Symplectic Plücker Transformations

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Dedicated to Hans Vogler on the occasion of his 60th birthday

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Abstract: *Plücker transformations of symplectic spaces with dimensions other than three are induced by orthogonality-preserving collineations. For three-dimensional symplectic spaces all Plücker transformations can be obtained - up to orthogonality-preserving collineations - by replacing some but not necessarily all non-isotropic lines by their absolute polar lines.*

1. Introduction

In this paper we discuss bijections of the set $\mathcal{L}$ of lines of a symplectic space, i.e. a (not necessarily finite-dimensional) projective space with orthogonality based upon an absolute symplectic[1] quasipolarity. Following [1], two lines are called related, if they are concurrent and orthogonal, or if they are identical. A bijection of $\mathcal{L}$ that preserves this relation in both directions is called a (symplectic[2]) Plücker transformation. We shall show that any bijection $\mathcal{L} \to \mathcal{L}$ taking related lines to related lines is already a Plücker transformation. Moreover, a complete description of all Plücker transformations (cf. the abstract above) will be given.

2. Symplectic spaces

Let $(\mathcal{P},\mathcal{L})$ be a projective space, $3 \leq \dim(\mathcal{P},\mathcal{L}) \leq \infty$. Assume that π is a symplectic quasipolarity [11], [12]. Thus π assigns to each point X of $\mathcal{P}$ a

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[1]Instead of 'symplectic' some authors are using the term 'null'.

[2]We shall omit the word 'symplectic', since we do not discuss other types of Plücker transformations in this paper. Cf., however, [1], [2], [4], [5], [8,p.80ff], [9], [10] for results on other Plücker transformations.

hyperplane X^π with $X \in X^\pi$; furthermore $Y \in X^\pi$ implies $X \in Y^\pi$ for all $X,Y \in \mathcal{P}$. Cf. also [6] for an axiomatic description of projective spaces endowed with a quasipolarity.

We define a mapping from the lattice of subspaces of $(\mathcal{P},\mathcal{L})$ into itself by setting

$$\mathcal{T} \mapsto \bigcap(X^\pi \mid X \in \mathcal{T}) \text{ for all subspaces } \mathcal{T} \neq \emptyset \text{ and } \emptyset \mapsto \mathcal{P}. \tag{1}$$

This mapping is again written as π and is also called a quasipolarity. If $(\mathcal{P},\mathcal{L})$ is finite-dimensional, then it is well known that π is an antiautomorphism of the lattice of subspaces of $(\mathcal{P},\mathcal{L})$. In case of infinite dimension the mapping (1) still has the properties

$$(\mathcal{T}_1 \vee \mathcal{T}_2)^\pi = \mathcal{T}_1{}^\pi \cap \mathcal{T}_2{}^\pi, \quad (\mathcal{T}_1 \cap \mathcal{T}_2)^\pi \supset \mathcal{T}_1{}^\pi \vee \mathcal{T}_2{}^\pi, \quad \mathcal{T} \subset \mathcal{T}^{\pi\pi}$$

$$\mathcal{T}_1 \subset \mathcal{T}_2 \Rightarrow \mathcal{T}_1{}^\pi \supset \mathcal{T}_2{}^\pi$$

for all subspaces $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T} \subset \mathcal{P}$. Note that in the last formula strict inclusions are not necessarily preserved, if $\mathcal{T}_1$ and $\mathcal{T}_2$ both have infinite dimension[3]. Moreover, it is an easy induction to show for all finite-dimensional subspaces $\mathcal{T} \subset \mathcal{P}$ that $\mathcal{T}^{\pi\pi} = \mathcal{T}$ and that every complement of $\mathcal{T}^\pi$ has the same finite dimension as $\mathcal{T}$.

$(\mathcal{P},\mathcal{L},\pi)$ is a **symplectic space** with **absolute quasipolarity** π [7,p.384ff], [11]. In terms of an underlying vector space $\mathbf{V}$ of $(\mathcal{P},\mathcal{L})$ the symplectic quasipolarity π can be described by a non-degenerate alternating bilinear form of $\mathbf{V}\times\mathbf{V}$ into the (necessarily commutative) ground field of $\mathbf{V}$. If $(\mathcal{P},\mathcal{L})$ is finite-dimensional, then it is well known that $\dim(\mathcal{P},\mathcal{L})$ is odd.

We are introducing two binary relations on $\mathcal{L}$: Given $a,b \in \mathcal{L}$ then define a and b to be **orthogonal** ($\perp$), if $a \cap b^\pi \neq \emptyset$. The lines a and b are called **related** ($\sim$), if $a \perp b$ and $a \cap b \neq \emptyset$, or if $a = b$. Given orthogonal lines a,b there exists a point $R \in a \cap b^\pi$. Therefore

$$R^\pi \supset (a \cap b^\pi)^\pi \supset a^\pi \vee b^{\pi\pi} = a^\pi \vee b.$$

The line b has a point in common with a^π, since R^π is a hyperplane and a^π is a co-line. Consequently, $\perp$ and $\sim$ are symmetric relations.

Each line $a \in \mathcal{L}$ either is contained in a^π or is a complement of a^π, since $a \cap a^\pi$ being a single point would imply that $X^\pi = a \vee a^\pi$ for all points $X \in a \setminus a^\pi$, in contradiction to $\pi|\mathcal{P}$ being injective. A line $a \in \mathcal{L}$ is **isotropic** (self-orthogonal) if, and only if, a is **totally isotropic**, i.e., $a \subset a^\pi$. We shall write $\mathcal{I}$ for the set of all isotropic lines.

If Q is a point, then $\mathcal{L}[Q]$ stands for the star of lines with centre Q and

[3]There are, e.g., hyperplanes $\mathcal{H} \subset \mathcal{P}$ with $\mathcal{H} \neq X^\pi$ for all $X \in \mathcal{P}$. For all such hyperplanes $\mathcal{H}^\pi = \mathcal{P}^\pi = \emptyset$, although $\mathcal{H} \neq \mathcal{P}$.

$\mathcal{I}[Q] := \mathcal{L}[Q] \cap \mathcal{I}$ for the set of all isotropic lines through Q. In the following Lemma 1 we state two simple properties of isotropic lines that are well known in case of finite dimension [3,p.181ff], [7,p.384ff] but hold as well for infinite dimension:

Lemma 1. *If $Q \in \mathcal{P}$, then all isotropic lines through Q are given by*

$$\mathcal{I}[Q] = \{x \in \mathcal{L} \mid Q \in x \subset Q^{\pi}\}.$$

Let $a \in \mathcal{L} \backslash \mathcal{I}$ be non-isotropic. The set of isotropic lines intersecting the line a equals the set of all lines intersecting both a and a^{π}.

Proof. Let a line x with $Q \in x \subset Q^{\pi}$ be given. This implies $Q^{\pi} \supset x^{\pi}$ so that x and x^{π} are in the same hyperplane Q^{π}. Since x^{π} is a co-line, x and x^{π} cannot be skew, i.e. $x \in \mathcal{I}[Q]$. On the other hand, from $x \in \mathcal{I}[Q]$ follows immediately that $x \subset x^{\pi} \subset Q^{\pi}$.

Next let $a \in \mathcal{L} \backslash \mathcal{I}$. If $b \in \mathcal{I}$ intersects a at a point Q, say, then $Q \in b \subset b^{\pi}$ implies $b^{\pi\pi} = b \subset Q^{\pi}$, whereas $Q \in a$ tells us $a^{\pi} \subset Q^{\pi}$. Thus, as before, b and a^{π} are not skew. Conversely, given points $Q \in a$ and $R \in a^{\pi}$ then $R \in a^{\pi} \subset Q^{\pi}$ and $Q \in a \subset R^{\pi}$, whence $Q \vee R \subset Q^{\pi} \cap R^{\pi} = (Q \vee R)^{\pi}$.■

We apply this result to show

Lemma 2. *Distinct lines $a,b \in \mathcal{L}$ with $a \cap b \neq \varnothing$ are related if, and only if, $a \in \mathcal{I}$ or $b \in \mathcal{I}$.*

Proof. If one of the given lines is isotropic, then $a \sim b$. Conversely, if $a \sim b$ and $a \notin \mathcal{I}$, say, then $b \in \mathcal{I}$ by Lemma 1.■

As an immediate consequence we obtain

Lemma 3. *Let $\mathcal{M}$ be a set of mutually related lines. Then at most one line of $\mathcal{M}$ is non-isotropic.*

Given lines $a,b \in \mathcal{L}$ then there is always a finite sequence

$$a \sim a_1 \sim \ldots \sim a_n \sim b:$$

This is trivial when $a = b$. If $a \cap b =: Q$ is a point, then there exists a line $a_1 \in \mathcal{I}[Q]$ so that $a \sim a_1 \sim b$ by Lemma 2. If a and b are skew then there exists a common transversal line of a and b, say c, whence repeating the previous construction for a,c and then for c,b gives the required sequence. Thus $(\mathcal{L},\sim)$

is a **Plücker space**[4] [1,p.199]. A (symplectic) **Plücker transformation** is a bijective mapping $\varphi : \mathcal{L} \to \mathcal{L}$ preserving the relation $\sim$ in both directions. We say that φ is **induced** by a mapping $\kappa : \mathcal{P} \to \mathcal{P}$, if

$$(A \vee B)^{\varphi} = A^{\kappa} \vee B^{\kappa} \text{ for all } A,B \in \mathcal{P} \text{ with } A \neq B.$$

The group $\mathrm{P\Gamma Sp}(\mathcal{P},\pi)$ consists of all collineations $\mathcal{P} \to \mathcal{P}$ commuting with π [7,p.388ff], [8,p.19]. Obviously, each $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$ is inducing a Plücker transformation.

If $\dim(\mathcal{P},\mathcal{L}) = 3$, then for each duality τ with $\mathcal{J}^{\tau} = \mathcal{J}$ the restriction $\tau|\mathcal{L} : \mathcal{L} \to \mathcal{L}$ is a Plücker transformation. Moreover, in the three-dimensional case there are always Plücker transformations not arising from collineations or dualities: Let $\mathcal{L}_1$ be any subset of $\mathcal{L}\setminus\mathcal{J}$ such that $\mathcal{L}_1{}^{\pi} = \mathcal{L}_1$. Then define

$$\delta : \mathcal{L} \to \mathcal{L}, \begin{cases} x \mapsto x & \text{if } x \in \mathcal{L}\setminus\mathcal{L}_1, \\ x \mapsto x^{\pi} & \text{if } x \in \mathcal{L}_1. \end{cases} \tag{2}$$

Such a bijection δ will be called **partial π-transformation** (with respect to $\mathcal{L}_1$); it is a Plücker transformation of $(\mathcal{L},\sim)$, since

$$a \sim b \iff a \sim b^{\pi} \iff a^{\pi} \sim b \iff a^{\pi} \sim b^{\pi} \text{ for all } a,b \in \mathcal{L},\ a \neq b.$$

The identity on $\mathcal{L}$ and the restriction of π to $\mathcal{L}$ are partial π-transformations, as follows from setting $\mathcal{L}_1 := \varnothing$ and $\mathcal{L}_1 := \mathcal{L}\setminus\mathcal{J}$, respectively. For every other choice of $\mathcal{L}_1$ (e.g., $\mathcal{L}_1 := \{a,a^{\pi}\}$) it is easily seen that there exist two non-isotropic concurrent lines $x \in \mathcal{L}\setminus\mathcal{L}_1$, $y \in \mathcal{L}_1$. Then $x^{\delta} = x$ and $y^{\delta} = y^{\pi}$ are skew lines. Such a Plücker transformation cannot arise from a collineation or duality.

3. The three-dimensional case

Theorem 1. *Let $(\mathcal{P},\mathcal{L},\pi)$ be a 3-dimensional symplectic space and let $\beta : \mathcal{L} \to \mathcal{L}$ be a bijection such that*

$$a \sim b \textit{ implies } a^{\beta} \sim b^{\beta} \textit{ for all } a,b \in \mathcal{L}.$$

Then there exists a partial π-transformation $\delta : \mathcal{L} \to \mathcal{L}$ such that $\delta\beta$ is induced by a collineation $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

Theorem 1 is a consequence of the subsequent Propositions 1.1 - 1.4 in which β and $(\mathcal{P},\mathcal{L},\pi)$ are given as above.

Proposition 1.1. *There exists an injective mapping $\kappa : \mathcal{P} \to \mathcal{P}$ such that*

[4]Alternatively, $\mathcal{L}$ may be seen as the set of vertices of a graph with two vertices joined by an edge if, and only if, the corresponding lines are distinct and related. We refrain, however, from using terminology of graph theory.

$$\mathcal{I}[Q]^{\beta} = \mathcal{I}[Q^{\kappa}] \text{ for all } Q \in \mathcal{P}. \tag{3}$$

Moreover, β is a Plücker transformation, since

$$\mathcal{I}^{\beta} = \mathcal{I}. \tag{4}$$

Proof. By the invariance of $\sim$ under β, the elements of $\mathcal{I}[Q]^{\beta}$ are mutually related. We infer from Lemma 3 that $\mathcal{I}[Q]^{\beta}$ contains at most one non-isotropic line. Thus $\mathcal{I}[Q]^{\beta} \cap \mathcal{I}$ has at least two distinct elements, whence it is a subset of a pencil of isotropic lines, say $\mathcal{I}[Q']$ with $Q' \in \mathcal{P}$.

We show $\mathcal{I}[Q'] \subset \mathcal{I}[Q]^{\beta}$: Assume, to the contrary, that there exists a line $x \notin \mathcal{I}[Q]$ with $x^{\beta} \in \mathcal{I}[Q']$. Recall that at most one line of $\mathcal{I}[Q]^{\beta}$ is non-isotropic. Therefore there is a point $X' \in x^{\beta}$ that is not incident with any line of $\mathcal{I}[Q]^{\beta}$. Thus we can draw a line $b' = b^{\beta}$ through X' that is not related to any line of $\mathcal{I}[Q]^{\beta}$. Hence b is not related to any line of $\mathcal{I}[Q]$. By $\dim(\mathcal{P},\mathcal{L}) = 3$, b and the plane Q^{π} have a common point lying on some line $c \in \mathcal{I}[Q]$, so that $c \sim b$, a contradiction.

Next $\mathcal{I}[Q]^{\beta} \subset \mathcal{I}[Q']$ will be established: Suppose there is a line $a \in \mathcal{I}[Q]$ such that $a^{\beta} \notin \mathcal{I}[Q']$. Then $a^{\beta} \sim \mathcal{I}[Q']$ forces that a^{β} is a non-isotropic line either through the point Q' or in the plane Q'^{π}. Let $d \in \mathcal{L}[Q]$ be non-isotropic, whence $\mathcal{I}[Q]^{\beta} \cup \{d^{\beta}\}$ is a set of mutually related lines containing the non-isotropic line a^{β}. Since $d^{\beta} \notin \mathcal{I}[Q]^{\beta}$ and $\mathcal{I}[Q'] \subset \mathcal{I}[Q]^{\beta}$, the line $d^{\beta} \neq a^{\beta}$ also has to be non-isotropic in contradiction to Lemma 3.

To sum up, there is a mapping κ satisfying formula (3). The injectivity of κ follows from the bijectivity of β together with (3).

Finally, we prove (4): $\mathcal{I}^{\beta} \subset \mathcal{I}$ is a consequence of (3). Conversely, assume that $e \in \mathcal{L}\backslash\mathcal{I}$. Choose a point $R \in e$. Then $\mathcal{I}[R]^{\beta} \cup \{e^{\beta}\} = \mathcal{I}[R^{\kappa}] \cup \{e^{\beta}\}$ is a set of mutually related lines. Therefore e^{β} is a non-isotropic line either through R^{κ} or in $R^{\kappa\pi}$. Lemma 2 and $\mathcal{I}^{\beta} = \mathcal{I}$ imply that β is a Plücker transformation.■

Proposition 1.2. *Let $a \in \mathcal{L}$. Then*

$$a^{\beta\pi} = a^{\pi\beta}, \tag{5}$$

$$Q^{\kappa} \in a^{\beta} \cup a^{\pi\beta} \text{ for all } Q \in a. \tag{6}$$

If $a \in \mathcal{L}\backslash\mathcal{I}$, then either

$$Q^{\kappa} \in a^{\beta} \text{ for all } Q \in a \tag{7}$$

or

$$Q^{\kappa} \in a^{\pi\beta} \text{ for all } Q \in a. \tag{8}$$

Proof. If $a \in \mathcal{I}$, then $a^{\beta} \in \mathcal{I}$, whence (5) follows from $a = a^{\pi}$ and $a^{\beta} = a^{\beta\pi}$. If $a \in \mathcal{L}\backslash\mathcal{I}$, then, by Lemma 1,

$$\mathscr{C} := \{x \in \mathscr{L} \mid x \neq a,\ x \sim a\}$$

is a hyperbolic linear congruence of lines with axes a and a^π; moreover $\mathscr{C} \subset \mathscr{I}$. We infer from β being a Plücker transformation and (4), that $\mathscr{C}^\beta \subset \mathscr{I}$ is also a hyperbolic linear congruence with $a^\beta, a^{\pi\beta}$ being its axes. Obviously, only a^β and $a^{\pi\beta}$ are meeting all lines of $\mathscr{C}^\beta$. On the other hand, by Lemma 1, the axes of $\mathscr{C}^\beta$ are a^β and $a^{\beta\pi}$. This completes the proof of (5).

If $a \in \mathscr{I}$, then (6) holds true, since $Q^\kappa \in a^\beta = a^{\pi\beta}$. If $a \in \mathscr{L}\setminus\mathscr{I}$ and $Q^\kappa \notin a$, then $a^\beta \sim \mathscr{I}[Q]^\beta = \mathscr{I}[Q^\kappa]$, whence $Q^{\kappa\pi} \supset a^\beta$ and therefore $Q^\kappa \in a^{\beta\pi} = a^{\pi\beta}$, as required to establish (6).

Now let $a \in \mathscr{L}\setminus\mathscr{I}$. Assume to the contrary that there exist points $Q_0, Q_1 \in a$ such that ${Q_0}^\kappa \in a^\beta$ and ${Q_1}^\kappa \in a^{\pi\beta}$. Then $a \notin \mathscr{I}$ implies $\mathscr{I}[Q_0] \cap \mathscr{I}[Q_1] = \varnothing$ whereas, by Lemma 1 and (3), ${Q_0}^\kappa \vee {Q_1}^\kappa \in \mathscr{I}[Q_0]^\beta \cap \mathscr{I}[Q_1]^\beta$. This is a contradiction to β being injective.■

Proposition 1.3. *Write $\mathscr{L}_1$ for the set of all lines $a \in \mathscr{L}\setminus\mathscr{I}$ satisfying (8). Then*

$$\delta : \mathscr{L} \to \mathscr{L}, \begin{cases} x \mapsto x & \text{if } x \in \mathscr{L}\setminus\mathscr{L}_1, \\ x \mapsto x^\pi & \text{if } x \in \mathscr{L}_1, \end{cases} \tag{9}$$

is a partial π-transformation. The Plücker transformation $\delta\beta : \mathscr{L} \to \mathscr{L}$ takes intersecting lines to intersecting lines.

Proof. In order to show that δ is a well-defined partial π-transformation, we just have to establish that $a \in \mathscr{L}_1$ implies $a^\pi \in \mathscr{L}_1$: Given $Q_0 \in a$ and $Q_1 \in a^\pi$ then $Q_0 \vee Q_1$ and $(Q_0 \vee Q_1)^\beta = {Q_0}^\kappa \vee {Q_1}^\kappa$ are isotropic lines. Therefore

$${Q_0}^\kappa \vee {Q_1}^\kappa \neq a^{\beta\pi} = a^{\pi\beta} \notin \mathscr{I}$$

so that ${Q_1}^\kappa \notin a^{\pi\beta}$. Now, by (8), $a^\pi \in \mathscr{L}_1$.

If distinct lines b and c intersect at a point R, then $b^{\delta\beta} \cap c^{\delta\beta} = R^\kappa$ follows from (7), (8) and (9).■

Proposition 1.4. *The mapping $\kappa : \mathscr{P} \to \mathscr{P}$ defined in (3) belongs to* $\mathrm{P\Gamma Sp}(\mathscr{P},\pi)$. *The Plücker transformation $\delta\beta$ is induced by this collineation κ.*

Proof. The bijection $\delta\beta$ is taking intersecting lines to intersecting lines. Every star of lines is mapped under $\delta\beta$ either onto a star of lines or onto a ruled plane [4], [10,Theorem 1]. The latter possibility does not occur, since $\delta\beta$ is induced by κ. Because of $\dim(\mathscr{P},\mathscr{L})$ being finite, the mapping κ is a collineation [10,Theorem 3]. Finally, $\mathscr{I}^\beta = \mathscr{I}$ implies $\kappa \in \mathrm{P\Gamma Sp}(\mathscr{P},\pi)$.■

4. The higher-dimensional case

Theorem 2. *Let* $(\mathcal{P},\mathcal{L},\pi)$ *be an* n*-dimensional symplectic space* $(5 \leq n \leq \infty)$ *and let* $\beta : \mathcal{L} \to \mathcal{L}$ *be a bijection such that*

$$a \sim b \text{ implies } a^\beta \sim b^\beta \text{ for all } a,b \in \mathcal{L}.$$

Then β *is induced by a collineation* $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

As before, the Theorem will be split into several Propositions subject to the assumptions stated above.

Proposition 2.1. *The bijection* β *takes intersecting lines to intersecting lines. There exists an injective mapping* $\kappa : \mathcal{P} \to \mathcal{P}$ *inducing* β. *This* κ *is preserving collinearity and non-collinearity of points. Moreover*

$$\mathcal{L}[Q]^\beta = \mathcal{L}[Q^\kappa] \text{ for all } Q \in \mathcal{P}. \tag{10}$$

Proof. Suppose that $a,b \in \mathcal{L}$ meet at a point Q. If $a \sim b$, then a^β and b^β are intersecting. Otherwise, by Lemma 2, $a \notin \mathcal{J}$ and $b \notin \mathcal{J}$. Then $\mathcal{J}[Q] \cup \{a\}$ and $\mathcal{J}[Q] \cup \{b\}$ are, respectively, sets of mutually related lines. Each line of $\mathcal{L}$ is related to at least one line in $\mathcal{J}[Q]$, since Q^π is a hyperplane covered by $\mathcal{J}[Q]$. If $\mathcal{J}[Q]^\beta$ would be a set of coplanar lines, then all lines in $\mathcal{L}$ would meet a fixed plane in contradiction to $n \geq 5$. Thus $\mathcal{J}[Q]^\beta$ is not contained in a plane, whence there exists a point Q' with $\mathcal{J}[Q]^\beta \subset \mathcal{L}[Q']$. Since the elements of $\mathcal{J}[Q]^\beta \cup \{a^\beta\}$ are mutually related, $Q' \in a^\beta$. Repeating this for b yields $Q' \in b^\beta$.

Now the assertions on κ follow from [10,Theorem 1].■

Proposition 2.2. *The bijection* β *is a Plücker transformation, since*

$$\mathcal{J}^\beta = \mathcal{J}. \tag{11}$$

Proof. Given $a \in \mathcal{J}$ then choose a point $Q \in a$. We observe that $a \sim \mathcal{L}[Q]$, whence $a^\beta \sim \mathcal{L}[Q^\kappa]$ by (10). Since $\mathcal{L}[Q^\kappa]$ contains more than one non-isotropic line, $a^\beta \in \mathcal{J}$ follows from Lemma 2.

Given $b \in \mathcal{L}\backslash\mathcal{J}$ then choose a point $R \in b$. Assume to the contrary that $b^\beta \in \mathcal{J}$. Then for each line

$$x \in \mathcal{L}[R] \setminus (\mathcal{J}[R] \cup \{b\})$$

there exists a line $\overline{x} \in \mathcal{J}[R]$ such that $b,x,\overline{x}$ are three distinct lines in one pencil. By the invariance of collinearity and non-collinearity of points under κ, as is stated in Proposition 2.1, $b^\beta,x^\beta,\overline{x}^\beta$ are again three distinct lines in one pencil. However, b^β and $\overline{x}^\beta$ are isotropic, so that

$$x^\beta \in \mathcal{J}[R^\kappa].$$

Hence $\mathcal{L}[R]^{\beta} \subset \mathcal{I}[R^{\kappa}]$ which is impossible by (10).

Now (11) and Lemma 2 show that β is a Plücker transformation.■

Proposition 2.3. *The mapping* $\kappa : \mathcal{P} \to \mathcal{P}$, *described in Proposition 2.1, is a collineation belonging to* $\mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

Proof. Since β is a Plücker transformation of $(\mathcal{L},\sim)$, Proposition 2.1 can be applied to β^{-1}. Therefore β and β^{-1} are preserving intersection of lines. By [10,Theorem 2], the mapping κ is a collineation and, by formula (11), $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.■

This completes the proof of Theorem 2.

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