Von Staudt's theorem revisited

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Abstract

We establish a version of von Staudt's theorem on mappings which preserve harmonic quadruples for projective lines over (not necessarily commutative) rings with "sufficiently many" units, in particular 2 has to be a unit.

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1 Introduction

The first edition of the seminal book *Geometrie der Lage* by Karl Georg Christian von Staudt appeared in 1847; see [27] for publication details. Projectivities are defined there by the invariance of harmonic quadruples [55, p. 49]: *"Zwei einförmige Grundgebilde heissen zu einander projektivisch* ($\barwedge$)*, wenn sie so auf einander bezogen sind, dass jedem harmonischen Gebilde in dem einen ein harmonisches Gebilde im andern entspricht."* Next, after defining perspectivities, the following theorem is established: Any projectivity is a finite composition of perspectivities and vice versa. (It was noticed later that there is a small gap in von Staudt's reasoning. A detailed exposition can be found in [57].) Any result in this spirit now is called a *von Staudt's theorem*.

In the present article we shall be concerned with *projective lines over rings* (associative with a unit element) and the algebraic description of their *harmonicity preservers*, i. e., mappings which take all harmonic quadruples of a first projective line to harmonic quadruples of a second one. There is a widespread literature on this topic. The following short review is rather sketchy, as it does not fully reflect the varying (often rather technical) assumptions on the underlying rings. Part of the presented material is related with mappings which reappear in a more general setting in the surveys [20] and [56].

All harmonicity preserving bijections of the projective line over any *commutative field* F of characteristic $\neq 2$ onto itself were determined by O. Schreier and E. Sperner [54, p. 191]. In terms of an underlying F-vector space V these transformations comprise precisely the *projective semilinear group* $\mathrm{P\Gamma L}(V)$. The case of a (not necessarily commutative) *field* of characteristic $\neq 2$ was settled in several steps by G. Ancochea [1], [2], [3] and L.-K. Hua [35] (see also [36]). For a proper skew field F one has to include mappings which arise from *antiautomorphisms* of F (provided that F admits any antiautomorphism). A. J. Hoffman [32] (F commutative) and R. Baer [4, p. 78] (F arbitrary) proved that similar results hold if the invariance of harmonic quadruples is replaced by the invariance of an arbitrary cross ratio $k \neq 0, 1$ in the centre of F. In this way the case of characteristic 2 need no longer be excluded. A detailed account with historical remarks is given in [40, pp. 56–57].

There are several outcomes for the projective line over a *ring R with stable rank* 2: Loosely speaking, in the case of a *commutative ring R* the result of Schreier and Sperner remains unaltered provided that R contains "sufficiently many" units, in particular 2 has to be a unit in R. Contributions (under varying additional assumptions) are due to W. Benz [8], [9, pp. 173–183], B. V. Limaye and N. B. Limaye [49], N. B. Limaye [50], [51], B. R. McDonald [52], and H. Schaeffer [53]. Little seems to be known for non-commutative rings: B. V. Limaye and N. B. Limaye ([47], [48]) treated the case of a (not necessarily commutative) *local ring R*. They determined all bijections of the projective line over R such that all quadruples with a given cross ratio k go over to quadruples with a given cross ratio k', where k, k' are elements in the centre of R other that $0, 1$. Here the algebraic description is more involved, since one has to use *Jordan automorphisms* (or, in a different terminology, *semiautomorphisms*) of R. More information can be retrieved from the surveys in [6], [10], and [11].

F. Buekenhout [21], St. P. Cojan [25], D. G. James [39], and B. Klotzek [41] characterised those (not necessarily injective) mappings between projective lines over fields which satisfy a much weaker form of cross ratio preservation than the one mentioned in the preceding paragraph. The link with ring geometry is achieved via a recoordinatisation of the domain projective line in terms of a *valuation ring* [39].

It was pointed out by C. Bartolone and F. Di Franco [7] that an algebraic description of all harmonicity preserving bijections of the projective line over an arbitrary ring is out of reach, even in the commutative case. They therefore initiated the study of mappings which preserve *generalised harmonic quadruples* and succeeded in describing all such mappings for commutative rings; see also M. Kulkarni [42]. However, this goes beyond the scope of the present article. With regard to the non-commutative case, we refer to the work of C. Bartolone and F. Bartolozzi [6], D. Chkhatarashvili [22], L. Cirlincione and M. Enea [23], and A. A. Lashkhi [44], [45], [46]. Take notice that some of the quoted papers are merely short communications without any proof. For harmonicity preserving mappings of other geometric structures see [12], [13], [26], and the references therein. It is also worth noting that the invariance of harmonic quadruples appears together with other conditions in an early paper [33] of L.-K. Hua on a characterisation of certain transformations of matrix spaces. However, as Hua pointed out in a subsequent note [34], the condition about harmonic quadruples is superfluous in that context, and it afterwards disappeared from the so-called *geometry of matrices*; cf. the monographs [37] and [58]. An analogous result for projective lines over certain semisimple rings is due to A. Blunck and the author [18].

The present article is organised as follows: In Section 2 we collect the relevant notions and we recall the definition of harmonicity preservers which arise from Jordan homomorphisms. Our main result is Theorem 1 in Section 3. It shows that under certain conditions there are no other harmonicity preservers between projective lines over rings, but those which arise from Jordan homomorphisms. A major tool in our proof is a lemma from [49] which characterises Jordan homomorphisms.

2 Basic notions and examples

All our rings are associative with a unit element 1 which is inherited by subrings and acts unitally on modules. The trivial case $1 = 0$ is excluded. The group of units (invertible elements) of a ring R, say, will be denoted by R^*.

Let R be a ring and let M be a free left R-module of rank 2. We say that $a \in M$ is *admissible* if there exists $b \in M$ such that (a, b) is a basis of M (with two elements). As a matter of fact, we do not require that all bases of M have the same number of elements; cf. [43, p. 3].

The following exposition is mainly taken from [31, p. 785]; see also [19, pp. 15–16] or [28, pp. 899–904]: The *projective line* over M is the set $\mathbb{P}(M)$ of all cyclic submodules Ra, where $a \in M$ is admissible. The elements of $\mathbb{P}(M)$ are called *points*. At times it will be convenient to use coordinates with respect to some basis (e_0, e_1) of M. Given any pair $(a, b) \in M^2$ let (x_0, x_1) and (y_0, y_1) be the coordinates of a and b, respectively. The matrix

$$\begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \tag{1}$$

will be called the *matrix of* (a, b) *w. r. t. the basis* (e_0, e_1). The pair (a, b) is a basis of M if, and only if, the matrix in (1) is invertible. Thus $(x_0, x_1) \in R^2$ is admissible (or, said differently, a coordinate pair of a point) precisely when it is the first (or second) row of a matrix in $\mathrm{GL}_2(R)$. One particular case deserves explicit mention, since it links the group R^* with the group $\mathrm{GL}_2(R)$: For all $x, y \in R$ holds

$$\begin{pmatrix} x & 1 \\ y & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \quad \text{if, and only if,} \quad x - y \in R^*. \tag{2}$$

This is immediate from

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x-y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} x & 1 \\ y & 1 \end{pmatrix}. \tag{3}$$

By definition, each point $p \in \mathbb{P}(M)$ has an admissible generator, say a. If there exist $x, y \in R$ with $xy = 1$ and $yx \neq 1$ then ya is a non-admissible generator of p, whereas xa is an admissible generator of a point other than p [14, Prop. 2.1 and Prop. 2.2]. We adopt from now on the following convention: *We only use admissible generators of points.* Two admissible elements of M generate the same point precisely when they are left-proportional by a unit in R.

Two points p and q are called *distant*, in symbols $p \vartriangle q$, if $M = p \oplus q$. For all $a, b \in M$ holds $Ra \vartriangle Rb$ precisely when the coordinate matrix of (a, b) w. r. t. any basis (e_0, e_1) of M is invertible. The graph of the relation $\vartriangle$, i. e. the pair $(\mathbb{P}(M), \vartriangle)$, is called the *distant graph* of $\mathbb{P}(M)$. It is an undirected graph without loops, and it need not be connected. In order to describe the *connected components* of the distant graph we need some prerequisites.

The *elementary linear group* $\mathrm{E}_2(R)$ is generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad t \in R;$$

see [24, p. 5]. Let $\mathcal{S}(R)$ be the set of all finite sequences in R (including the empty sequence). We adopt the shorthand notation

$$E(T) := E(t_1) \cdot E(t_2) \cdots E(t_n) \quad \text{where} \quad T = (t_1, t_2, \ldots, t_n) \in \mathcal{S}(R).$$

(Note that $n \geq 0$, the length of T, is arbitrary.) From $E(t)^{-1} = E(0, -t, 0)$ follows that all matrices $E(T)$ with $T \in \mathcal{S}(R)$ comprise the entire group $\mathrm{E}_2(R)$. The subgroup of $\mathrm{GL}_2(R)$, which is generated by $\mathrm{E}_2(R)$ and the set of all invertible diagonal matrices, is denoted by $\mathrm{GE}_2(R)$. By definition, a GE_2*-ring* R is characterised by $\mathrm{GL}_2(R) = \mathrm{GE}_2(R)$.

If (e_0, e_1) is a basis of M then the connected component of the point $Re_0 \in \mathbb{P}(M)$ is given by the set of all points $p = R(x_0e_0 + x_1e_1)$, where (x_0, x_1) is the first row of some matrix $E(T)$ with $T \in \mathcal{S}(R)$ or, said differently,

$$(x_0, x_1) = (1, 0) \cdot E(T) \quad \text{for some} \quad T \in \mathcal{S}(R). \tag{4}$$

Furthermore, the distant graph $(\mathbb{P}(M), \triangle)$ is connected precisely when R is a GE_2-ring [15, Thm. 3.2].

A quadruple $(p_0, p_1, p_2, p_3) \in \mathbb{P}(M)^4$ is *harmonic* if its cross ratio [31, p. 787] equals $-1 \in R$, i. e., there exists a basis (g_0, g_1) of M such that

$$p_0 = Rg_0, \quad p_1 = Rg_1, \quad p_2 = R(g_0 + g_1), \quad p_3 = R(g_0 - g_1). \tag{5}$$

In this case we write $\mathrm{H}(p_0, p_1, p_2, p_3)$. In terms of coordinates w. r. t. some basis (e_0, e_1) of M there is an alternative description: $\mathrm{H}(p_0, p_1, p_2, p_3)$ holds if, and only if, there is a matrix $G \in \mathrm{GL}_2(R)$ such that

$$(1,0)\cdot G, \quad (1,0)\cdot E(0)\cdot G, \quad (1,0)\cdot E(1)\cdot G, \quad (1,0)\cdot E(-1)\cdot G \tag{6}$$

are coordinates of the points p_0, p_1, p_2, p_3, respectively. Indeed, if (5) holds for some basis (g_0, g_1) we can take as G the coordinate matrix of (g_0, g_1) w. r. t. (e_0, e_1) in order to obtain (6). Conversely, the rows of G provide the coordinates w. r. t. (e_0, e_1) of an appropriate basis of M to guarantee $\mathrm{H}(p_0, p_1, p_2, p_3)$.

From $\mathrm{H}(p_0, p_1, p_2, p_3)$ follows $p_0 \triangle p_1$ and $p_i \triangle p_j$ for all $i \in \{0, 1\}$ and all $j \in \{2, 3\}$. *Therefore all four points belong to the same connected component of the distant graph* $(\mathbb{P}(M), \triangle)$. By virtue of (2), we have

$$p_2 \triangle p_3 \quad \text{if, and only if,} \quad 2 \in R^*. \tag{7}$$

The inequality $p_2 \neq p_3$ holds precisely when $-1 \neq 1 \in R$. (In [17, 4.7] these two conditions erroneously got mixed up.)

If p_0, p_1, p_2 are three mutually distant points of $\mathbb{P}(M)$ then there is a unique point of $\mathbb{P}(M)$, say p_3 with $\mathrm{H}(p_0, p_1, p_2, p_3)$. This is the well known *uniqueness of the fourth harmonic point*. Since $\mathrm{H}(p_0, p_1, p_2, p_3)$ is equivalent to $\mathrm{H}(p_0, p_1, p_3, p_2)$, there holds as well the *uniqueness of the third harmonic point*. The latter (less prominent) property will be used when proving Lemma 2.

Let M' be a free left module of rank 2 over a ring R'. A mapping $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ will be called a *harmonicity preserver* if it takes all harmonic quadruples of $\mathbb{P}(M)$ to harmonic quadruples of $\mathbb{P}(M')$. No further assumptions, like injectivity or surjectivity of μ are made here. A simple, though important, property is that any harmonicity preserver $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ is *distant preserving*, i. e.,

$$p_0 \triangle p_1 \quad \text{implies} \quad p_0^\mu \triangle p_1^\mu \quad \text{for all} \quad p_0, p_1 \in \mathbb{P}(M). \tag{8}$$

This follows readily from the existence of points p_2 and p_3 with $\mathrm{H}(p_0, p_1, p_2, p_3)$.

We close this section by quoting several examples of harmonicity preservers $\mathbb{P}(M) \to \mathbb{P}(M')$.

Example 1. Let $\alpha : R \to R'$ be a Jordan homomorphism, i. e. a mapping satisfying

$$(x+y)^\alpha = x^\alpha + y^\alpha, \quad 1^\alpha = 1', \quad (xyx)^\alpha = x^\alpha y^\alpha x^\alpha \quad \text{for all} \quad x, y \in R.$$

See, among others, [31, p. 832] or [38, p. 2]. Also, let C be any connected component of the distant graph $(\mathbb{P}(M), \triangle)$. We select bases (e_0, e_1) and (e'_0, e'_1) of M and M', respectively, subject to the condition $Re_0 \in C$. According to a result of A. Blunck and the author [17, Thm. 4.4] the following (rather cumbersome) construction gives a well defined mapping

$$\mu : C \to \mathbb{P}(M') : p \mapsto p^\mu. \tag{9}$$

By (4), any point $p \in C$ can be written in the form $p = R(x_0e_0 + x_1e_1)$ with

$$(x_0, x_1) = (1, 0) \cdot E(T)$$

for some $T \in \mathcal{S}(R)$, say $T = (t_1, t_2, \ldots, t_n)$ with $n \geq 0$. We use the shorthand $T^\alpha := (t_1^\alpha, t_2^\alpha, \ldots, t_n^\alpha) \in \mathcal{S}(R')$ and let

$$(x'_0, x'_1) := (1, 0) \cdot E(T^\alpha). \tag{10}$$

The point p^μ is defined as $R'(x'_0e'_0 + x'_1e'_1)$. By [17, Prop. 4.8], $\mathrm{H}(p_0, p_1, p_2, p_3)$ implies $\mathrm{H}(p_0^\mu, p_1^\mu, p_2^\mu, p_3^\mu)$ for all $p_0, p_1, p_2, p_3 \in C$.

The previous construction can be repeated for all connected components of the distant graph on $\mathbb{P}(M)$. Thereby is not necessary to stick to a fixed Jordan homomorphism. Altogether this gives a globally defined harmonicity preserver $\mathbb{P}(M) \to \mathbb{P}(M')$.

One particular case, due to C. Bartolone [5], deserves special mention: Let R be a ring of *stable rank* 2 [56, p. 1039]. Then $(\mathbb{P}(M), \triangle)$ has a single connected component, each of its points can be described in terms of at least one finite sequence $T = (t_1, t_2) \in R^2$, and μ can be rewritten as

$$\mu : \mathbb{P}(M) \to \mathbb{P}(M') : R((t_1t_2 - 1)e_0 + t_1e_1) \mapsto R'((t_1^\alpha t_2^\alpha - 1)e'_0 + t_1^\alpha e'_1).$$

Example 2. We adopt the settings of Example 1, but we make the extra assumption that α is a homomorphism of rings. Then

$$\sigma : M \to M' : x_0e_0 + x_1e_1 \mapsto x_0^\alpha e'_0 + x_1^\alpha e'_1 \quad \text{for all} \quad x_0, x_1 \in R$$

is an α-semilinear mapping and

$$\alpha_* : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R') : X \mapsto X^\alpha,$$

i. e., α is applied to each entry of X, is a homomorphism of groups. Thus for any basis (a, b) of M the image (a^σ, b^σ) is a basis of M'. Consequently, the mapping

$$\lambda : \mathbb{P}(M) \to \mathbb{P}(M') : Ra \mapsto R'(a^\sigma) \quad \text{(with } a \in M \text{ admissible)}$$

is well defined, and it preserves harmonicity. The mapping μ from (9) is the *restriction* of λ to C. The matrix $E(T^\alpha)$ from (10) now can be expressed as $E(T)^{\alpha_*}$, since $E(t)^{\alpha_*} = E(t^\alpha)$ for all $t \in R$.

Example 3. We adopt the settings of Example 1, but we make the extra assumption that α is an antihomomorphism of rings. We have the homomorphism

$$\alpha_{**} : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R') : X \mapsto E(0)^{-1} \cdot ((X^{-1})^{\mathrm{T}})^\alpha \cdot E(0), \tag{11}$$

where $(X^{-1})^{\mathrm{T}}$ denotes the transpose of X^{-1} and α is applied entrywise. (We must not use α_* in (11), since $(X^{-1})^{\mathrm{T}}$ need not be invertible.) A straightforward calculation shows $E(t)^{\alpha_{**}} = E(t^\alpha)$ for all $t \in R$. Hence the matrix $E(T^\alpha)$ from (10) now can be expressed as $E(T)^{\alpha_{**}}$. This leads us to the definition of a mapping

$$\delta : \mathbb{P}(M) \to \mathbb{P}(M') : R(x_0 e_0 + x_1 e_1) \mapsto R'(x_0' e_0' + x_1' e_1')$$

which runs as follows: (x_0, x_1) is chosen as the first row of any matrix $X \in \mathrm{GL}_2(R)$ and (x_0', x_1') is defined as the first row of the matrix $X^{\alpha_{**}}$. By [17, Ex. 4.8] this mapping is well defined. An equivalent (and more lucid) definition of δ in terms of the dual module of M can be read off from [16, Rem. 5.4] or [30, Prop. 3.3]. Formula (6) provides an easy direct proof for δ being a harmonicity preserver. The mapping μ from (9) is the *restriction* of δ to C.

It may happen that $\alpha : R \to R'$ is a homomorphism and an antihomomorphism. Then R^α is a commutative subring of R' and we have $(\det X^{\alpha_*}) X^{\alpha_{**}} = X^{\alpha_*}$ for all $X \in \mathrm{GL}_2(R)$. So in this case the mappings λ and δ coincide.

3 Von Staudt's theorem

We already noted in Section 2 that the distant graph on $\mathbb{P}(M)$ has a single connected component if, and only if, R is a GL_2-ring. In this case the following version of von Staudt's theorem provides a unified algebraic description of harmonicity preservers, otherwise it gives only a description on an arbitrarily chosen connected component.

Theorem 1. *Let M and M' be free modules of rank 2 over rings R and R', respectively. Furthermore, let R satisfy the two conditions:*

(i) *Given $x_1, x_2, \ldots, x_5 \in R$ there exists $x \in R$ such that $x - x_1, x - x_2, \ldots, x - x_5$ are units in R.*

(ii) 2 *is a unit in R.*

Let $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ be a harmonicity preserver. Choose any connected component, say C, of the distant graph $(\mathbb{P}(M), \triangle)$. Then there exist a basis (a_0, a_1) of M, a basis (a'_0, a'_1) of M', and a Jordan homomorphism $\alpha : R \to R'$ such that the restriction of μ to C admits the following description:

$$\mu|C : C \to \mathbb{P}(M') : R(x_0 a_0 + x_1 a_1) \mapsto R(x'_0 a'_0 + x'_1 a'_1),$$

where

$$(x_0, x_1) = (1, 0) \cdot E(T), \quad (x'_0, x'_1) = (1, 0) \cdot E(T^\alpha), \tag{12}$$

and T is any finite sequence of elements in R.

We postpone the proof until we have established four auxiliary results. In all of them we tacitly adopt the assumptions of Theorem 1. Lemma 1 is self-explanatory. In Lemma 2 we exhibit a mapping $\beta : R \to R'$ which can be viewed as a "local coordinate representation of μ". Next, in Lemma 3, we establish that "new local coordinates" (describing other parts of the given projective lines) can be chosen in such a way that the "new local coordinate representations" of μ coincides with the "old" one. This observation is the backbone of our demonstration. Afterwards, in Lemma 4, the mapping β is shown to be a Jordan homomorphism. The actual proof Theorem 1 amounts then to verifying that the given mapping $\mu|C$ coincides with the harmonicity preserver which arises from β according to Example 1. It goes without saying that part of our demonstration follows the same lines as previous work by other authors. Condition (i) is taken from [49]. It is equivalent to the following property of the projective line $\mathbb{P}(M)$:

(i') Given points $p_1, p_2, \ldots, p_5 \in \mathbb{P}(M)$, all of which are distant to some point $p_0 \in \mathbb{P}(M)$, there exists $p \in \mathbb{P}(M)$ which is distant to $p_0, p_1, \ldots, p_5$.

The equivalence follows easily from (2) upon choosing a basis (e_0, e_1) of M with $p_0 = Re_0$. Then $p_i = R(x_i e_0 + e_1)$ for $i \in \{1, 2, \ldots, 5\}$ and $p = R(xe_0 + e_1)$. Take notice that neither the elements $x_1, x_2, \ldots, x_5$ nor the points $p_1, p_2, \ldots, p_5$ are assumed to be distinct.

Lemma 1. 2 *is a unit in R'.*

Proof. Since M is free of rank 2, there exists a harmonic quadruple (p_0, p_1, p_2, p_3) in $\mathbb{P}(M)^4$. We read off $p_2 \triangle p_3$ from (7) and (ii). Application of μ yields $p_2^\mu \triangle p_3^\mu$ by virtue of (8). Now (7) in turn shows that 2 is a unit in R'. □

Lemma 2. *Given bases* (e_0, e_1) *of* M *and* (e'_0, e'_1) *of* M' *such that*

$$(Re_0)^\mu = R'e'_0, \quad (Re_1)^\mu = R'e'_1, \quad (R(e_0 \pm e_1))^\mu = R'(e'_0 \pm e'_1) \tag{13}$$

there exists a unique mapping $\beta : R \to R'$ *with the property*

$$(R(xe_0 + e_1))^\mu = R'(x^\beta e'_0 + e'_1) \quad \textit{for all} \quad x \in R. \tag{14}$$

This β *is additive and satisfies* $1^\beta = 1$.

Proof. For any $x \in R$ the point $p := R(xe_0 + e_1)$ is distant from Re_0. From (8) follows $p^\mu \vartriangle R'e'_0$ so that the point p^μ has a unique generator of the form $x'e'_0 + e'_1$ with $x' \in R'$. We therefore can define a unique mapping $\beta : R \to R'$ satisfying condition (14) by $x^\beta := x'$.

By (2), for all $x, y \in R$ with $x - y \in R^*$ the points

$$q_0 := R(xe_0 + e_1), \qquad q_1 := R(ye_0 + e_1),$$
$$q_2 := R((x + y)e_0 + 2e_1), \qquad q_3 := R((x - y)e_0) = Re_0$$

satisfy $\mathrm{H}(q_0, q_1, q_2, q_3)$. From (14), condition (ii), and (13) follows

$$\begin{aligned} q_0^\mu &= R(x^\beta e'_0 + e'_1), & q_1^\mu &= R(y^\beta e'_0 + e'_1), \\ q_2^\mu &= R\left(\left(\frac{x+y}{2}\right)^\beta e'_0 + e'_1\right), & q_3^\mu &= Re'_0. \end{aligned} \tag{15}$$

We infer from (8) that $q_0^\mu \vartriangle q_1^\mu$, and so $(x^\beta e'_0 + e'_1, y^\beta e'_0 + e'_1)$ is a basis of M'. Now (2) yields that $x^\beta - y^\beta$ is a unit in R', whence $q_3^\mu = R'((x^\beta - y^\beta)e'_0)$. By defining

$$q'_2 := R'((x^\beta + y^\beta)e'_0 + 2e'_1) \tag{16}$$

we obtain $\mathrm{H}(q_0^\mu, q_1^\mu, q'_2, q_3^\mu)$. The uniqueness of the third harmonic point (see Section 2) shows $q'_2 = q_2^\mu$. Comparing (15) with (16) and taking into account Lemma 1 gives

$$\left(\frac{x+y}{2}\right)^\beta = \frac{x^\beta + y^\beta}{2} \quad \text{for all} \quad x, y \in R \quad \text{with} \quad x - y \in R^*. \tag{17}$$

Also $(Re_1)^\mu = R'e'_1$ implies $0^\beta = 0$.

Due to the last observation, condition (i), Lemma 1, and (17), we can apply the first part of [49, Lemma 1]. This establishes that β is additive. Moreover, (13) implies $1^\beta = 1$. $\square$

Lemma 3. *Let* (e_0, e_1), (e'_0, e'_1) *and* β *be given as in Lemma* 2. *Let* $t \in R$ *be fixed. Then*

$$(f_0, f_1) := (te_0 + e_1, -e_0) \quad \textit{and} \quad (f'_0, f'_1) := (t^\beta e'_0 + e'_1, -e'_0) \tag{18}$$

are bases of M *and* M', *respectively, and there holds*

$$(R(xf_0 + f_1))^\mu = R'(x^\beta f'_0 + f'_1) \quad \textit{for all} \quad x \in R. \tag{19}$$

Proof. (a) The matrix of (f_0, f_1) w. r. t. (e_0, e_1) is $E(t) \in \mathrm{E}_2(R)$. So (f_0, f_1) is a basis of M. Likewise, $E(t^\beta) \in \mathrm{E}_2(R')$ shows that (f_0', f_1') is a basis of M'. We deduce $(Rf_0)^\mu = R'f_0'$ from (14), whereas (13) yields $(Rf_1)^\mu = R'f_1'$. Now the additivity of β together with $1^\beta = 1$ gives

$$(R(f_0 \pm f_1))^\mu = \Big(R((t \mp 1)e_0 + e_1)\Big)^\mu = R'((t^\beta \mp 1)e_0' + e_1') = R'(f_0' \pm f_1'). \qquad (20)$$

Consequently, as in Lemma 2, there is a unique mapping $\gamma : R \to R'$ such that

$$(R(xf_0 + f_1))^\mu = R'(x^\gamma f_0' + f_1') \quad \text{for all} \quad x \in R. \qquad (21)$$

Also, as before, γ turns out to be additive with $1^\gamma = 1$.

(b) Consider a fixed $x \in R$ such that $1 + x$ and $1 - x$ are units. We define

$$\begin{aligned}
g_0 &:= (t + 1)e_0 + e_1 = f_0 - f_1,\\
g_1 &:= (t - 1)e_0 + e_1 = f_0 + f_1,\\
g_2 &:= 2((t + x)e_0 + e_1) = 2(f_0 - xf_1),\\
g_3 &:= 2((1 + xt)e_0 + xe_1) = 2(xf_0 - f_1).
\end{aligned}$$

The matrix of (g_0, g_1) w. r. t. (f_0, f_1) is in $\mathrm{GL}_2(R)$ due to $2 \in R^*$ and (2), whence (g_0, g_1) is a basis. The equations $(1+x)g_0+(1-x)g_1 = g_2$ and $(1+x)g_0-(1-x)g_1 = g_3$ yield that the points $p_i := Rg_i$, $i \in \{0, 1, 2, 3\}$, satisfy $\mathrm{H}(p_0, p_1, p_2, p_3)$. We define

$$\begin{aligned}
g_0' &:= ((t^\beta + 1)e_0' + e_1'),\\
g_1' &:= ((t^\beta - 1)e_0' + e_1'),\\
g_2' &:= 2((t^\beta + x^\beta)e_0' + e_1'),\\
g_3' &:= 2((1 + x^\beta t^\beta)e_0' + x^\beta e_1'),
\end{aligned}$$

whence Lemma 2 gives

$$p_0^\mu = R'g_0', \quad p_1^\mu = R'g_1', \quad p_2^\mu = R'g_2'. \qquad (22)$$

Now $\mathrm{H}(p_0^\mu, p_1^\mu, p_2^\mu, p_3^\mu)$ implies $p_0^\mu \vartriangle p_2^\mu \vartriangle p_1^\mu$ so that

$$\begin{pmatrix} t^\beta \pm 1 & 1 \\ t^\beta + x^\beta & 1 \end{pmatrix} \in \mathrm{GL}_2(R')$$

which in turn, by (2), gives that $1 + x^\beta$ and $1 - x^\beta$ are units in R'. We therefore are in a position to proceed as above in order to establish $\mathrm{H}(R'g_0', R'g_1', R'g_2', R'g_3')$. By (22) and the uniqueness of the fourth harmonic point, we obtain

$$p_3^\mu = R'g_3' = R'((1 + x^\beta t^\beta)e_0' + x^\beta e_1') = R'(x^\beta f_0' - f_1').$$

On the other hand, writing $p_3 = R((-x)f_0 + f_1)$ allows us to apply (21) which gives $p_3^\mu = R'((-x)^\gamma f_0' + f_1')$. The additivity of γ yields

$$x^\beta = x^\gamma \quad \text{for all} \quad x \in R \quad \text{with} \quad 1 + x \quad \text{and} \quad 1 - x \quad \text{units.} \tag{23}$$

(c) If x is any element of R then, by condition (i), there exists $y \in R$ with $1 + y$, $1 - y$, $1 + (x + y)$, and $1 - (x + y)$ units. We infer $y^\beta = y^\gamma$ and $(x + y)^\beta = (x + y)^\gamma$ from (23) whence, by the additivity of β and γ, we obtain

$$x^\beta = x^\gamma \quad \text{for all} \quad x \in R.$$

This completes the proof of (19). □

Lemma 4. *The mapping β from Lemma 2 is a Jordan homomorphism.*

Proof. We make use of Lemma 3 in the special case $t = 0$, i. e., $(f_0, f_1) = (e_1, -e_0)$ and $(f_0', f_1') = (e_1', -e_0')$. Given any $x \in R^*$ we calculate the image of $R(xe_0 + e_1) = R(-x^{-1}f_0 + f_1)$ according to (14) and (19). This gives

$$R'(x^\beta e_0' + e_1') = R'((-x^{-1})^\beta f_0' + f_1') = R'((-x^{-1})^\beta e_1' - e_0').$$

Since $x^\beta e_0' + e_1'$ and $(-x^{-1})^\beta e_1' - e_0'$ are admissible generators of the same point, there exists a unit $u' \in R'$ with $u'(x^\beta e_0' + e_1') = (-x^{-1})^\beta e_1' - e_0'$. Now $u'x^\beta = -1$ implies that x^β is a unit in R' and, by the additivity of β, we obtain

$$(x^\beta)^{-1} = (x^{-1})^\beta \quad \text{for all} \quad x \in R^*. \tag{24}$$

Due to $1^\beta = 1$ and (24) we are in a position to apply also the second part of [49, Lemma 1] which establishes that β satisfies

$$(xy + yx)^\beta = x^\beta y^\beta + y^\beta x^\beta \quad \text{for all} \quad x, y \in R. \tag{25}$$

Recall that 2 is a unit in R by condition (ii), and also a unit in R' by Lemma 1. Moreover, from Lemma 2, β is additive and satisfies $1^\beta = 1$. It is well known that under these circumstances (25) characterises β as being a Jordan homomorphism; see, e. g., [29, p. 47] or [36, p. 320]. □

Proof of Theorem 1. Choose any point of the connected component C, say $p_0 = Ra_0$, and any $a_1 \in M$ such that (a_0, a_1) is a basis of M. Let $p_1 := Ra_1$, $p_2 := R(a_0 + a_1)$, and $p_3 := R(a_0 - a_1)$. Then $\mathrm{H}(p_0, p_1, p_2, p_3)$ implies $\mathrm{H}(p_0^\mu, p_1^\mu, p_2^\mu, p_3^\mu)$ so that there exists a basis (a_0', a_1') of M' satisfying

$$(Ra_0)^\mu = R'a_0', \quad (Ra_1)^\mu = R'a_1', \quad (R(a_0 \pm a_0))^\mu = R'(a_0' \pm a_1'). \tag{26}$$

We apply Lemma 2 to the bases (a_0, a_1) and (a'_0, a'_1), but relabel the mapping β from there as α. So, by (14) and Lemma 4, there exists a Jordan homomorphism $\alpha : R \to R'$ with

$$\big(R(xa_0 + a_1)\big)^\mu = R'(x^\alpha a'_0 + a'_1) \quad \text{for all} \quad x \in R. \tag{27}$$

By (4), a point $p \in \mathbb{P}(M)$ belongs to C precisely when there is at least one sequence $T \in \mathcal{S}(R)$ such that $p = R(x_0a_0 + x_1a_1)$ with $(x_0, x_1) = (1, 0) \cdot E(T)$. It therefore remains to verify that for all finite sequences $T \in \mathcal{S}(R)$ the coordinate rows (x_0, x_1) and (x'_0, x'_1) from (12) define points which correspond under μ. We proceed by induction on the length of T which will be denoted by n.

For $n = 0$ the sequence T is empty and $E()$ is the identity matrix. Now (12) reads $(x_0, x_1) = (1, 0)$, $(x'_0, x'_1) = (1, 0)$, and indeed $(Ra_0)^\mu = R'a'_0$ according to (26).

For $n = 1$ we have $T = (t_1)$ with $t_1 \in R$. The assertion follows from (27), since (12) now takes the form $(x_0, x_1) = (t_1, 1)$, $(x'_0, x'_1) = (t_1^\alpha, 1)$.

Let $n \geq 2$ and suppose $T = (t_1, t_2, \ldots, t_n) \in \mathcal{S}(R)$. There is a unique basis of M, say (e_0, e_1), with $E(t_3, \ldots, t_n)$ being its matrix w. r. t. (a_0, a_1). We proceed analogously in M' and obtain a basis (e'_0, e'_1) with $E(t_3^\alpha, \ldots, t_n^\alpha)$ being its matrix w. r. t. (a'_0, a'_1). The following table displays for all $x \in R$ the coordinates of certain elements of M and M':

Coordinates w. r. t. (a_0, a_1)		Coordinates w. r. t. (a'_0, a'_1)	
e_0	$(1, 0) \cdot E(t_3, \ldots, t_n)$	e'_0	$(1, 0) \cdot E(t_3^\alpha, \ldots, t_n^\alpha)$
$xe_0 + e_1$	$(1, 0) \cdot E(x, t_3, \ldots, t_n)$	$x^\alpha e'_0 + e'_1$	$(1, 0) \cdot E(x^\alpha, t_3^\alpha, \ldots, t_n^\alpha)$

(28)

Those elements of M and M' which appear in the same row of table (28) generate corresponding points under μ due to the induction hypothesis. In particular, as x ranges in $\{0, 1, -1\}$, we get

$$(Re_0)^\mu = R'e'_0, \quad (Re_1)^\mu = R'e'_1, \quad \big(R(\pm e_0 + e_1)\big)^\mu = R'(\pm e'_0 + e'_1).$$

Hence the bases (e_0, e_1) and (e'_0, e'_1) satisfy (13) so that Lemma 2 can be applied to them (without any notational changes). We claim that α, as defined via (27), coincides with the Jordan homomorphism β appearing in Lemma 2: Indeed, α satisfies the defining equation (14) according to the second row of table (28) in conjunction with the induction hypothesis. We now introduce bases (f_0, f_1) of M and (f'_0, f'_1) of M' as in Lemma 3, but replace the arbitrary $t \in R$ from there by the given $t_2 \in R$. This gives a second table of coordinates:

Coordinates w. r. t. (a_0, a_1)		Coordinates w. r. t. (a'_0, a'_1)	
f_0	$(1, 0) \cdot E(t_2, \ldots, t_n)$	f'_0	$(1, 0) \cdot E(t_2^\alpha, \ldots, t_n^\alpha)$
$t_1f_0 + f_1$	$(1, 0) \cdot E(t_1, t_2, \ldots, t_n)$	$t_1^\alpha f'_0 + f'_1$	$(1, 0) \cdot E(t_1^\alpha, t_2^\alpha, \ldots, t_n^\alpha)$

(29)

Since $\alpha = \beta$, we can read off from (19) that $(R(t_1 f_0 + f_1))^\mu = R'(t_1^\alpha f_0' + f_1')$. Hence the coordinates from the last row of table (29) describe points which correspond under μ. □

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