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Giuseppe Veronese and Ernst Witt – Neighbours in $\mathrm{PG}(5,3)$

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Dedicated to János Aczél on the occasion of his 75th birthday

Summary. Let P be a point of the Veronese surface $\mathcal{V}$ in $\mathrm{PG}(5,3)$. Then there are four conics of $\mathcal{V}$ through P. We show that the internal points of those conics form a 12–cap which is a point model for Witt's 5–$(12,6,1)$ design. In fact, this construction is "dual" to a similar construction that has been established in [6] recently. We give an explicit parametrization of the cap $\mathcal{K}$; the domain is a dual affine plane which arises from $\mathrm{PG}(2,3)$ by removing one point. Thus, as a by–product, we obtain an easy approach to the extended ternary Golay code G_{12}. Finally, we discuss some other procedures that yield 12–sets of points from the Veronese surface.

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1 Introduction

A construction of a cap $\mathcal{K}$, in $\mathrm{PG}(5,3)$, which is a point model for *Witt's* 5–$(12,6,1)$ *design* W_{12} (see, among others, [3, Chapter IV]) has been found by H.S.M. COXETER [5] and, independently, by G. PELLEGRINO [9]. The cap $\mathcal{K}$ has exactly twelve points and any five distinct points of $\mathcal{K}$ span a prime (hyperplane) of $\mathrm{PG}(5,3)$ which contains exactly six points of $\mathcal{K}$. Such a $\mathcal{K}$ is projectively unique. The group of collineations fixing $\mathcal{K}$, as a set, is the automorphism group of W_{12}, i.e. the *Mathieu group* M_{12}. Also, J.A. TODD [10] has shown that there are exactly twelve primes of $\mathrm{PG}(5,3)$ carrying no point of $\mathcal{K}$. Those primes gives rise to a point model $\mathcal{K}^*$ of W_{12} in the dual space of $\mathrm{PG}(5,3)$.

The *Veronese surface* $\mathcal{V}$ in $\mathrm{PG}(5,3)$ is a set of thirteen points; cf. e.g. [8, Chapter 25]. It determines uniquely its *dual Veronese surface* $\mathcal{V}^*$ in the dual space of $\mathrm{PG}(5,3)$. As has been pointed out in [6], the following holds true: If one conic of the Veronese surface $\mathcal{V}$ is replaced with the set formed by the internal points of that conic, then a point model $\mathcal{K}$ of W_{12} is obtained. Figure 1 illustrates a conic in a projective plane of order three: The four points of the conic, its three internal points, and its six external points are drawn as squares, triangles, and hexagons, respectively.

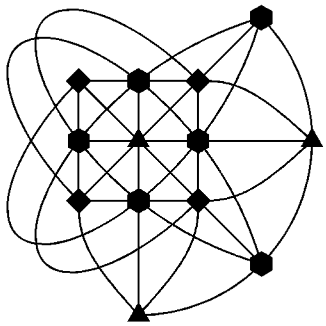

Figure 1: A conic in PG$(2,3)$

Clearly, now the question about the connection between the dual Veronese surface $\mathcal{V}^*$ and the dual point model $\mathcal{K}^*$ arises. For our purposes it will be convenient to adopt a dual point of view from the very beginning: So we start with $\mathcal{V}$, then go over to its dual $\mathcal{V}^*$, next apply the procedure of [6] to obtain a $\mathcal{K}^*$ from $\mathcal{V}^*$, and finally go back to $\mathcal{K}^{**} =: \mathcal{K}$.

In Theorem 1 we give a direct description (avoiding the dual space) of that construction. It turns out that the present construction is different from the one given in [6], since now we do not have a distinguished conic of $\mathcal{V}$, but a distinguished point $P \in \mathcal{V}$. In contrast to [6], $\mathcal{V}$ and $\mathcal{K}$ do not have common points, but there is a bijection $\rho : \mathcal{V} \setminus \{P\} \to \mathcal{K}$ such that $Y \in \mathcal{V} \setminus \{P\}$, $Y^\rho \in \mathcal{K}$, and P are always collinear.

The *Veronese mapping* φ is a bijection of PG$(2,3)$ onto $\mathcal{V}$. By combining it with the above–mentioned bijection ρ, we find a bijection of a dual affine plane (a PG$(2,3)$ with one point removed) onto $\mathcal{K}$. Using homogeneous coordinates in PG$(2,3)$ and PG$(5,3)$ then gives an explicit parametric representation ψ of $\mathcal{K}$ described in Theorem 2. It is a peculiar feature that one of the coordinate functions of ψ contains an *inhomogeneous* term, something which usually does not make sense. But here it is meaningful, since 1 is the only non–zero square in GF(3).

In Section 4 we discuss procedures, similar to that of Theorem 1, which yield 12–sets of points from the Veronese surface $\mathcal{V}$. Some of them give point models of W_{12}, others do not. However, one should keep in mind that each point model of W_{12} arises in some way or another from a fixed Veronese surface; so our discussion is far from being complete. In fact, all our 12–sets are "close neighbours" of the Veronese surface $\mathcal{V}$, as they belong to the algebraic hypersurface (of order three) which is formed by all points of $\mathrm{PG}(5,3)$ that are on a chord of $\mathcal{V}$.

2 The Veronese Surface and its Dual

Let us recall some properties of the Veronese surface $\mathcal{V}$ in $\mathrm{PG}(5,3)$ [4, Kapitel V], [7], [8, Chapter 25]: The term *conic plane* is used for a plane which meets $\mathcal{V}$ in a conic. Any two distinct conic planes have one and only one point in common. This point belongs to $\mathcal{V}$. Let $\mathcal{C}$ be the set of all conics of $\mathcal{V}$. Then $(\mathcal{V}, \mathcal{C}, \in)$ is a $\mathrm{PG}(2,3)$.

For each $m \in \mathcal{C}$ there is a unique prime which meets $\mathcal{V}$ exactly in m; this *osculating prime* (or *contact prime*) of $\mathcal{V}$ along m will be denoted by $\mathcal{H}_m\mathcal{V}$. A line l is called a *tangent* of $\mathcal{V}$, if l is a tangent of a conic $m \subset \mathcal{V}$. Given $P \in \mathcal{V}$ the *tangent plane* of $\mathcal{V}$ at P is the union of all tangents of $\mathcal{V}$ which are running through P. It is written as $\mathcal{T}_P\mathcal{V}$. Another description of osculating primes and tangent planes is given by

$$\mathcal{H}_m\mathcal{V} \quad = \quad \operatorname{span}\left(\bigcup_{X \in m} \mathcal{T}_X\mathcal{V}\right) \quad (m \in \mathcal{C}), \tag{1}$$

$$\mathcal{T}_P\mathcal{V} \quad = \quad \bigcap_{P \in m \in \mathcal{C}} \mathcal{H}_m\mathcal{V} \quad (P \in \mathcal{V}). \tag{2}$$

Any two distinct tangent planes have a unique common point; this point is not on $\mathcal{V}$.

If $\mathcal{S}$ is a subspace of $\mathrm{PG}(5,3)$, then let $[\mathcal{S}]^*$ be the star of primes through $\mathcal{S}$. All osculating primes of $\mathcal{V}$ form the *dual Veronese surface* $\mathcal{V}^*$, i.e. a Veronese surface in the dual space, since $\mathrm{Char}\,\mathrm{GF}(3) \neq 2$. There is a one–one correspondence between the tangent and conic planes of $\mathcal{V}$ with the "conic" and "tangent planes" of $\mathcal{V}^*$, respectively:

Each tangent plane $\mathcal{T}_P\mathcal{V}$ yields the "conic plane" $[\mathcal{T}_P\mathcal{V}]^*$ of $\mathcal{V}^*$. The osculating primes of $\mathcal{V}$ that are passing through P comprise the corresponding "conic" $c^* \subset \mathcal{V}^*$. If we choose one "point" of the "conic" c^*, say $\mathcal{H}_m\mathcal{V}$, then its "tangent" is given by the pencil $[\mathcal{T}_P\mathcal{V} \vee \operatorname{span} m]^*$. An "internal point" of c^* is a prime $\mathcal{I} \in [\mathcal{T}_P\mathcal{V}]^*$ which is on no "tangent" of c^*, i.e.

$$\operatorname{span} m \cap \mathcal{I} = \operatorname{span} m \cap \mathcal{T}_P\mathcal{V} \text{ for all } m \in \mathcal{C} \text{ with } P \in m. \tag{3}$$

Alternatively, an "internal point" of c^* may be characterized as a prime $\mathcal{I}$ of $\mathrm{PG}(5,3)$ satisfying

$$\mathcal{I} \cap \mathcal{V} = \{P\}, \tag{4}$$

since (4) implies that $\mathcal{I}$ corresponds to a quadric in $\mathrm{PG}(3,2)$ consisting of one double point only, so that $\mathcal{T}_P\mathcal{V} \subset \mathcal{I}$ (cf. [4, p. 168, Satz 1]).

Likewise, each conic plane $\operatorname{span} m$ ($m \in \mathcal{C}$) yields the "tangent plane" $[\operatorname{span} m]^*$ of $\mathcal{V}^*$ at the "point" $\mathcal{H}_m\mathcal{V} \in \mathcal{V}^*$.

3 Point Models of W_{12}

Let $\mathcal{K}$ be a set of twelve points in a $\mathrm{PG}(5,3)$. Define a *block* of $\mathcal{K}$ as hyperplane section of $\mathcal{K}$ which contains exactly six points. Write $\mathcal{B}$ for the set of all such blocks. If $(\mathcal{K}, \mathcal{B}, \in)$ is Witt's 5–(12, 6, 1) design W_{12}, then $\mathcal{K}$ is called a *point model* of W_{12} in $\mathrm{PG}(5,3)$.

In what follows we put $\mathrm{GF}(3) = \{0,1,2\} =: F$.

Theorem 1 *Let P be a point of the Veronese surface $\mathcal{V}$ in $\mathrm{PG}(5,3)$. The four conics of $\mathcal{V}$ through P are denoted by m_k ($k \in F \cup \{\infty\}$). The set of internal points of each m_k is written as Δ_k. Also let c^* be the set of osculating primes of $\mathcal{V}$ through P and Δ^* the set of all primes that meet $\mathcal{V}$ in P only. Then the following holds true:*

1. *The set*
$$\mathcal{K} := \bigcup_{k \in F \cup \{\infty\}} \Delta_k \tag{5}$$
is a point model of the Witt design W_{12}.

2. *No point of $\mathcal{K}$ is incident with a prime belonging to*
$$\mathcal{K}^* := (\mathcal{V}^* \setminus c^*) \cup \Delta^*. \tag{6}$$

Proof. According to Section 2, c^* is a "conic" of the dual Veronese surface and Δ^* is the set of its "internal points". Hence $\mathcal{K}^*$ is a point model of W_{12} in the dual space of $\mathrm{PG}(5,3)$ [6, Remark 3]. By a result of J.A. Todd [10, p. 408], applied to $\mathcal{K}^*$, there are exactly twelve points of $\mathrm{PG}(5,3)$ which are not in any prime belonging to $\mathcal{K}^*$. Moreover, those points form a point model of W_{12}.

Each conic m_k has exactly three internal points. The planes of two distinct conics of the Veronese surface do not have common internal points. Thus $\#\mathcal{K} = 12$ and the theorem follows, if we can show that no point of $\mathcal{K}$ lies in a prime belonging to $\mathcal{K}^*$.

Let $\mathcal{I}$ be one of the three "internal points" of c^*. By (3), the prime $\mathcal{I}$ meets the plane of each m_k ($k \in F \cup \{\infty\}$) in the tangent of m_k at P, so that
$$\mathcal{I} \cap \mathcal{K} = \emptyset. \tag{7}$$

Any of the remaining nine primes in $\mathcal{K}^*$ is an osculating prime $\mathcal{H}_c\mathcal{V}$ along a conic $c \in \mathcal{C} \setminus \{m_0, m_1, m_2, m_\infty\}$. Put $\{S_k\} := c \cap m_k$. By (2), $\mathcal{T}_{S_k}\mathcal{V} \subset \mathcal{H}_c\mathcal{V}$, whence $\mathcal{H}_c\mathcal{V} \cap \operatorname{span} m_k$ is the tangent of m_k at S_k. It follows that
$$\mathcal{H}_c\mathcal{V} \cap \mathcal{K} = \emptyset, \tag{8}$$

which completes the proof. □

Now we introduce coordinates in order to obtain a parametric representation of $\mathcal{K}$. Assume that PG(2,3) and PG(5,3) are projective spaces $\mathcal{P}(F^3)$ and $\mathcal{P}(F^6)$, respectively. The Veronese mapping is given as

$$\varphi : \mathcal{P}(F^3) \to \mathcal{P}(F^6),\ F(x_0, x_1, x_2) \mapsto F(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2). \tag{9}$$

We fix the points

$$U := F(1,0,0) \text{ and } P := U^\varphi = F(1,0,0,0,0,0). \tag{10}$$

Let $m_k \in \mathcal{C}$ be a conic through P. Each bisecant of m_k contains exactly one internal point of m_k; see Figure 1. Thus the mapping

$$\rho : \mathcal{V} \setminus \{P\} \to \mathcal{K},\ Y \mapsto Y^\rho \text{ with } P, Y, Y^\rho \text{ collinear} \tag{11}$$

is a well–defined bijection. Putting

$$\mathcal{W} := \mathcal{P}(F^3) \setminus \{U\} \tag{12}$$

yields the bijection

$$\psi : \mathcal{W} \to \mathcal{K},\ X \mapsto X^{\varphi\rho} \tag{13}$$

whose domain is a dual affine plane.

Theorem 2 *In terms of homogeneous coordinates the mapping (13) takes the form*

$$F(x_0, x_1, x_2) \overset{\psi}{\mapsto} F(x_0^2 + 1, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2) \tag{14}$$

Proof. At first we note that the use of the inhomogeneous term $x_0^2 + 1$ is not ambiguous: In fact, if $q : F^{n+1} \to F$ is a quadratic form, then

$$(2x_0, \dots, 2x_n)^q = 2^2 \cdot (x_0, \dots, x_n)^q = 1 \cdot (x_0, \dots, x_n)^q. \tag{15}$$

Next choose a fixed pair $(x_1, x_2) \in F^2 \setminus \{(0,0)\}$. The Veronese image of the line joining U with the point $F(0, x_1, x_2) \in \mathcal{P}(F^3)$ is a conic m through P. Then m comprises those four points which are spanned by the vectors

$$\mathbf{v}_u := (u^2, ux_1, ux_2, x_1^2, x_1x_2, x_2^2) \quad (u \in F), \quad \mathbf{v}_\infty := (1,0,0,0,0,0). \tag{16}$$

The three internal points of the conic m are the three diagonal points of the planar quadrangle m. We observe

$$\mathbf{v}_u + \mathbf{v}_\infty = 2\mathbf{v}_{u+1} + 2\mathbf{v}_{u+2} \text{ for all } u \in F, \tag{17}$$

whence $F(u, x_1, x_2)^\psi = (F\mathbf{v}_u)^\rho = F(\mathbf{v}_u + \mathbf{v}_\infty)$ for all $u \in F$.

As (x_1, x_2) varies in $F^2 \setminus \{(0,0)\}$, we obtain all four conics $\{m_0, m_1, m_2, m_\infty\}$, since those conics can be relabelled in such a way that (x_1, x_2) yields the conic m_k with $k = x_2/x_1$. □

Remark 1 The dual Veronese mapping φ^* assigns to each line l of $\mathrm{PG}(2,3)$ with homogeneous coordinates $F(a_0, a_1, a_2)$ the prime of $\mathrm{PG}(5,3)$ with homogeneous coordinates

$$F(a_0^2, 2a_0a_1, 2a_0a_2, a_1^2, 2a_1a_2, a_2^2), \tag{18}$$

since l^{φ^*} equals the osculating prime of $\mathcal{V}$ along the conic l^{φ}. The image of φ^* is the dual Veronese surface $\mathcal{V}^*$. The nine lines which are not running through U are characterized by $a_0 \neq 0$, whence their images under φ^* are immediate from (18). By [6, Remark 1], the remaining three primes in $\mathcal{K}^*$ have coordinates

$$F(0,0,0,1,0,1),\ F(0,0,0,2,2,1),\ F(0,0,0,2,1,1). \tag{19}$$

Note that the 01–, 02–, and 12–coordinates in [6] have to multiplied by 2 in order to match (18). By virtue of (14), (18), and (19) it is easy to verify in terms of coordinates that no point of $\mathcal{K}$ is incident with a prime belonging to $\mathcal{K}^*$. This gives an alternative proof of Theorem 1.

Remark 2 With the help of formula (14) one may immediately write down twelve vectors of F^6 representing the points of $\mathcal{K}$. If those vectors are regarded as columns of a 6×12 matrix over F, then a generator matrix of the *extended ternary Golay code* G_{12} arises. We refer to [1], [2, 8.6], and [6] for further details on the connections between the Witt design W_{12} and coding theory.

4 Replacing Conics of the Veronese Surface

In what follows we shall stick to the terminology introduced in Theorem 1, as we aim at generalizing the construction given there.

Choose one conic m_k ($k \in F \cup \{\infty\}$). Let t_k be the tangent of m_k at P, $\Delta_{k,0} := m_k \setminus \{P\}$, and $\Delta_{k,1} := \Delta_k$. Denote by $\Delta_{k,2}$ the set of all external points of m_k that are off the tangent t_k. It is easily seen from Figure 1 that there is a unique elation κ_k of the plane $\operatorname{span} m_k$ with centre P and axis t_k such that

$$(\Delta_{k,j})^{\kappa_k} = \Delta_{k,j+1} \text{ for all } j \in F. \tag{20}$$

Also, each $\Delta_{k,j+1}$ is the set of internal points of the conic $\Delta_{k,j} \cup \{P\}$, whence the restrictions of κ_k and ρ to the conic m_k are coinciding.

All four collineations κ_k do not simultaneously extend to a collineation of $\mathrm{PG}(5,3)$, since $\mathcal{V} \setminus \{P\}$ contains four distinct coplanar points, whereas $\mathcal{K}$ does not. However, if we choose three collineations, e.g. κ_1, κ_2, and κ_∞, then they extend to a unique perspective collineation μ_0 of $\mathrm{PG}(5,3)$ with centre P and axis $\mathcal{H}_{m_0}\mathcal{V}$: If the numbering of the conics m_k is done according to the proof of Theorem 2, then $0 = 0/1$ implies that μ_0 is given by

$$F(y_{00}, y_{01}, y_{02}, y_{11}, y_{12}, y_{22}) \mapsto F(y_{00} + y_{22}, y_{01}, y_{02}, y_{11}, y_{12}, y_{22}). \tag{21}$$

Clearly, all points of the conic plane span $m_0 \subset \mathcal{H}_{m_0}\mathcal{V}$ are fixed under μ_0, i.e. μ_0 extends $\kappa_0^0 = \mathrm{id}$.

One may assign to each $(p,q,r,s) \in F^4$ the quadruple $(\kappa_0^p, \kappa_1^q, \kappa_2^r, \kappa_\infty^s)$ of collineations. This mapping is an isomorphism of the group $(F^4, +)$ onto the direct sum of the collineation groups generated by the κ_k's. In addition, each $(p,q,r,s) \in F^4$ yields the point set

$$\Delta_{0,p} \cup \Delta_{1,q} \cup \Delta_{2,r} \cup \Delta_{\infty,s} \tag{22}$$

consisting of twelve points in $\mathrm{PG}(5,3)$.

Each permutation of the four conics m_0, m_1, m_2, m_∞ arises from an automorphic projective collineation of the Veronese surface $\mathcal{V}$. So, if two elements of F^4 differ only in the arrangement of their entries, then they yield projectively equivalent 12–sets.

The subgroup S of $(F^4, +)$ generated by $(1,1,1,0)$, $(1,1,0,1)$, $(1,0,1,1)$, and $(0,1,1,1)$ consists of all (p,q,r,s) with $p+q+r+s=0$. Each element of S yields a quadruple of planar collineations that extend to a collineation of $\mathrm{PG}(5,3)$ by (21). Hence all corresponding 12–sets are projectively equivalent to $\mathcal{V} \setminus \{P\}$.

The elements of the coset $(1,0,0,0)+S$ are characterized by $p+q+r+s=1$ and yield 12–sets that are projectively equivalent to $\mathcal{K}$.

We mention without proof some results about the 12–sets that arise from the elements of the remaining coset ($p+q+r+s=2$). Each such point set, say $\mathcal{R}$, is neither projectively equivalent to $\mathcal{K}$ nor projectively equivalent to a subset of $\mathcal{V}$. Among the 364 primes of $\mathrm{PG}(5,3)$ there are exactly 42 which meet $\mathcal{R}$ in precisely six points. Those 42 primes have P as their only common point. Thus P is invariant under the group of automorphic collineations of $\mathcal{R}$. So it seems natural to project $\mathcal{R}$ through the point P to a prime $\mathcal{H}$ of $\mathrm{PG}(5,3)$ not containing P: The conic planes m_k ($k \in F \cup \{\infty\}$) are projected to four mutually skew lines $l_k \subset \mathcal{H}$, the tangent plane $\mathcal{T}_P\mathcal{V}$ of the Veronese surface goes over to the only transversal line of the l_k's, and the set $\mathcal{R}$ is mapped onto those twelve points of the four lines l_k that are not on their common transversal line.

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