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SPREADS OF RIGHT QUADRATIC SKEW FIELD EXTENSIONS

ABSTRACT. Let L/K be a right quadratic (skew) field extension and let $\tilde{\mathcal{P}}$ be a 3-dimensional projective space over K which is embedded in a 3-dimensional projective space $\mathcal{P}$ over L. Moreover let $\mathcal{I}$ be a line of $\mathcal{P}$ which carries no point of $\tilde{\mathcal{P}}$. The main result is that - even when L or K is a skew field - the following holds true: A desarguesian spread of $\tilde{\mathcal{P}}$ is given by the set of all lines of $\tilde{\mathcal{P}}$ which are indicated by the points of $\mathcal{I}$. A spread of $\tilde{\mathcal{P}}$ arises in this way if, and only if, there exists an isomorphism of L onto the kernel of the spread such that K is elementwise invariant. Furthermore a geometric characterization of right quadratic extensions with a left degree other than two and of quadratic Galois extensions is given.

1. INTRODUCTION

The complex numbers $\mathbb{C}$ and the real numbers $\mathbb{R}$ form the classical example of a quadratic field extension. There are various geometric interpretations, but we shall only be concerned with the following ones:

Firstly, we may think of $\mathbb{C}$ as being a euclidian plane over $\mathbb{R}$. Here multiplication in $\mathbb{C}$ gives rise to the group of dilative rotations fixing 0.

Secondly, we may take the projective line over $\mathbb{C}$, viz. the projective space $\mathcal{P}(\mathbb{C}^2)$. Putting $\mathbb{C}^2 = \mathbb{R}^4$ results in a map of $\mathcal{P}(\mathbb{C}^2)$ (as a set of points) into the set of lines of the 3-dimensional projective space $\mathcal{P}(\mathbb{R}^4)$. The image set of this map is an elliptic linear congruence of lines or, in other words, a regular spread.

But there is one more feature, namely complexification of real spaces: This turns the euclidian plane $\mathbb{C}$ (over $\mathbb{R}$) into a minkowskian plane (over $\mathbb{C}$) with two isotropic directions which are invariant under all dilative rotations and pointing to the absolute circular points on the line at infinity. Every elliptic linear congruence of lines in $\mathcal{P}(\mathbb{R}^4)$ extends to a subset of a hyperbolic linear congruence in $\mathcal{P}(\mathbb{C}^4)$ with two skew conjugate imaginary focal lines. On the other hand the set of all real lines of $\mathcal{P}(\mathbb{C}^4)$ which intersect a fixed line $\mathcal{I}$, without any real point, determines an elliptic linear congruence of lines in $\mathcal{P}(\mathbb{R}^4)$.

These interpretations are well known. Cf. e.g. [20,205 f], [24,119 ff], [24,281 ff], [25]. For further references and historical remarks see [2], [10], [25].

In this paper L/K is an arbitrary field[1] extension of right degree two. Firstly, we investigate the affine plane given by the right vector space L over K. Left multiplication in L ($x \mapsto ax$) is described via 2 by 2 matrices with entries in K. The crucial result is a characterization of these matrices in terms of a common eigenvector in $L^2 \supset K^2$.

Secondly, the points of the projective line over L [3,320 ff], i.e. the 1-dimensional subspaces of the right vector space L^2 over L, yield the lines of a spread in the 3-dimensional projective space on the right vector space L^2 over K. (Cf. also [12]). As is known from the theory of translation planes, the kernel of this spread is isomorphic[2] to L [19,6].

We establish a geometric construction of such spreads. Every 3-dimensional projective space $\tilde{\mathcal{P}}$ over K can be embedded in a 3-dimensional projective space $\mathcal{P}$ over L. It is easily seen that there exists at least one line $\mathcal{I}$ of $\mathcal{P}$ such that $\mathcal{I} \cap \tilde{\mathcal{P}} = \emptyset$. Every line $\mathcal{I} \subset \mathcal{P}$ with this property determines a spread $\tilde{\mathcal{G}}(\mathcal{I})$ of $\tilde{\mathcal{P}}$ as follows: The spread is formed by all lines $\tilde{\ell}$ of $\tilde{\mathcal{P}}$ which are indicated by some point X of $\mathcal{I}$; this means that X belongs to the extended line ℓ which is spanned by $\tilde{\ell} \subset \tilde{\mathcal{P}}$ within $\mathcal{P}$. It turns out that a spread of $\tilde{\mathcal{P}}$ arises in this way if, and only if, there exists an isomorphism of L onto the kernel of the spread with the property that, after a natural identification, every element of K is left fixed. However, in contrast to the classical case, only the points of some subline $\hat{\mathcal{I}}$ of $\mathcal{I}$ actually indicate lines of $\tilde{\mathcal{G}}(\mathcal{I})$. It is shown that the left degree of L/K is greater than two if, and only if, $\tilde{\mathcal{G}}(\mathcal{I})$ is not a dual spread of $\tilde{\mathcal{P}}$. This in turn is equivalent to $\mathcal{I} \neq \hat{\mathcal{I}}$. As a corollary to an algebraic result (existence of field extensions with right degree two but different left degree) we obtain the existence of desarguesian spreads which are not dual spreads. Finally, we prove that there are at least two different lines which indicate, respectively, the same spread $\tilde{\mathcal{G}}(\mathcal{I})$ if, and only if, L/K is a Galois extension.

Similar results on spreads arising from field extensions L/K, where L is a commutative or even finite field and L/K is a finite or even Galois extension, have been established by various authors. Cf. [2], [4], [5], [6], [15], [16], [17], [18], [21], [22], [23].

2. The plane given by L/K

Let K be a field and L an extension field of K. We assume throughout this

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[1]The term field is to mean a not necessarily commutative field.

[2]Some authors use reverse multiplication and obtain an antiisomorphic field.

paper that L is right quadratic over K, i.e.

$$|L:K|_{\text{right}} = 2.$$

The left degree of L over K may be different from 2; cf. section 4.

Take any element $i \in L\backslash K$. Then $\{1,i\}$ is a basis of the right vector space L over K. Every element of L can be written uniquely in the form $\alpha+i\beta$ with $\alpha,\beta \in K$. Hence there are $\lambda,\mu \in K$ such that

$$i^2+i\lambda+\mu = 0,\ \mu \neq 0, \tag{1}$$

and there exist mappings $S:K \to K$ and $D:K \to K$ such that

$$\alpha i = i\alpha^S+\alpha^D \text{ for all } \alpha \in K.$$

This S is an injective endomorphism of K and D is an S-derivation of K, i.e.

$$(\alpha+\beta)^D = \alpha^D+\beta^D \text{ and } (\alpha\beta)^D = \alpha^D\beta^S+\alpha\beta^D \text{ for all } \alpha,\beta \in K.$$

Cf. [8], [9,56]. We may identify elements $\alpha+i\beta \in L$ with columns $(\alpha,\beta)^{\mathsf{T}} \in K^2$ thus turning K^2 into field $(K^2,+,\circ)$. Defining matrices

$$M_{\alpha\beta} := \begin{pmatrix} \alpha & \alpha^D-\mu\beta^S \\ \beta & \alpha^S-\lambda\beta^S+\beta^D \end{pmatrix} \quad (\alpha,\beta \in K) \tag{2}$$

yields

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \circ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = M_{\alpha\beta} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{for all } \alpha,\beta,\xi,\eta \in K. \tag{3}$$

This generalizes the elementary concept of dilative rotations.

THEOREM 1. *The matrices $M_{\alpha\beta}$, given by (2), are characterized within the set of 2×2 matrices over K by the property that the column $(-i,1)^{\mathsf{T}} \in L^2$ is an eigenvector.*

Proof. We read off from

$$\begin{pmatrix} \alpha & \alpha^D-\mu\beta^S \\ \beta & \alpha^S-\lambda\beta^S+\beta^D \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha i+\alpha^D-\mu\beta^S \\ -\beta i+\alpha^S-\lambda\beta^S+\beta^D \end{pmatrix} \tag{4}$$

and

$$\begin{aligned} -i(-\beta i+\alpha^S-\lambda\beta^S+\beta^D) &= i^2\beta^S+i\beta^D-i\alpha^S+i\lambda\beta^S-i\beta^D \\ &= -i\lambda\beta^S-\mu\beta^S-i\alpha^S+i\lambda\beta^S \\ &= -\mu\beta^S-\alpha i+\alpha^D \end{aligned}$$

that $-\beta i+\alpha^S-\lambda\beta^S+\beta^D$ is a right eigenvalue of $M_{\alpha\beta}$ and that $(-i,1)^{\mathsf{T}} \in L^2$ is a corresponding eigenvector[3] for all $\alpha,\beta \in K$. Conversely, suppose that

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} c$$

[3]So we have at least one "absolute point" on the line at infinity of L^2.

for $\alpha,\beta,\gamma,\delta \in K$ and $c \in L$. Then $-\alpha i+\gamma = -ic$, $-\beta i+\delta = c$, whence

$$-ic = -i\alpha^S-\alpha^D+\gamma = (-i\lambda-\mu)\beta^S+i\beta^D-i\delta$$

and

$$\gamma = \alpha^D-\mu\beta^S, \quad \delta = \alpha^S-\lambda\beta^S+\beta^D,$$

as required.■

3. Spreads of lines arising from indicator sets which are subspaces

Let $\tilde{\mathbf{V}}$ be a right vector space over K. Then (with $1 \in K \subset L$) we have the canonical embedding $\mathbf{v} \mapsto \mathbf{v}\otimes 1$ of $\tilde{\mathbf{V}}$ in the right vector space $\tilde{\mathbf{V}}\otimes_K L$ over L. This yields an embedding of the projective space $\tilde{\mathcal{P}}$ on $\tilde{\mathbf{V}}$ in the projective space $\mathcal{P}$ on $\tilde{\mathbf{V}}\otimes_K L$ and $\mathcal{P},\tilde{\mathcal{P}}$ will be called *projective spaces over the right quadratic field extension L/K.*

Subspaces of $\tilde{\mathcal{P}}$ or $\mathcal{P}$ will be regarded as sets of points. Every subspace $\tilde{\mathcal{M}}$ of $\tilde{\mathcal{P}}$ extends to a subspace of $\mathcal{P}$, say $\mathcal{M}$, which is the span of $\tilde{\mathcal{M}}$ within $\mathcal{P}$. Note that $\tilde{\mathcal{M}} = \tilde{\mathcal{P}}\cap\mathcal{M}$ and $\dim\tilde{\mathcal{M}} = \dim\mathcal{M}$. By abuse of language $\mathcal{M}$ will be called a subspace of $\tilde{\mathcal{P}}$. This notation ($\tilde{\mathcal{M}} \subset \tilde{\mathcal{P}}$, $\mathcal{M} \subset \mathcal{P}$) will be kept until the end of this note.

We state three lemmata on projective spaces $\mathcal{P},\tilde{\mathcal{P}}$ over L/K. The proof of Lemma 1 is a straightforward generalization of results in [4], [6,522] and is left to the reader; cf. also section 3 in [13]. Lemma 2 has been proved in [4] (Proposition 2.7,a) under the additional assumption that $\mathcal{P}$ is pappian.

LEMMA 1. *Let* $\dim\mathcal{P} \geq 1$. *Then every hyperplane $\mathcal{H}$ of $\mathcal{P}$ which is not a hyperplane of $\tilde{\mathcal{P}}$ contains a unique co-line of $\tilde{\mathcal{P}}$.*

LEMMA 2. *Let $\mathcal{T}$ be an $(n-d)$-dimensional subspace of $\mathcal{P}$,* $1 \leq \dim\mathcal{P} = n < \infty$, $n-2d \geq -1$. *Then $\mathcal{T}$ contains an at least $(n-2d)$-dimensional subspace of $\tilde{\mathcal{P}}$.*

Proof. If $\mathcal{T} = \mathcal{P}$ then the result is obvious; otherwise $\mathcal{T}$ may be written as intersection of $d \geq 1$ independent hyperplanes $\mathcal{H}_1,\dots,\mathcal{H}_d$ of $\mathcal{P}$. If such an $\mathcal{H}_k$ is a hyperplane of $\tilde{\mathcal{P}}$ then put $\mathcal{H}_k =: \mathcal{H}_k' =: \mathcal{H}_k''$. If $\mathcal{H}_k$ does not belong to $\tilde{\mathcal{P}}$ then, by Lemma 1, there exist two different hyperplanes $\mathcal{H}_k',\mathcal{H}_k''$ of $\tilde{\mathcal{P}}$ such that $\mathcal{H}_k'\cap\mathcal{H}_k'' \subset \mathcal{H}_k$. Write

$$\mathcal{U} := \mathcal{H}_1'\cap\mathcal{H}_1''\cap\dots\cap\mathcal{H}_d'\cap\mathcal{H}_d'' \subset \mathcal{H}_1\cap\dots\cap\mathcal{H}_d = \mathcal{T}.$$

Then $\dim\mathcal{U} \geq n-2d$ and $\mathcal{U}$ is a subspace of $\tilde{\mathcal{P}}$.■

Given a point $X \in \mathcal{P}\setminus\tilde{\mathcal{P}}$ then there is at most one line ℓ of $\tilde{\mathcal{P}}$ which is incident with X. If such a line ℓ exists then it is said to be *indicated* by X.

A subset $\mathcal{I}$ of $\mathcal{P}\setminus\tilde{\mathcal{P}}$ will be called an *indicator set*[4] *(with respect to* $\tilde{\mathcal{P}}$*)* if the following conditions hold true:

(I) Whenever $X,X' \in \mathcal{I}$ indicate lines of $\tilde{\mathcal{P}}$ with non-empty intersection[5] then $X = X'$.

(II) The set $\mathcal{S}(\mathcal{I})$, formed by all lines ℓ of $\tilde{\mathcal{P}}$ which are indicated by the points of $\mathcal{I}$, is covering $\tilde{\mathcal{P}}$.

In using this terminology even when the left degree of L over K is different from 2, we have to be aware of the fact that a point of $\mathcal{I}$ may not indicate any line of $\tilde{\mathcal{P}}$ at all. See the results in section 4.

When $\mathcal{I}$ is an indicator set then write

$$\tilde{\mathcal{S}}(\mathcal{I}) := \{\tilde{\ell} = \ell\cap\tilde{\mathcal{P}} \mid \ell \in \mathcal{S}(\mathcal{I})\}.$$

It is immediate from (I) and (II) that $\tilde{\mathcal{S}}(\mathcal{I})$ is a 1-spread of $\tilde{\mathcal{P}}$, i.e. a partition of $\tilde{\mathcal{P}}$ into lines. We shall also say that $\mathcal{I}$ is an indicator set of $\tilde{\mathcal{S}}(\mathcal{I})$ or that $\tilde{\mathcal{S}}(\mathcal{I})$ is indicated by $\mathcal{I}$.

The following Lemma 3 is a generalization of the description of elliptic linear congruences of lines mentioned in section 1, of Theorem 3.6 in [4], where $\mathcal{P}$ is assumed to be pappian, and in part[6] of Theorem 3 in [13].

LEMMA 3. *Let $\mathcal{I}$ be an m-dimensional subspace of $\mathcal{P}$,* $1 \leq \dim\mathcal{P} = 2m+1 < \infty$, *which carries no point of $\tilde{\mathcal{P}}$. Then $\mathcal{I}$ is an indicator set.*

Proof. Fix any point $X \in \tilde{\mathcal{P}}$. Every line of $\tilde{\mathcal{P}}$ which goes through X and is indicated by a point of $\mathcal{I}$ necessarily is contained in the subspace

$$\mathcal{T}_X := X\vee\mathcal{I}.$$

By Lemma 2 and $\dim\mathcal{T}_X = m+1 = 2m+1-m$, there is at least one line of $\tilde{\mathcal{P}}$ which lies in $\mathcal{T}_X$. Now suppose that there exists a plane $\mathcal{E}$ of $\tilde{\mathcal{P}}$ satisfying $\mathcal{E} \subset \mathcal{T}_X$: Then, by the dimension formula, $\mathcal{E}\cap\mathcal{I}$ would be a line which, by Lemma 1, would have a common point with $\tilde{\mathcal{P}}$ in contradiction to $\mathcal{I}\cap\tilde{\mathcal{P}} = \emptyset$. We deduce that X is on every line ℓ of $\tilde{\mathcal{P}}$ which is contained in $\mathcal{T}_X$, since $\ell\vee X$ cannot be a plane. On the other hand there cannot be two distinct lines ℓ,ℓ' of $\tilde{\mathcal{P}}$ passing through X, since $\ell\vee\ell'$ cannot be a plane either. Hence there is a unique line of $\tilde{\mathcal{P}}$, say ℓ_X, such that $X \in \ell_X \subset \mathcal{T}_X$. Furthermore this ℓ_X is indicated by a point of $\mathcal{I}$,

[4] This is slightly more general than the definition in [6,523]. Cf. also the concept of *spazio direttore* [21,29].

[5] Two lines $\tilde{\ell},\tilde{\ell}' \subset \tilde{\mathcal{P}}$ have a common point within $\tilde{\mathcal{P}}$ if, and only if, the extended lines ℓ,ℓ' have a common point in $\mathcal{P}$.

[6] That result, in contrast to the present one, is not restricted to finite-dimensional spaces, but it is subject to other assumptions which do not appear in Lemma 3.

since $\dim\mathcal{I} = \dim\mathcal{T}_X - 1$. Therefore $\mathcal{I}$ is an indicator set.■

In the sequel we shall confine our attention to 3-dimensional projective spaces $\tilde{\mathcal{P}}$ and $\mathcal{P}$. Here a 1-spread briefly will be called a spread. Every spread gives rise to a translation plane and we are going to use the term desarguesian spread if this plane is desarguesian. Recall that the kernel of a spread $\tilde{\mathcal{S}}$ of $\tilde{\mathcal{P}}$ is the set formed by all endomorphisms of the abelian group $(\tilde{\mathbf{V}},+)$ which map every line[7] $\tilde{\ell} \in \tilde{\mathcal{S}}$ into itself. This kernel is a field and will be written as $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}})$. Cf. e.g. [19,3]. Every $\alpha \in K$ gives rise to an element of $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}})$, again denoted by α, by putting

$$\mathbf{x}^{\alpha} := \mathbf{x}\alpha \text{ for all } \mathbf{x} \in \tilde{\mathbf{V}}.$$

This identification of $\alpha \in K$ and $\alpha \in \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}})$ turns K into a subfield of $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}})$.

THEOREM 2. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be 3-dimensional projective spaces over a right quadratic field extension L/K. There exists a line of $\mathcal{P}$ which carries no point of $\tilde{\mathcal{P}}$. Every line $\mathcal{J}$ with this property is an indicator set of a desarguesian spread $\tilde{\mathcal{S}}(\mathcal{J})$ of $\tilde{\mathcal{P}}$. There exists an isomorphism of the field L onto the kernel* $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}(\mathcal{J}))$ *of this spread such that K is elementwise invariant.*

Proof. (a) In $\tilde{\mathcal{P}}$ there exist three mutually skew lines $\tilde{\ell}_{01},\tilde{\ell}_{23},\tilde{\ell}$. Draw within $\tilde{\mathcal{P}}$ three different transversal lines $\tilde{t}_{02},\tilde{t}_{13},\tilde{t}$, say. Then put

$$P_0 := \tilde{\ell}_{01}\cap\tilde{t}_{02},\ P_1 := \tilde{\ell}_{01}\cap\tilde{t}_{13},\ P_2 := \tilde{\ell}_{23}\cap\tilde{t}_{02},\ P_3 := \tilde{\ell}_{23}\cap\tilde{t}_{13},\ P := \tilde{\ell}\cap\tilde{t}.$$

Hence there exists a coordinatization of $\tilde{\mathcal{P}}$ with (P_0,P_1,P_2,P_3,P) as frame of reference and this extends to a coordinatization of $\mathcal{P}$. Choose any $i \in L\setminus K$ and define $\mathcal{J}$ as the line of $\mathcal{P}$ which joins the points with coordinates

$$\left(-i,1,0,0\right)^{\mathsf{T}},\ \left(0,0,-i,1\right)^{\mathsf{T}}.$$

This line has no common points with $\tilde{\mathcal{P}}$.

(b) Now let $\mathcal{J}$ be a line of $\mathcal{P}$ such that $\mathcal{J}\cap\tilde{\mathcal{P}} = \emptyset$. By Lemma 3, $\mathcal{J}$ is an indicator set. Choose three different lines of $\tilde{\mathcal{S}}(\mathcal{J})$, label them $\tilde{\ell}_{01},\tilde{\ell}_{23},\tilde{\ell}$ and introduce coordinates as above. The points of intersection $\ell_{01}\cap\mathcal{J}$, $\ell\cap\mathcal{J}$, $\ell_{23}\cap\mathcal{J}$ are collinear, whence they have coordinate vectors

$$\left(-i,1,0,0\right)^{\mathsf{T}},\ \left(-i,1,-i,1\right)^{\mathsf{T}},\ \left(0,0,-i,1\right)^{\mathsf{T}}, \tag{5}$$

respectively, for some $i \in L\setminus K$. All results of section 2 hold true for this specific element $i \in L\setminus K$.

Let g be any line of $\tilde{\mathcal{P}}$ which is skew to ℓ_{23}. There is a unique 2×2 matrix

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[7]Whenever it is conveniant, we shall use the same symbol to denote a line, i.e. a range of points in $\tilde{\mathcal{P}}$ (or $\mathcal{P}$), as well as the associated 2-dimensional subspace of the vector space $\tilde{\mathbf{V}}$ (or $\mathbf{V}$).

$M_{\mathcal{g}}$ with entries in K such that the coordinates of vectors in $\mathcal{g}$ have the form

$$\left(x,y,u,v\right)^{\mathsf{T}} \text{ with } \begin{pmatrix} u \\ v \end{pmatrix} = M_{\mathcal{g}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} \in L^2.$$

By definition of $\mathcal{S}(\mathcal{I})$ this line $\mathcal{g}$ belongs to $\mathcal{S}(\mathcal{I})$ if, and only if, $\left(-i,1\right)^{\mathsf{T}}$ is an eigenvector of the matrix $M_{\mathcal{g}}$. By Theorem 1 this is equivalent to

$$M_{\mathcal{g}} = M_{\alpha\beta} \text{ for some } \alpha,\beta \in K.$$

This is a well known description of a spread by means of matrices [19,7ff], but we use it in an extended way, since it is applied to all points of $\mathcal{g} \subset \mathcal{P}$ rather than $\tilde{\mathcal{g}} \subset \tilde{\mathcal{P}}$.

We infer that formula (3) gives the multiplication rule in an underlying quasifield $(Q,+,\circ) = (K^2,+,\circ)$ of the spread $\tilde{\mathcal{S}}(\mathcal{I})$. Hence Q is even a field which is isomorphic to L. Thus $\tilde{\mathcal{S}}(\mathcal{I})$ is desarguesian and the kernel of $\tilde{\mathcal{S}}(\mathcal{I})$ also is isomorphic to L.

(c) We declare a mapping $E : L \to \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}(\mathcal{I}))$ as follows: For every $\alpha+i\beta \in L$ $(\alpha,\beta \in K)$ the action of the endomorphism $(\alpha+i\beta)^E : \tilde{\mathbf{V}} \to \tilde{\mathbf{V}}$ is defined in terms of coordinates by

$$\begin{pmatrix} \xi_0 \\ \eta_0 \\ \xi_1 \\ \eta_1 \end{pmatrix} \mapsto \begin{pmatrix} \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \circ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \circ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \end{pmatrix} \text{ for all } \begin{pmatrix} \xi_0 \\ \eta_0 \\ \xi_1 \\ \eta_1 \end{pmatrix} \in K^4.$$

A straightforward calculation shows that this definition does make sense and that E is an injective endomorphism of fields which leaves K elementwise invariant. But $K \subset L^E \subset \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}(\mathcal{I}))$ implies $L^E = \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}(\mathcal{I}))$, since both L^E and $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}(\mathcal{I}))$ are right quadratic extensions of K.■

Now we investigate all spreads of $\tilde{\mathcal{P}}$ which arise according to Theorem 2.

THEOREM 3. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be 3-dimensional projective spaces over a right quadratic field extension L/K, let $\tilde{\mathcal{S}}_1$ be a spread of $\tilde{\mathcal{P}}$ which is indicated by a line $\mathcal{I}_1$ of $\mathcal{P}$ and let $\tilde{\mathcal{S}}_2$ be a spread of $\tilde{\mathcal{P}}$. Then the following assertions are equivalent:*

(a) *There exists a projective collineation $\tilde{\kappa} : \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ which maps $\tilde{\mathcal{S}}_1$ onto $\tilde{\mathcal{S}}_2$.*

(b) *$\tilde{\mathcal{S}}_2$ is indicated by a line $\mathcal{I}_2$ of $\mathcal{P}$.*

(c) *There exists an isomorphism of the kernel of $\tilde{\mathcal{S}}_1$ onto the kernel of $\tilde{\mathcal{S}}_2$ which fixes K elementwise.*

Proof. (a) ⇒ (b) The given collineation $\tilde{\kappa}$ is induced by an $\tilde{f} \in \mathrm{GL}(\tilde{\mathbf{V}})$ which extends to a mapping $f \in \mathrm{GL}(\mathbf{V}\otimes_K L)$. This f in turn induces a projective collineation κ of $\mathcal{P}$ which is an extension of $\tilde{\kappa}$. Then $\mathcal{I}_2 := \mathcal{I}_1{}^{\kappa}$ is a line with the required properties.

(b) ⇒ (c) This is an immediate consequece of Theorem 2.

(c) ⇒ (a) Suppose that an isomorphism $E : \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1) \to \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_2)$ fixes every element of K. On one hand $\tilde{\mathbf{V}}$ will be regarded as vector space over $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1)$ and, on the other hand, as vector space over $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_2)$. So given $a \in \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1)$ or $a \in \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_2)$ and $\mathbf{x} \in \tilde{\mathbf{V}}$ we may write $\mathbf{x}a$ instead of $\mathbf{x}^a$.

There exists an E-semilinear bijection $\tilde{f}$ of $\tilde{\mathbf{V}}$ over $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1)$ onto $\tilde{\mathbf{V}}$ over $\mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_2)$. This $\tilde{f}$ operates linearly on $\tilde{\mathbf{V}}$ over K. Fix a line $\tilde{\ell} \in \tilde{\mathcal{S}}_1$ and a non-zero vector $\mathbf{x} \in \tilde{\ell}$. Then

$$\tilde{\ell} = \{\mathbf{x}a \,|\, a \in \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1)\}$$

and a similar description holds true for the lines of $\tilde{\mathcal{S}}_2$. We deduce from

$$(\mathbf{x}a)^{\tilde{f}} = (\mathbf{x}^{\tilde{f}})a^E \text{ for all } \mathbf{x} \in \mathbf{V},\ a \in \mathrm{K}(\tilde{\mathbf{V}},\tilde{\mathcal{S}}_1),$$

and from the bijectivity of $\tilde{f}$ that the projective collineation of $\tilde{\mathcal{P}}$ which is induced by $\tilde{f}$ maps $\tilde{\mathcal{S}}_1$ onto $\tilde{\mathcal{S}}_2$.■

In general it is not true that *every* desarguesian spread of $\tilde{\mathcal{P}}$ with an underlying field isomorphic to L is of the form $\tilde{\mathcal{S}}(\mathcal{I})$ for some line $\mathcal{I} \subset \mathcal{P}$. This is illustrated by the following simple counterexample:

Put $K := \mathbb{Q}(\sqrt{2})$. Let L, L' be the splitting fields of the polynomials $X^2-\sqrt{2}$, $X^2+\sqrt{2} \in \mathbb{Q}(\sqrt{2})[X]$, respectively. There exists an automorphism F of $\mathbb{Q}(\sqrt{2})$ such that $\sqrt{2}$ is interchanged with $-\sqrt{2}$. Thus L and L' are isomorphic fields. But there does not exist an isomorphism $L \to L'$ which leaves $\mathbb{Q}(\sqrt{2})$ element-wise invariant, since $\sqrt{2}$ is a square in L but a non-square in L'. Define $\tilde{\mathcal{P}}$ and $\mathcal{P}$ over K and L, respectively, and take a spread $\tilde{\mathcal{S}}_1 = \tilde{\mathcal{S}}(\mathcal{I})$ which exists according to Theorem 2. There is a non-projective collineation $\tilde{\kappa}$ of $\tilde{\mathcal{P}}$ belonging to $F \in \mathrm{Aut}(K)$. The image of $\tilde{\mathcal{S}}_1$ under $\tilde{\kappa}$ is a spread $\tilde{\mathcal{S}}_2$ of $\tilde{\mathcal{P}}$. The kernel of $\tilde{\mathcal{S}}_1$ is isomorphic to the kernel of $\tilde{\mathcal{S}}_2$, but condition (c) of Theorem 3 is not met, whence $\tilde{\mathcal{S}}_2$ is not indicated by a line of $\mathcal{P}$. Clearly, in some *other* extension of $\tilde{\mathcal{P}}$, namely the projective space on $\tilde{\mathbf{V}}\otimes_K L'$, such a line exists.

It seems to be an open problem to give an intrinsic characterization of all desarguesian spreads of $\tilde{\mathcal{P}}$ in terms of $\tilde{\mathcal{P}}$ alone. However, when K is part of the centre of L then results stated in [14] can be applied.

4. Geometric aspects of the left degree

The left degree of a right quadratic extension L/K is either two or strictly greater than two, since it cannot be equal to one. The following result tells us, how to see from different geometric points of view, if this left degree equals two or not. One condition is based upon the embedding of a projective space $\tilde{\mathcal{P}}$ in a projective space $\mathcal{P}$ (cf. the beginning of section 3), while the other uses a three dimensional projective space $\tilde{\mathcal{P}}$ together with the concept of dual spread, i.e. a set of lines with exactly one line in every plane of $\tilde{\mathcal{P}}$.

THEOREM 4. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be projective spaces over a right quadratic field extension L/K. The following assertions are equivalent:*

(a) The left degree of K/L equals two.

(b) Provided that $\dim\mathcal{P} \geq 2$, every point of $\mathcal{P}$ which is not a point of $\tilde{\mathcal{P}}$ is on a unique line of $\tilde{\mathcal{P}}$.

(c) If $\dim\mathcal{P} = 3$ then every spread of $\tilde{\mathcal{P}}$ which is indicated by a line $\mathcal{I}$ of $\mathcal{P}$ is also a dual spread of $\tilde{\mathcal{P}}$.

Proof. (a) $\Leftrightarrow$ (b) If the left degree of L over K is two then through every point of $\mathcal{P}\setminus\tilde{\mathcal{P}}$ there goes a unique line of $\tilde{\mathcal{P}}$. This is easily shown in terms of coordinates; cf. the references with respect to Lemma 1.

If this left degree is > 2 then there are elements $a,b \in L$ such that

$$\xi+\eta a+\zeta b = 0 \text{ with } \xi,\eta,\zeta \in K \text{ implies } \xi = \eta = \zeta = 0. \tag{6}$$

Take any plane $\mathcal{E}$ of $\tilde{\mathcal{P}}$ and introduce projective coordinates with respect to a quadrangle of $\tilde{\mathcal{P}}\cap\mathcal{E}$. There exists a point Z which has a coordinate vector $(1,a,b)^{\mathsf{T}}$. If a line of $\tilde{\mathcal{P}}$ goes through Z then it has to be a part of $\mathcal{E}$, since $Z \in \mathcal{E}\setminus\tilde{\mathcal{P}}$. However, by (6), such a line cannot exist within the plane $\mathcal{E}$.

(a) $\Leftrightarrow$ (c) We adopt the notations of the proof of Theorem 2. There is a line of $\mathcal{S}(\mathcal{I})$ in a given plane $\mathcal{E}$ of $\tilde{\mathcal{P}}$ if, and only if, $\mathcal{E}\cap\mathcal{I}$ is on a line of $\mathcal{S}(\mathcal{I})$, because of $\mathcal{E}\cap\mathcal{I} \notin \tilde{\mathcal{P}}$.

If $|L:K|_{\text{left}} = 2$ then, by (b), every point of $\mathcal{I}$ is on a line of $\tilde{\mathcal{P}}$ and this line belongs to $\mathcal{S}(\mathcal{I})$ by definition. Hence $\tilde{\mathcal{S}}(\mathcal{I})$ is a dual spread.

Conversely, if $|L:K|_{\text{left}} \neq 2$ then

$$|L:K|_{\text{left}} = |K:K^S|_{\text{left}} + 1$$

(cf. [8,540] or [9,57]) so that S is not surjective. We obtain all points of $\mathcal{I}$ which are incident with a member of $\mathcal{S}(\mathcal{I})\setminus\{\ell_{23}\}$ via formula (3). By (4) the right eigenvalue of $M_{\alpha\beta}$ belonging to $(-i,1)^{\mathsf{T}}$ is

$$v_{\alpha\beta} := -\beta i+\alpha^S-\lambda\beta^S+\beta^D = -i\beta^S-\beta^D+\alpha^S-\lambda\beta^S+\beta^D = \alpha^S-(\lambda+i)\beta^S \tag{7}$$

and this yields, by (5), the point of $\mathcal{I}$ with coordinates

$$(-i,1,0,0)^{\mathsf{T}}+(0,0,-i,1)^{\mathsf{T}}v_{\alpha\beta}.$$

There exists an element $\gamma \in K\setminus K^S$. Moreover $\gamma \neq v_{\alpha\beta}$ for all $\alpha,\beta \in K^2$, since $\gamma = v_{\alpha\beta}$ would force $\beta = 0$ and $\gamma = \alpha^S$, an absurdity. Hence the point $G \in \mathcal{I}$ with coordinates

$$(-i,1,0,0)^{\mathsf{T}}+(0,0,-i,1)^{\mathsf{T}}\gamma$$

does not indicate a line of $\mathcal{S}(\mathcal{I})$. On the other hand it follows from

$$(-i,1,-i\gamma,\gamma)^{\mathsf{T}} = (1,0,0,0)^{\mathsf{T}}(-i)+(0,1,0,\gamma)^{\mathsf{T}}+(0,0,1,0)^{\mathsf{T}}(-i\gamma)$$

that there is a plane of $\tilde{\mathcal{P}}$ which goes through G. Therefore $\tilde{\mathcal{S}}(\mathcal{I})$ is not a dual spread.■

The existence of right quadratic extensions of left degree greater than two [8], [9,123 ff] establishes:

COROLLARY 1. *There exist* 3-*dimensional projective spaces with a desarguesian spread which is not a dual spread.*

It is well known that a spread of a projective 3-space of infinite order need not be a dual spread [7], [11], but it seems that so far no attention has been paid to the question whether such a spread is desarguesian or not. Another question related with Theorem 4 is the definition of Baer subplanes in projective planes of infinite order which has been discussed in [1]. We are in a position to contribute the following

COROLLARY 2. *There exist desarguesian projective planes $\mathcal{P}$ with a subplane $\tilde{\mathcal{P}}$ such that every line of $\mathcal{P}$ is incident with at least one point of $\tilde{\mathcal{P}}$, but not every point of $\mathcal{P}$ is on a line of $\tilde{\mathcal{P}}$.*

Next we show that, whenever $\tilde{\mathcal{S}}(\mathcal{I})$ is not a dual spread of $\tilde{\mathcal{P}}$ ($\dim\tilde{\mathcal{P}} = 3$), we may easily describe those points of $\mathcal{I}$ which contain a line of $\mathcal{S}(\mathcal{I})$. An indicator set $\hat{\mathcal{I}} \subset \mathcal{P}\backslash\tilde{\mathcal{P}}$ will be called *minimal* if every proper subset of $\hat{\mathcal{I}}$ is not an indicator set.

THEOREM 5. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be 3-dimensional projective spaces over a right quadratic field extension L/K. Assume that $\tilde{\mathcal{S}}(\mathcal{I})$ is a spread of $\tilde{\mathcal{P}}$ which is indicated by a line $\mathcal{I}$ of $\mathcal{P}$. There exists a subline of $\mathcal{I}$ which is a minimal indicator set of $\tilde{\mathcal{S}}(\mathcal{I})$. This subline is a proper subline* of $\mathcal{I}$ *if, and only if, $\tilde{\mathcal{S}}(\mathcal{I})$ is not a dual spread of $\tilde{\mathcal{P}}$.*

Proof. According to [8,535] the endomorphism $S : K \to K$ may be extended to an endomorphism of L (again denoted by S) by putting $i \mapsto -(\lambda+i)$; cf. formula (1). Moreover $K^S = K$ is shown to be equivalent to $L^S = L$. In this notation formula (7) reads

$$v_{\alpha\beta} = (\alpha+i\beta)^S,$$

whence those points of $\mathcal{I}$ which are on a line of $\mathcal{S}(\mathcal{I})$ form an L^S-subline $\hat{\mathcal{I}}$ of $\mathcal{I}$. This $\hat{\mathcal{I}}$ is a minimal indicator set and $\tilde{\mathcal{S}}(\hat{\mathcal{I}}) = \tilde{\mathcal{S}}(\mathcal{I})$. But $\mathcal{I} \neq \hat{\mathcal{I}}$ is equivalent, by the proof of Theorem 4, to $\tilde{\mathcal{S}}(\mathcal{I})$ not being a dual spread.■

5. Galois extensions

There arises the question if there are two or even more lines of $\mathcal{P}$, such that each of these lines is indicating a given spread $\tilde{\mathcal{S}}(\mathcal{I})$. When L is commutative then similar questions have been discussed in [4], [5,439], [16], [17], [21,29].

THEOREM 6. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be 3-dimensional projective spaces over a right quadratic field extension L/K and let $\tilde{\mathcal{S}}(\mathcal{I})$ be a spread of $\tilde{\mathcal{P}}$ which is indicated by a line $\mathcal{I} \subset \mathcal{P}$. There exist two different lines of $\mathcal{P}$ such that each of them is an indicator set of $\tilde{\mathcal{S}}(\mathcal{I})$ if, and only if, L/K is a Galois extension.*

Proof. We shall use the settings of Theorem 2.

If L/K is a Galois extension, i.e. there is a non-trivial automorphism R of L which leaves K elementwise fixed, then necessarily $i^R \neq i$. In geometric terms we obtain the existence of a collineation $\kappa : \mathcal{P} \to \mathcal{P}$ which fixes every point of $\tilde{\mathcal{P}}$ but actually moves the point with coordinates $(-i,1,0,0)^{\mathsf{T}}$ along the invariant line ℓ_{01}, so that $\mathcal{I}$ cannot be κ-invariant. Thus the line $\mathcal{I}^{\kappa}$ is another indicator set of the spread $\tilde{\mathcal{S}}(\mathcal{I})$.

Now let a line $\mathcal{J} \neq \mathcal{I}$ be an indicator set of $\tilde{\mathcal{S}}(\mathcal{I})$, whence $\ell_{01}\cap\mathcal{J}$, $\ell\cap\mathcal{J}$, $\ell_{23}\cap\mathcal{J}$ have coordinate vectors

$$(-j,1,0,0)^{\mathsf{T}},\ (-j,1,-j,1)^{\mathsf{T}},\ (0,0,-j,1)^{\mathsf{T}},$$

respectively, with $i \neq j \in L\backslash K$; cf. formula (5). Since $(-j,1)^{\mathsf{T}}$ has to be an eigenvector of every matrix $M_{\alpha\beta}$ appearing in formula (2), we may replace i by j in formula (4). Evaluating the right eigenvalue of $M_{\alpha\beta}$ which belongs to $(-j,1)^{\mathsf{T}}$ yields the identity

$$-j(-\beta j+\alpha^S-\lambda\beta^S+\beta^D) = -\mu\beta^S-\alpha j+\alpha^D \text{ for all } \alpha,\beta \in K. \tag{8}$$

Now we substitute in (8) as follows: Putting $\beta := 0$ establishes

$$\alpha j = j\alpha^S + \alpha^D \text{ for all } \alpha \in K, \tag{9}$$

whereas $\alpha := 0$, $\beta := 1$, together with $1^D = 0$, shows

$$j^2+j\lambda+\mu = 0. \tag{10}$$

By (9), (10) and (1) the mapping

$$R : L \to L,\ \alpha+i\beta \mapsto \alpha+j\beta\ (\alpha,\beta \in K)$$

is an automorphism of L with the required properties.■

We remark that any right quadratic Galois extension L/K has left degree two [9,49]. So we are able to state the following

COROLLARY 3. *Let $\mathcal{P},\tilde{\mathcal{P}}$ be 3-dimensional projective spaces over a right quadratic field extension L/K and let $\tilde{\mathcal{S}}$ be a spread of $\tilde{\mathcal{P}}$. If there are two different lines of $\mathcal{P}$ such that each of them is an indicator set of $\tilde{\mathcal{S}}$ then $\tilde{\mathcal{S}}$ is also a dual spread of $\tilde{\mathcal{P}}$.*

If L is commutative then the Galois group of the quadratic extension L/K is of order ≤ 2, whence there cannot be more than two different lines which indicate a spread. But even when K is commutative and L is not commutative this need no longer be true.

This is illustrated, for example, by taking the real quaternions as L and any subfield of L which is isomorphic to the complex numbers as K. Every element of $K\backslash\{0\}$ gives rise to an inner automorphism of L. This implies for every spread $\tilde{\mathcal{S}}(\mathcal{I})$ that there is an infinite number of lines such that each of them indicates $\tilde{\mathcal{S}}(\mathcal{I})$. Cf. also [13], where a relationship to Segre manifolds has been established.

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