Divisible designs from twisted dual numbers

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Dedicated to Helmut Mäurer on the occasion of his 70th birthday

Abstract

The generalized chain geometry over the local ring $K(\varepsilon;\sigma)$ of twisted dual numbers, where K is a finite field, is interpreted as a divisible design obtained from an imprimitive group action. Its combinatorial properties as well as a geometric model in 4-space are investigated.
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1 Preliminaries

This paper deals with a special class of *divisible designs*, namely, those that are chain geometries over certain finite local rings, and their representation in projective space.

A finite geometry $\Sigma = (\mathcal{P}, \mathcal{B}, \parallel)$, consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{B}$ of *blocks*, and an equivalence relation $\parallel$ (*parallel*) on $\mathcal{P}$, is called a t-(s,k,λ_t)-*divisible design* (t-DD for short), if there exist positive integers t, s, k, λ_t such that the following axioms hold:

- Each block B is a subset of $\mathcal{P}$ containing k pairwise non-parallel points.
- Each parallel class consists of s points.
- For each set Y of t pairwise non-parallel points there exist exactly λ_t blocks containing Y.

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- $t \le k \le v/s$, where $v := |\mathcal{P}|$.

Note that sometimes DDs are called "group divisible designs".

A DD with trivial parallel relation, i.e. with $s = 1$, is an ordinary *design.* A DD with $k = v/s$ is called *transversal.* In the subsequent sections we shall deal with transversal 3-DDs.

A method to construct DDs with a large group of automorphisms is due to A.G. Spera [11], using imprimitive group actions: Let G be a group acting on a (finite) set $\mathcal{P}$ of "points" and leaving invariant an equivalence relation $\|$ ("parallel"). Let t be a positive integer such that there are at least t parallel classes, and let B_0 be a set of $k \ge t$ pairwise non-parallel points (the "base block"). Assume that G acts transitively on the set of t-tuples of pairwise non-parallel points. Let $\mathcal{B}$ be the orbit of B_0 under G, i.e. $\mathcal{B} = \{B_0^g \mid g \in G\}$. Then $\Sigma = (\mathcal{P}, \mathcal{B}, \|)$ is a t-DD with

$$\lambda_t = \frac{|G|}{|G_{B_0}|} \cdot \frac{\binom{k}{t}}{\binom{v/s}{t} s^t} \tag{1}$$

where G_{B_0} is the (setwise) stabilizer of B_0 in G (see [11, Prop. 2.3)]).

The projective line $\mathbb{P}(R)$ over a finite local ring R is endowed with an equivalence relation (usually denoted by $\|$). It is invariant under the action of the general linear group $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$. Since $\mathrm{GL}_2(R)$ acts transitively on the set of triples of non-parallel points, any k-set ($k \ge 3$) of mutually non-parallel points of $\mathbb{P}(R)$ can be chosen as base block B_0 in order to apply Spera's construction of a DD. This is, of course, a very general approach. Therefore, it is not surprising that not too much can be said about the corresponding 3-DDs. It is straightforward to express their parameters v and s, as well as the order of the group $\mathrm{GL}_2(R)$, in terms of $|R|$ and $|I|$, i.e. the cardinality of the unique maximal ideal I of the given ring R. However, in order to calculate the parameter λ_3 by virtue of (1), one needs to know the order of the stabilizer of B_0 in $\mathrm{GL}_2(R)$. But it seems hopeless to calculate this order without further information about the base block B_0.

If R is even a finite local algebra over a field F, say, then the projective line $\mathbb{P}(F)$ over F can be considered as a subset of $\mathbb{P}(R)$, and it can be chosen as a base block. All 3-DDs obtained in this way satisfy $\lambda_3 = 1$; they are—up to notational differences—precisely the (classical) *chain geometries* $\Sigma(F, R)$; see [1], [6] or [9]. This was pointed out by Spera [11, Example 2.5]. In the cited paper also a series of interesting DDs are constructed from base blocks which are certain subsets of $\mathbb{P}(F)$. See also [7] for similar results. We mention in passing that higher-dimensional projective spaces over local algebras give rise to 2-DDs [12].

The divisible designs which are constructed in the present paper arise also from chain geometries. However, we use this term in a more general form which was introduced in [3] just a few years ago. The essential difference is as follows: We consider a finite local ring R containing a subfield K which is not necessarily in the centre of R. Thus R *need not be a K-algebra*, but of course it is an algebra over some subfield of K. As before, we can define $\mathbb{P}(K) \subset \mathbb{P}(R)$ to be the base block. This gives a 3-DD which coincides with the (generalized) chain geometry $\Sigma(K, R)$. It is possible to express the parameter λ_3 of this DD in algebraic terms (see [3, Theorem 2.4]), but this is not very explicit in the general case. Therefore, we focus our attention on a particular class of local rings, namely twisted dual numbers. If the "twist" is non-trivial, then 3-DDs with parameter $\lambda_3 = |K|$ are obtained.

In Section 4 we present an alternative description of our 3-DDs in a finite projective space over K.

2 Twisted dual numbers

Let R be a (not necessarily commutative) local ring containing a (not necessarily central) subfield K. In view of our objective to construct DDs, we will later restrict ourselves to finite rings and fields, and hence we assume from the beginning that K is commutative. As usual, we denote by R^* the group of units (invertible elements) of R. We set $I := R \setminus R^*$; since R is local we have that I is an ideal.

The ring R is in a natural way a left vector space over K, sometimes written as ${}_KR$. We assume that $\dim({}_KR) = 2$. Moreover, we assume that R is not a field. We want to determine the structure of R. The ideal I is a non-trivial subspace of the vector space ${}_KR$. So $\dim({}_KI) = 1$, and $I = K\varepsilon$ for some $\varepsilon \in R \setminus K$. Then $1, \varepsilon$ is a basis of ${}_KR$, and we may write $R = K + K\varepsilon$.

In order to describe the multiplication in R we first observe that $\varepsilon^2 \in I$, so $\varepsilon^2 = b\varepsilon$ for some $b \in K$. This implies $(\varepsilon - b)\varepsilon = 0$, whence also $\varepsilon - b \in I$ and so $b = 0$. For each $x \in K$ we have $\varepsilon x \in I$, so there is a unique $x' \in K$ such that $\varepsilon x = x'\varepsilon$. One can easily check that $\sigma : x \mapsto x'$ is an injective field endomorphism.

Conversely, given a field K and an injective endomorphism σ of K we obtain a ring of *twisted dual numbers* $R = K(\varepsilon; \sigma) = K + K\varepsilon$ with multiplication

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc^\sigma)\varepsilon.$$

In the special case that $\sigma = \mathrm{id}$ this is the well known commutative ring $K(\varepsilon)$ of *dual numbers* over K.

The subfield $\mathrm{Fix}(\sigma)$ of K fixed elementwise by σ will be called F. So $F = K$ if, and only if, $\sigma = \mathrm{id}$.
The units of R are exactly the elements of $R \setminus I = K^* + K\varepsilon$. One can easily check that the inverse of a unit $u = a + b\varepsilon$ (with $a, b \in K, a \neq 0$) is

$$u^{-1} = a^{-1} - a^{-1}b(a^{\sigma})^{-1}\varepsilon. \tag{2}$$

Later we shall need the following algebraic statements on $R = K(\varepsilon; \sigma)$.

2.1 Lemma. *The multiplicative group R^* is the semi-direct product of K^* and the normal subgroup*

$$U = 1 + K\varepsilon = \{1 + b\varepsilon \mid b \in K\}. \tag{3}$$

Proof: Direct computation, using (2) for showing that U is normal in R^*. □

2.2 Lemma. *Let N be the normalizer of K^* in R^*, i.e.,*

$$N = \{n \in R^* \mid n^{-1}K^*n = K^*\}. \tag{4}$$

Then $N = R^*$ if $\sigma = \mathrm{id}$ and $N = K^*$ otherwise.

Proof: For $\sigma = \mathrm{id}$ the assertion is clear. So let $\sigma \neq \mathrm{id}$ and $n = a + b\varepsilon \in N$. Take an element $x \in K \setminus F$. Using (2) we get $n^{-1}xn = x + a^{-1}b(x - x^{\sigma})\varepsilon$, which must belong to K since $n \in N$. Because of our choice of x we have $x - x^{\sigma} \neq 0$, whence $b = 0$, as desired. □

3 The associated DD

In this section we construct a 3-DD using the ring $R = K(\varepsilon; \sigma)$. The construction is a special case of Spera's construction method described in Section 1 (see also [8, Section 2.3]). On the other hand, the resulting DD is nothing else than the (generalized) *chain geometry* over (K, R) (compare [3], for details on ordinary chain geometries see [6], [9]).
From now on we assume that R, and hence also K and F, are finite. Then $F = \mathrm{GF}(m)$ for some prime power m, and $K = \mathrm{GF}(q)$ with q a power of m. Moreover, σ now is an automorphism of K, namely, $\sigma : x \mapsto x^m$.
The construction is based on the action of the group $G = \mathrm{GL}_2(R)$ of invertible 2×2-matrices with entries in R on the *projective line* over R, i.e., on the set

$$\mathbb{P}(R) = \{R(a, b) \leq R^2 \mid \exists\, c, d \in R : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G\}. \tag{5}$$

Since R is local, each pair (a, b) as in (5) has the property that at least one of the two elements a, b is invertible, because otherwise the existence of an inverse matrix $\left(\begin{smallmatrix} x & * \\ y & * \end{smallmatrix}\right)$ would lead to the contradiction $1 = ax + by \in I$. So $\mathbb{P}(R)$ is the disjoint union

$$\mathbb{P}(R) = \{R(x, 1) \mid x \in R\} \cup \{R(1, z) \mid z \in I\}. \tag{6}$$

On $\mathcal{P} = \mathbb{P}(R)$ we have an equivalence relation $\parallel$ given by

$$R(a, b) \parallel R(c, d) :\Longleftrightarrow \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \notin G. \tag{7}$$

More explicitly, this means for arbitrary $x, y \in R, z, w \in I$:

$$R(1, z) \parallel R(1, w);\ R(x, 1) \nparallel R(1, z);\ \big(R(x, 1) \parallel R(y, 1) \Leftrightarrow x - y \in I\big). \tag{8}$$

Using the description in (8) one can see that $\parallel$ in fact is an equivalence relation.

Let us recall two facts (see [9, 1.2.2] and [9, Prop. 1.3.3], where non-parallel points are called “distant”): The group G acts on $\mathcal{P}$ leaving $\parallel$ invariant. Moreover, G acts transitively on the set of triples of pairwise non-parallel points of $\mathcal{P}$. By virtue of this action of G and (8), any two parallel classes have the same cardinality $s = |I|$.

In order to apply Spera’s method we now need a *base block* consisting of pairwise non-parallel points. As usual for chain geometries, we use the projective line over K.

Since K is a subfield of R, the projective line $\mathbb{P}(K)$ can be seen as a subset B_0 of $\mathcal{P} = \mathbb{P}(R)$ as follows:

$$B_0 = \mathbb{P}(K) = \{R(x, 1) \mid x \in K\} \cup \{R(1, 0)\}. \tag{9}$$

Let $\mathcal{B} = B_0^G$. Then we get the following.

3.1 Theorem. *The structure $\Sigma = (\mathcal{P}, \mathcal{B}, \parallel)$ is a transversal 3-DD with parameters $v = q^2 + q$, $s = q$, $k = q + 1 (= v/s)$ and*

$$\lambda_3 = \begin{cases} 1 & \text{if } \sigma = \mathrm{id}, \\ q & \text{if } \sigma \neq \mathrm{id}. \end{cases}$$

Proof: From Spera’s theorem we know that Σ is a DD. The values of v, s, and k are obtained from (6), (8), and (9), respectively. By [3, Theorem 2.4], we have $\lambda_3 = |R^*|/|N|$, where N is the normalizer defined in (4). By Lemma 2.2 we have two cases: If $\sigma = \mathrm{id}$, the normalizer N coincides with R^* and so $\lambda_3 = 1$. If $\sigma \neq \mathrm{id}$, the normalizer N equals K^*, whence $\lambda_3 = |R^*|/|N| = (q - 1)q/(q - 1) = q$. $\square$

The equation $\sigma = \mathrm{id}$ holds precisely when K lies in the centre of R; in this case our DD is an ordinary chain geometry, namely, the *Miquelian Laguerre plane* over the algebra of dual numbers (see [1, I.2, II.4]).

We mention here that the parameter λ_3 could also be computed directly using the formula

$$\lambda_3 = \frac{|G|}{|G_{B_0}|} \cdot \frac{1}{s^3} \tag{10}$$

(see (1), note that $k = v/s$).

We add without proof that the 3-DD Σ can also be described as a *lifted DD* in the sense of [5, Theorem 2.5], using the point set $\mathcal{P}$, the equivalence relation $\|$, the group

$$H = \left\{ \begin{pmatrix} 1+a\varepsilon & b\varepsilon \\ c\varepsilon & 1+d\varepsilon \end{pmatrix} \mid a, b, c, d \in K \right\}$$

which acts on $\mathcal{P}$, and the base block B_0 as (trivial) base DD. However, this alternative approach does not immediately show the large group of automorphisms given by the action of G on $\mathcal{P}$.

We now have a closer look at the case $\sigma \neq \mathrm{id}$. We want to determine the q blocks through three given pairwise non-parallel points more explicitly. Because of the transitivity properties of G it suffices to consider the points $\infty = R(1,0)$, $0 = R(0,1)$, $1 = R(1,1)$. From [3, Theorem 2.4] we know the following: The blocks through $\infty, 0, 1$ are exactly the images of B_0 under the group

$$\widehat{R^*} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in R^* \right\}, \tag{11}$$

and two elements $\omega = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ and $\omega' = \begin{pmatrix} u' & 0 \\ 0 & u' \end{pmatrix}$ of $\widehat{R^*}$ determine the same block if, and only if, $Nu = Nu'$, with N as in (4). So from Lemmas 2.2 and 2.1 we obtain:

3.2 Lemma. *Let $\sigma \neq \mathrm{id}$. Then the blocks containing $\infty = R(1,0)$, $0 = R(0,1)$, $1 = R(1,1)$ are exactly the q sets*

$$B_0^\omega, \text{ with } \omega = \begin{pmatrix} 1+b\varepsilon & 0 \\ 0 & 1+b\varepsilon \end{pmatrix}, \ b \in K. \tag{12}$$

We now give an explicit description of the action of the group

$$\widehat{U} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in U \right\} = \left\{ \begin{pmatrix} 1+b\varepsilon & 0 \\ 0 & 1+b\varepsilon \end{pmatrix} \mid b \in K \right\}, \tag{13}$$

associated to U (see (3)), on $\mathcal{P} = \mathbb{P}(R)$.

A direct calculation shows that each $\omega = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$, with $u \in R^*$, acts on $\mathcal{P}$ via "conjugation" as follows:

$$\omega : R(x,1) \mapsto R(u^{-1}xu, 1), \quad R(1,z) \mapsto R(1, u^{-1}zu), \tag{14}$$

where, as before, $x \in R, z \in I$. For $u = 1 + b\varepsilon \in U$ this yields, using (2),

$$\omega : R(x,1) \mapsto R(x + b(x_1 - x_1^\sigma)\varepsilon, 1), \quad R(1,z) \mapsto R(1,z), \tag{15}$$

where $x = x_1 + x_2\varepsilon$. So the mapping $\omega \in \widehat{U}$ of (15) maps each point to a parallel one. Moreover, it fixes exactly those elements of the base block $B_0 = \mathbb{P}(K)$ that belong to the subset $\mathbb{P}(F)$. This subset in turn is the intersection of all blocks through $\infty, 0, 1$ (compare (12)); such intersections are also called *traces* (in German: "Fährten", see [1], [3]).

We consider a parallel class on which $\widehat{U}$ does not act trivially. By (15) this is the parallel class of some point $p = R(x_1, 1)$, where $x_1 \in K \setminus F$ and consequently $p \in B_0 \setminus \mathbb{P}(F)$. Then $\widehat{U}$ acts regularly on the parallel class under consideration. As a matter of fact, for each p' parallel to p, which has the form $p' = R(x_1 + x_2\varepsilon, 1)$, there is a unique $b \in K$ with $x_2 = b(x_1 - x_1^\sigma)$, so $p^\omega = p'$, with ω as in (15). This means that for each $p' \parallel p$ there is exactly one block through $\infty, 0, 1$ that contains p' (and each block through $\infty, 0, 1$ is obtained in this way, as each block meets all parallel classes).

All these results can be carried over to an arbitrary triple of pairwise non-parallel points, using the action of G. So we have the following.

3.3 Proposition. *Let $\sigma \neq \mathrm{id}$. Let $p_1, p_2, p_3 \in \mathcal{P}$ be pairwise non-parallel. Let T be the intersection of all blocks through p_1, p_2, p_3, and let C be a parallel class not meeting T. Then the following hold.*

(a) *There is a $g \in G$ such that $T = \mathbb{P}(F)^g$.*

(b) *Each block through p_1, p_2, p_3 meets C, and for each $x \in C$ there is a (unique) block through p_1, p_2, p_3, x.*

3.4 Corollary. *Let p_1, p_2, p_3 be pairwise non-parallel, let T be the intersection of all blocks through p_1, p_2, p_3, and let $x \nparallel p_1, p_2, p_3$. Then the number of blocks through p_1, p_2, p_3, x is*

- *q, if $x \in T$,*
- *0, if $x \notin T$, but $x \parallel x'$ for some $x' \in T$,*
- *1, otherwise.*

Finally, let us point out a particular case:

3.5 Corollary. *Let q be even and let $m = 2$, i.e., $x^\sigma = x^2$ for all $x \in K$. Then $\Sigma = (\mathcal{P}, \mathcal{B}, \|)$ is a 4-divisible design with parameter $\lambda_4 = 1$.*

This result is immediate from Corollary 3.4, since $F = \mathrm{GF}(2)$ implies now $|T| = |\mathbb{P}(F)| = 3$.

4 A geometric model

Now we are looking for a geometric point model of the DD Σ defined above, i.e. a DD isomorphic to Σ whose points are points of a suitable projective space. We find such a model on the Klein quadric $\mathcal{K}$ in $\mathrm{PG}(5, K)$ by using H. Hotje's representation [10].

4.1 Remark. One could also first find a *line model* of Σ in $\mathrm{PG}(3, K)$ (where the points of Σ are certain lines in 3-space) and then apply the Klein correspondence. For details on such line models see [4], in particular Examples 5.2 and 5.4, and [2].

We embed the ring $R = K(\varepsilon; \sigma)$ in the ring $M = \mathrm{M}(2, K)$ of 2×2-matrices with entries in K via the ring monomorphism

$$a + b\varepsilon \mapsto \left(\begin{smallmatrix} a & b \\ 0 & a^\sigma \end{smallmatrix}\right). \tag{16}$$

From now on we identify the ring R with its image under this embedding.

The projective line $\mathbb{P}(M)$ is defined, mutatis mutandis, according to (5). The points of $\mathbb{P}(M)$ are of the form $M(A, B)$, where (A, B) are the first two rows of an invertible 4×4-matrix over K, because (up to notation) $\mathrm{GL}_2(M)$ equals $\mathrm{GL}_4(K)$. Then (16) allows to identify the point set $\mathbb{P}(R)$ of Σ with a subset of $\mathbb{P}(M)$.

Now we establish the existence of a bijection Φ from $\mathbb{P}(M)$ onto the Klein quadric $\mathcal{K}$. For this we notice that M is a K-algebra, with K embedded in M via $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, and that this algebra is *kinematic*, i.e., each element of M satisfies a quadratic equation over K. Note that this embedding of K in M is different from the one obtained from (16), unless $\sigma = \mathrm{id}$. In [10] Hotje embeds the projective line over an arbitrary kinematic algebra in an appropriate quadric. For the matrix algebra M this quadric is $\mathcal{K}$, and the embedding, which here is a bijection, is the following:

$$\Phi : \mathbb{P}(M) \to \mathcal{K} : M(A, B) \mapsto K(\widetilde{B}A, \det A, \det B), \tag{17}$$

where A, B are matrices in M, and for $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we set $\widetilde{B} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. The image of Φ is indeed the Klein quadric, because $M \times K \times K$ is a 6-dimensional vector space over K endowed with the hyperbolic quadratic form $(C, x, y) \mapsto \det C - xy$.

We need the following additional statements:

4.2 Proposition. *Consider the bijection $\Phi : \mathbb{P}(M) \to \mathcal{K}$ given in (17), and its restriction to $\mathbb{P}(R)$. Then*

(a) *The bijection Φ induces a homomorphism of group actions, mapping $\mathrm{GL}_2(M)$, acting on $\mathbb{P}(M)$, to a subgroup of the group of collineations of $\mathrm{PG}(5, K)$ leaving $\mathcal{K}$ invariant.*

(b) *This homomorphism maps the subgroup $\mathrm{GL}_2(R)$, acting on $\mathcal{P} = \mathbb{P}(R)$, to a subgroup of the group of collineations of $\mathrm{PG}(5, K)$ leaving $\mathcal{P}^\Phi$ invariant.*

(c) *Two points of $\mathbb{P}(R)$ are parallel if, and only if, their Φ-images are joined by a line contained in $\mathcal{K}$.*

Proof: For (a) see [10, (7.1/2/3)]; (b) follows from (a).
(c): This follows from [10, (7.5)] and [4, Prop. 3.2]. □

Writing $K(x_1, x_2, x_3, x_4, x_5, x_6)$ instead of $K(\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right), x_5, x_6)$, we obtain by a direct computation that the mapping Φ given in (17) acts on the points of $\mathcal{P} = \mathbb{P}(R) \subseteq \mathbb{P}(M)$ as follows:

$$R(a + b\varepsilon, 1) \mapsto K(a, b, 0, a^\sigma, aa^\sigma, 1); \; R(1, c\varepsilon) \mapsto K(0, -c, 0, 0, 1, 0) \qquad (18)$$

We shall identify the elements of $\mathbb{P}(M)$ with their Φ-images. Then, in particular, we have

$$B_0 = \{K(a, 0, 0, a^\sigma, aa^\sigma, 1) \mid a \in K\} \cup \{K(0, 0, 0, 0, 1, 0)\}. \qquad (19)$$

In the next lemma we collect some observations, which can be seen directly using (18) and (19).

4.3 Lemma. *Let $\mathcal{P}$ and B_0 be the point sets in* $\mathrm{PG}(5, K)$ *from above. Then the following hold:*

(a) *$\mathcal{P} = \mathcal{C} \setminus \{S\}$, where $\mathcal{C}$ is the* cone *with* vertex *$S = K(0, 1, 0, 0, 0, 0)$ over B_0, i.e. the union of all lines joining S with B_0.*

(b) $\mathcal{P}$ *is entirely contained in the hyperplane* $\boldsymbol{H}$ *with equation* $x_3 = 0$*, which is the tangent hyperplane to* $\mathcal{K}$ *at* S*.*

(c) *Two points of* $\mathcal{P}$ *are parallel if, and only if, they lie on a* generator *of* $\mathcal{C}$*, i.e. a line through* S *contained in* $\mathcal{C}$*.*

Now we describe the (image of) the base block B_0 more closely:

4.4 Lemma. *Let* B_0 *be as in (19). Then the following hold:*

(a) B_0 *is a* cap*, i.e. a set of points no three of which are collinear.*

(b) *If* $\sigma = \mathrm{id}$*, then* B_0 *is a regular conic; in particular,* B_0 *is contained in a plane.*

(c) *If* $\sigma \neq \mathrm{id}$*, then* B_0 *spans the* 3*-space* $\boldsymbol{U}_0$*, given by* $x_2 = 0 = x_3$*, complementary to* S *in* $\boldsymbol{H}$*.*

Proof: (a): Assume that the line L carries three points of B_0. Then $L \subseteq \mathcal{K}$. From Proposition 4.2(c) we see that the three points are pairwise parallel, a contradiction.
(b): Here $B_0 = \{K(a,0,0,a,a^2,1) \mid a \in K\} \cup \{K(0,0,0,0,1,0)\}$, which obviously is a regular conic in the plane spanned by the points $K(1,0,0,1,0,0)$, $K(0,0,0,0,1,0)$, $K(0,0,0,0,0,1)$ (namely, the intersection of this plane with the Klein quadric).
(c): In this case, the four vectors

$$(0,0,0,0,1,0),\ (0,0,0,0,0,1),\ (1,0,0,1,1,1),\ \text{and}\ (a,0,0,a^\sigma,aa^\sigma,1),$$

with $a \in K \setminus F$, are linearly independent, so the point set B_0 spans $\boldsymbol{U}_0$. $\square$

In case $\sigma = \mathrm{id}$, our geometric model is nothing else than the "cylinder model" of the Miquelian Laguerre plane Σ: The points are the points of a cylinder in 3-space (a quadratic cone minus its vertex), and the blocks are the regular conics on the cylinder (the intersections with planes complementary to the vertex). See, e.g., [1, I.2] for the real case.
We have a closer look at the special case that $\sigma^2 = \mathrm{id}$, $\sigma \neq \mathrm{id}$. Then $q = m^2$ and K is a quadratic extension of F. In this case there are *Baer subspaces*, i.e. spaces coordinatized by F, in each projective space over K.

4.5 Proposition. *Let* $\sigma^2 = \mathrm{id}$, $\sigma \neq \mathrm{id}$*. Then* B_0 *is an elliptic quadric in the Baer subspace* $\mathbb{B} \cong \mathrm{PG}(3,m)$ *of* $\boldsymbol{U}_0 \cong \mathrm{PG}(3,q)$ *defined by the* F*-subspace*

$$\{(x,0,0,x^\sigma,f_1,f_2) \mid x \in K, f_i \in F\}. \tag{20}$$

Proof: Obviously, the set in (20) is a 4-dimensional subspace of K^6, seen as a vector space over F, satisfying the equations $x_2 = 0 = x_3$ and hence giving rise to a Baer subspace $\mathbb{B}$ of $\boldsymbol{U}_0$. The elements of B_0 all lie in $\mathbb{B}$. Moreover, by (19), B_0 equals the quadric in $\mathbb{B}$ determined by $N(x) = f_1 f_2$, where $N(x) = xx^\sigma$ is the norm of $x \in K$ with respect to the field extension $K : F$ and, in particular, N is a quadratic form on the vector space ${}_F K$. Since B_0 is a cap by 4.4 (a), the quadric must be elliptic. □

The quadratic form used in the above is just the restriction to $\mathbb{B}$ of the quadratic form describing the Klein quadric. The intersection of the Klein quadric and $\boldsymbol{U}_0$ is a hyperbolic quadric.

For the rest of this section we consider the case that $\sigma \neq \mathrm{id}$. We try to describe the geometric model of the DD Σ more explicitly. From the above we know that our base block B_0 is a certain cap that spans a 3-space $\boldsymbol{U}_0$ complementary to S in the tangent hyperplane $\boldsymbol{H} \cong \mathrm{PG}(4, K)$ of $\mathcal{K}$ at S. In the next proposition we describe all blocks. Together with Lemma 4.3 this gives a description of Σ in terms of $\mathrm{PG}(4, K)$.

4.6 Proposition. *Let $\sigma \neq \mathrm{id}$. Then the blocks of Σ are exactly the intersections of the cone $\mathcal{C}$ with the 3-spaces complementary to S in $\boldsymbol{H}$.*

Proof: We know that $B_0 = \mathcal{C} \cap \boldsymbol{U}_0$, with $\boldsymbol{U}_0$ complementary to S in $\boldsymbol{H}$. Let B be any block. Then $B = B_0^g$ for some $g \in G = \mathrm{GL}_2(R) \leq \mathrm{GL}_2(M)$. By Proposition 4.2(b), g induces a collineation, say $\widetilde{g}$, of $\mathrm{PG}(5, K)$ leaving $\mathcal{K}$ and $\mathcal{P}$ invariant. This collineation fixes S (which is the intersection of the lines corresponding to parallel classes) and its tangent hyperplane $\boldsymbol{H}$. So B, seen as a set of points in $\boldsymbol{H}$, is $B = B_0^{\widetilde{g}} = \mathcal{C} \cap \boldsymbol{U}_0^{\widetilde{g}}$, where $\boldsymbol{U}_0^{\widetilde{g}}$ is a 3-space complementary to S, as desired. The 3-space $\boldsymbol{U}_0^{\widetilde{g}}$ is independent of the choice of g, as it is nothing else than the span of B.

So we have a mapping from the set of blocks to the set of complements of S in $\boldsymbol{H}$, which is injective since each complement contains exactly one point of each generator of $\mathcal{C}$, i.e. of each parallel class of $\mathcal{P}$, and hence cannot belong to more than one block. A simple counting argument shows that the mapping is also surjective: The number of blocks is $b = |G|/|G_{B_0}| = q^4$ (this can be computed directly, or from (10) using $\lambda_3 = q$), and the number of complements of S in $\boldsymbol{H}$ also is q^4, because they form an affine 4-space of order $|K| = q$. □

4.7 Remark. The projective model of Σ studied in this section is a special case of the lifted t-DDs described in [5, Cor. 3.3]. There, the following geometries are described as t-DDs obtained via the lifting process: Consider an arbitrary finite projective space $\mathrm{PG}(n,q)$ and a set B_0 of k points spanning a subspace $\boldsymbol{U}_0$ and having the property that any t points of B_0 are independent. Let $\boldsymbol{S}$ be a complement of B_0. The point set of the t-DD is the cone with basis B_0 and vertex $\boldsymbol{S}$, minus $\boldsymbol{S}$. The blocks are the intersections of the cone with subspaces complementary to $\boldsymbol{S}$, and two points are parallel if, and only if, together with $\boldsymbol{S}$ they span the same subspace.

The following is an obvious geometric analogue of Proposition 3.3 and Corollary 3.4.

4.8 Corollary. *Let p_1, p_2, p_3 be pairwise non-parallel, let T be the intersection of all blocks through p_1, p_2, p_3, and let $x \nparallel p_1, p_2, p_3$. Then*

(a) *T is the intersection of the cone $\mathcal{C}$ with the plane $\boldsymbol{E}$ spanned by p_1, p_2, p_3.*

(b) *The blocks through p_1, p_2, p_3, x are exactly the intersections of $\mathcal{C}$ with 3-spaces through $\boldsymbol{E}$ complementary to S. The number of such 3-spaces is*

- *q, if $x \in T$,*
- *0, if $x \notin T$, but $x \parallel x'$ for some $x' \in T$,*
- *1, otherwise.*

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