of traces within $\mathcal{S}_{L/K}$ are exactly the 1-spheres on $\mathbf{S}_{L/K}$ and that Lemma 1 can be applied on the σ-image of every chain. This gives a geometric characterization of case 1 and case 2 in terms of the σ-image of any chain.

We infer from (13_1), (13_2), (14_1) and (14_2) that $\mathbf{S}_{L/K}$ is identical with the Hotje model of the chain geometry $\Sigma(Z,L)$, because the quadratic form appearing at the coordinate p_{23} is just the norm of $(\alpha+a\beta)+i(\gamma+a\delta) \in L$; cf. [13], [14].

4. Automorphisms

4.1. We restrict our attention to cases 1 and 2, i.e. L is a skew field. Every automorphic collineation of the Klein quadric which leaves $\mathbf{S}_{L/K}$ invariant maps 2-spheres onto 2-spheres. This may be reversed as follows[6] (cf. also [21]):

Theorem 4. *Let $\varphi : \mathbf{S}_{L/K} \to \mathbf{S}_{L/K}$ be a bijection such that both φ and φ^{-1} map 2-spheres onto 2-spheres. Then φ extends to a collineation $\psi : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ which leaves the Klein quadric invariant.*

Proof. We can go back from $\mathbf{S}_{L/K}$ to the projective line over L via the Klein map and the spread $\mathcal{S}_{L/K}$. Hence φ gives rise to an automorphism of $\Sigma(K,L)$. By [1], [2,343] and [15], every automorphism of $\Sigma(K,L)$ is induced by a product of bijections $f : L^2 \to L^2$ of the following three types:

1. f is an L-linear mapping.
2. $(l_0,l_1)^f = (l_0{}^J,l_1{}^J)$, $J \in \mathrm{Aut}(L)$ and $K^J = K$.
3. $(l_0,l_1)^f = \big((l_1{}^{-1})^J,(l_0{}^{-1})^J\big)$, J an antiautomorphism of L and $K^J = K$.

Mappings of first and second type are semilinear bijections of the vector space L^2 over K, whence we obtain corresponding automorphic collineations of the spread $\mathcal{S}_{L/K}$. When f is of third type then define

$$\tau : L^2 \times L^2 \to L,\ \big((l_0,l_1),(m_0,m_1)\big) \mapsto -l_1{}^J m_0 + l_0{}^J m_1.$$

This τ is a non-degenerate sesquilinear form on L^2 (over L or K). Moreover

[6]The assumptions of Theorem 4 may be weakened by virtue of "Satz 2" in [15].

$$\big((l_0,l_1),(m_0,m_1)\big)^\tau = 0 \iff (m_0,m_1) \in \big((l_1{}^{-1})^J,(l_0{}^{-1})^J\big)L.$$

Thus τ gives rise to a duality of $\mathcal{P}$ which leaves $\mathcal{S}_{L/K}$ invariant and transforms the lines of the spread in the required way. But every automorphic collineation or automorphic duality of $\mathcal{S}_{L/K}$ induces an automorphic collineation of the KLEIN quadric which leaves $\mathbf{S}_{L/K}$ invariant. So there exists a collineation ψ with the required properties. ■

4.2. We did not assert the uniqueness of ψ in Theorem 4. On the other hand $\psi|\Pi$ is uniquely determined by φ, since the identity mapping of Π is the only collineation extending the identity of $\mathbf{S}_{L/K}$. In case 1, by (3_1), the mapping

$$h : L^2 \to L^2,\ (l_0,l_1) \mapsto (l_0 i, l_1 i)$$

is a semilinear bijection of L^2 over K with respect to $(\bar{\ })\in \mathrm{Aut}(K)$. This h yields a non-trivial collineation of $\mathcal{P}$ which fixes every line of the spread $\mathcal{S}_{L/K}$, but no other line, no point and no plane of $\mathcal{P}$. The σ-transform of this collineation is a BAER involution of $\hat{\mathcal{P}}$ fixing Π elementwise; cf. also [19,IV,177]. In case 2, ψ is unique, since K/Z is not GALOIS.

4.3. Suppose that L is arbitrary. We close with the following

Corrollary. *Let L/K be a quadratic field extension, where K is commutative. The spread $\mathcal{S}_{L/K}$ admits a symplectic polarity fixing every line of this spread if, and only if, L is commutative.*

Proof. The σ-transforms of symplectic polarities of $\mathcal{P}$ (regarded as transformations on the set of lines) are exactly the involutory automorphic perspective collineations of the KLEIN quadric. When L is commutative then $\mathbf{S}_{L/K}$ spans a 3-dimensional subspace of $\mathcal{T}$ and therefore admits such a perspective collineation fixing $\mathbf{S}_{L/K}$ pointwise. When L is non-commutative then $\mathbf{S}_{L/K}$ is spanning $\hat{\mathcal{P}}$ and no such perspective collineation exists. ■

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