Geometries on σ-Hermitian Matrices

Modern Algebra and Its Applications
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Part 1

Square Matrices

The first part deals with some notions and results from ring geometry and the geometry of square matrices.

The presentation is not given in the most general form, but in a way which is tailored for our needs.

Basic Notation

Throughout this lecture we adopt the following notation:

- K ... a (not necessarily commutative) field.
- n ... an integer > 1. (Many results hold trivially for $n = 1$.)
- R ... the ring of $n \times n$ matrices with entries in K.
- R^2 ... considered as free left R-module over R. (We use row notation).
- $\mathrm{GL}_2(R) = \mathrm{GL}_{2n}(K)$ ... the group of invertible 2×2-matrices with entries in R.

The Projective Line over a Ring

Below we follow Herzer [10]; see also Blunck and Herzer [7].

- $(A, B) \in R^2$ is called an *admissible pair* if there exists a matrix in $\mathrm{GL}_2(R)$ with (A, B) being its first row.

- The *projective line over R*, in symbols $\mathbb{P}(R)$ is the set of cyclic submodules $R(A, B)$ for all admissible pairs $(A, B) \in R^2$.

- Let $(A', B'), (A, B) \in R^2$ with (A, B) admissible.

$$R(A', B') = R(A, B) \quad \Leftrightarrow \quad (A', B') = U(A, B) \ \text{ for some } \ U \in \mathrm{GL}_n(K).$$

  In this case (A', B') is admissible too.

The results from the last item do not hold over any ring; see [3].

A Link with Grassmannians

The projective line over our matrix ring R allows the following description (see Blunck [2]) which is not available for arbitrary rings, as it makes use of the left row rank of a matrix X over K (in symbols: $\operatorname{rank} X$):

$$\mathbb{P}(R) = \{R(A,B) \mid A,B \in R,\ \operatorname{rank}(A,B) = n\}. \tag{1}$$

Here $(A,B) \in R^2$ has to be interpreted as $n \times 2n$ matrix over K.

Because of (1), $\mathbb{P}(R)$ is in bijective correspondence with the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ comprising all n-dimensional subspaces of the left K-vector space K^{2n} via

$$\mathbb{P}(R) \to \mathrm{Gr}_{2n,n}(K) : R(A,B) \mapsto \text{left row space of } (A,B). \tag{2}$$

R has Stable Rank 2

Our matrix ring $R = K^{n\times n}$ has *stable rank* 2. (See Veldkamp [13].) Viz. for each $(A, B) \in R^2$ which is *unimodular*, i. e., there are $X, Y \in R$ with $AX + BY = I$, there exists $W \in R$ such that

$$A + BW \in \mathrm{GL}_n(K).$$

Consequently, two important results hold:

- Any unimodular pair $(A, B) \in R^2$ is admissible.
  (Unimodularity is in general much easier to check than admissibility.)
- *Bartolone's parametrisation*

$$R^2 \to \mathbb{P}(R) : (T_1, T_2) \mapsto R(T_2T_1 - I, T_2) \tag{3}$$

  is well defined and surjective (Bartolone [1]). Hence

$$\mathbb{P}(R) = \{R(T_2T_1 - I, T_2) \mid T_1, T_2 \in R\}.$$

R has Stable Rank 2 (cont.)

We have $\mathbb{P}(R) = R(I,0)^{\mathrm{GL}_2(R)}$. (This holds over an arbitrary ring.)

The elementary subgroup $\mathrm{E}_2(R)$ of $\mathrm{GL}_2(R)$ is generated by the set of all elementary matrices

$$B_{12}(T) := \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \quad \text{and} \quad B_{21}(T) := \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \quad \text{with} \quad T \in R.$$

$\mathrm{E}_2(R)$ is also generated by the set of all matrices

$$E(T) := \begin{pmatrix} T & I \\ -I & 0 \end{pmatrix} \quad \text{with} \quad T \in R.$$

Indeed, $\mathbb{P}(R) = R(I,0)^{\mathrm{E}_2(R)}$ follows from

$$(T_2T_1 - I, T_2) = (I,0) \cdot E(T_2) \cdot E(T_1) \quad \text{for all} \quad T_2, T_1 \in R.$$

See [4] and Veldkamp [13].

Projective Matrix Spaces

- The point set $\mathbb{P}(K^{n\times n}) = \mathbb{P}(R)$ can be identified with the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ according to (2).
- All pairs (A, I) and (I, A) with $A \in R$ are admissible, because $\mathrm{rank}(A, I) = \mathrm{rank}(I, A) = n$.
- The Grassmannian $\mathrm{Gr}_{2n,n}(K)$ is also called the *projective space of $n \times n$ matrices over K*. See Wan [14]; cf. also Dieudonné [9].
- The bijection from (2) turns (3) into a surjective parametric representation of the Grassmannian $\mathrm{Gr}_{2n,n}(K)$, namely

$$R^2 \to \mathrm{Gr}_{2n,n}(K) : (T_1, T_2) \mapsto \text{left row space of } (T_2T_1 - I, T_2).$$

- Many authors (like Wan [14]) adopt the projective point of view for $\mathrm{Gr}_{2n,n}(K)$: $(n-1)$-dimensional subspaces of an $(2n-1)$-dimensional projective space.

Additional Structure

A major difference concerns the additional structure on $\mathbb{P}(R) = \mathrm{Gr}_{2n,n}(K)$:

Ring Geometry

- $\mathbb{P}(R)$ is endowed with the symmetric and anti-reflexive relation *distant* ($\triangle$) defined by

$$R(A,B) \triangle R(C,D) \quad \Leftrightarrow \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(R).$$

- Being distant is equivalent to the complementarity of the n-dimensional subspaces of K^{2n} which correspond via (2).
- Given points $p, q \in \mathbb{P}(R)$ there exists some point $r \in \mathbb{P}(R)$ such that $p \triangle r \triangle q$.

  This property holds, more generally, over any ring of stable rank 2. It provides another way of understanding Bartolone's parametrisation, as

$$R(I,0) \triangle R(T_1, I) \triangle R(T_2T_1 - I, T_2) \quad \text{for all} \quad T_2, T_1 \in R.$$

- $\big(\mathbb{P}(R), \triangle\big)$ is called the *distant graph*.

Additional Structure (cont.)

Matrix Geometry

- Two n-dimensional subspaces of K^{2n} are called *adjacent* ($\sim$) if, and only if, their intersection has dimension $n-1$.

- $\big(\mathrm{Gr}_{2n,n}(K), \sim\big)$ is called the *Grassmann graph*.

Adjacency can be expressed in terms of being distant and vice versa; see [5].

Therefore the distant graph and the Grassmann graph have—up to the identification from (2)—the same automorphism group .

An Application: Chow's Theorem

There is no need to distinguish in the following description (like, e. g., in Wan [14]) between those automorphisms of the Grassmann graph which arise from semilinear bijections of K^{2n} and those which arise from non-degenerate sesquilinear forms on K^{2n}.

Theorem 1 (Chow (1949), [6]). *A mapping $\Phi : \mathrm{Gr}_{2n,n}(K) \to \mathrm{Gr}_{2n,n}(K)$ is an automorphism of the Grassmann graph if, and only if, it can be written in the form*

$$\text{left row space of } (T_2T_1 - I, T_2) \mapsto \text{left row space of } (T_2^{\varphi}T_1^{\varphi} - I, T_2^{\varphi}) \cdot A,$$

where $\varphi : R \to R$ is an automorphism or antiautomorphism of R and $A \in \mathrm{GL}_{2n}(K)$.

The above theorem describes the full automorphism group of the Grassmann graph and—up to the identification with $\mathbb{P}(R)$ from (2)—also the full automorphism group of the distant graph.

Part 2

σ-Hermitian Matrices

The second part deals with geometries on σ-Hermitian matrices.

The situation is more complicated here, because the σ-Hermitian matrices do not comprise a subring of the ring of square matrices.

σ-Transposition

We suppose from now on that the field K admits an *involution*, i. e. an antiautomorphism σ, say, such that $\sigma^2 = \mathrm{id}_K$. As before, we let $R = K^{n\times n}$ with $n > 1$.

- σ determines an antiautomorphism of R, namely the *σ-transposition*

$$M = (m_{ij}) \mapsto (M^\sigma)^{\mathrm{T}} := (m_{ji}^\sigma).$$

- The elements of $H_\sigma := \{X \in R \mid X = (X^\sigma)^{\mathrm{T}}\}$ are the *σ-Hermitian matrices* of R.

- In the special case that $\sigma = \mathrm{id}_K$ the field K is commutative, and we obtain the subset of *symmetric matrices* of $K^{n\times n}$.

Algebraic Properties

Below we adopt the terminology from Blunck and Herzer [7]: We consider $R = K^{n\times n}$ as an algebra over $F = \mathrm{Fix}\,\sigma \cap Z(K)$, where $\mathrm{Fix}\,\sigma = \{x \in K \mid x = x^\sigma\}$ and $Z(K)$ denotes the centre of K.

- H_σ is a *Jordan system* of R. This means:
  1. H_σ is a subspace of the F-vector space R.
  2. $I \in H_\sigma$.
  3. $A^{-1} \in H_\sigma$ for all $A \in \mathrm{GL}_n(K) \cap H_\sigma$.
- H_σ is *Jordan closed*, i. e., it satisfies the condition

$$ABA \in H_\sigma \quad \text{for all} \quad A, B \in H_\sigma.$$

- The set H_σ is not closed under matrix multiplication.

Ring Geometry . . .

The *projective line over H_σ*, in symbols $\mathbb{P}(H_\sigma)$, is defined as

$$\mathbb{P}(H_\sigma) = \{R(T_2T_1 - I, T_2) \mid T_1, T_2 \in H_\sigma\}. \quad (4)$$

One motivation to exhibit such structures came from the theory of *chain geometries*. These generalise the classical circle geometry of Möbius by replacing the $\mathbb{R}$-algebra $\mathbb{C}$ with an arbitrary algebra over a commutative field (here: the F-algebra R). See Blunck and Herzer [7].

- From Bartolone's parametrisation (3), $\mathbb{P}(H_\sigma)$ is indeed a subset of $\mathbb{P}(R)$.

- $\mathbb{P}(H_\sigma)$ is not defined as the set of all cyclic submodules $R(A, B)$ with (A, B) admissible and $A, B \in H_\sigma$.

- Nevertheless, all points $R(A, I)$ and $R(I, A)$ with $A \in H_\sigma$ belong to $\mathbb{P}(H_\sigma)$.

. . . vs. Matrix Geometry

Below we follow Wan [14]: Let $\beta : K^{2n} \times K^{2n} \to K$ be the non-degenerate σ-anti-Hermitian sesquilinear form given by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathrm{GL}_{2n}(K).$$

This form β is trace-valued and has Witt index n.

The subset of $\mathrm{Gr}_{2n,n}(K)$ comprising all maximal totally isotropic (m. t. i.) subspaces of β is the point set of the *projective space of σ-Hermitian matrices*.
(Or: the point set of the *dual polar space* given by β; see also Cameron [8].)

- An admissible pair $(A, B) \in R^2$ gives rise to a m. t. i. subspace if, and only if,

$$A(B^\sigma)^{\mathrm{T}} = B(A^\sigma)^{\mathrm{T}}. \tag{5}$$

- All pairs (A, I) and (I, A) with $A \in H_\sigma$ give rise to m. t. i. subspaces.

Remarks

- Cf. Blunck and Herzer [7, 3.1.5].

  Note that our Jordan system H_σ need not be strong in the sense of the authors (in German: “starkes Jordan-System”). We do not assume any richness conditions, like the strongness from *loc. cit.*

- Cf. Wan [14, p. 306].

  When dealing with σ-Hermitian matrices extra assumptions on the set $\mathrm{Fix}\,\sigma$, the centre of K, and the trace map $K \to \mathrm{Fix}\,\sigma : x \mapsto x + x^\sigma$ are adopted. None of them is not used here.

Question

The set H_σ of σ-Hermitian $n \times n$ matrices over K gives rise to two subsets of the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ (which has to be identified with the projective line $\mathbb{P}(R)$ according to (2)).

- In the ring-geometric setting the subset is given in terms of the parametric representation (4).
- In the matrix-geometric setting there is the defining matrix equation (5).

Question: Do these two subsets coincide or not?

Example

Let K be a commutative field, $\sigma = \mathrm{id}_K$, and $n = 2$.

Hence β is a symplectic form on K^4 and H_σ is the set of symmetric 2×2 matrices over K.

- In this case the answer to our previous question is affirmative.
- In projective terms we have:

  $\mathbb{P}(R)$ ... the Grassmannian of lines of a 3-dimensional projective space over K.

  $\mathbb{P}(H_\sigma)$ ... a general linear complex, i. e., the set of null-lines of a symplectic polarity.

Main Theorem

Theorem 2 ([6]). *Let K be any field admitting an involution σ. Then the following sets coincide:*

- *the point set of the projective line over the Jordan system H_σ of all σ-Hermitian $n \times n$ matrices over K;*

- *the point set of the projective space of σ-Hermitian $n \times n$ matrices over K.*

Our proof of this theorem uses two auxiliary results about dual polar spaces. So we work in the realm of matrix geometry, *viz.* the Grassmannian $\mathrm{Gr}_{2n,n}(K)$ and the sesquilinear form β, rather than in a ring-theoretic setting.

Two Auxiliary Results

The first result is rather technical.

Lemma 1 ([6]). *Let $U = V \oplus W$ be a maximal totally isotropic subspace of (K^{2n}, β) which is given as direct sum of subspaces V and W. Then there exists a maximal totally isotropic subspace, say X, such that $X \cap V^{\perp} = W$.*

Proof (sketched). Our proof amounts to changing from the standard basis to a new basis of K^{2n} such that the matrix of the sesquilinear form β with respect to this new basis has a particular block form from which the assertion is immediate. Starting from the given matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

for β, this task can be accomplished by several elementary row and column transformations. $\square$

Two Auxiliary Results (cont.)

Lemma 2 ([6]). *Let U_1 and U_2 be maximal totally isotropic subspaces of (K^{2n}, β). Then there exists a maximal totally isotropic subspace X which is a common complement of U_1 and U_2.*

Proof (sketched). Let $V := U_1 \cap U_2$. The first step is to find a t. i. subspace W such that $V \oplus W$ is a m. t. i. subspace and $U_i \cap W = 0$ for $i = 1, 2$. Then Lemma 1, applied to $U := V \oplus W$, establishes the existence of a subspace X with the required properties. $\square$

Lemma 2 can be reformulated in ring-theoretic language as follows:

Corollary. *Given points $p, q \in \mathbb{P}(H_\sigma)$ there exists some point $r \in \mathbb{P}(H_\sigma)$ with the property $p \bigtriangleup r \bigtriangleup q$.*

Proof of the Main Theorem

Proof (sketched). The proof of one inclusion simply amounts to substituting the parametrisation (4) into the matrix equation (5).
Conversely, let the left row space of (A, B) be a m. t. i. subspace. By Lemma 2, there exists a m. t. i. subspace of K^{2n} which is a common complement of the left row spaces of $(I, 0)$ and (A, B). In matrix form it can be written as

$$(C, I) \quad \text{with} \quad C \in H_\sigma.$$

So, in terms of $\mathbb{P}(H_\sigma) \subset \mathbb{P}(R)$, we have

$$R(I, 0) \vartriangle R(C, I) \vartriangle R(A, B).$$

Defining

$$T_1 := C \quad \text{and} \quad T_2 := (BC - A)^{-1} B$$

gives after some calculations that $R(A, B) = R(T_2 T_1 - I, T_2)$ and $T_1, T_2 \in H_\sigma$. Hence, finally $R(A, B) \in \mathbb{P}(H_\sigma)$. $\square$

Remark

In view of Theorem 2 one may carry over results from $\mathbb{P}(H_\sigma)$ which are based on the parametrisation (4) to the projective space of σ-Hermitian matrices.

See [6] for further details.

Open Problems

1. *Is it possible to express the adjacency relation on a projective space of σ-Hermitian matrices in terms of the distant relation on $\mathbb{P}(H_\sigma)$?*

   An affirmative answer would extend our result from a structural point of view.

   See [6], Kwiatkowski and Pankov [11], and Pankov [12, 4.7.1] for further details.

2. *Is it possible to extend the present results from the matrix ring $R = K^{n\times n}$ to other rings which admit an anti-automorphism?*

   An affirmative answer would give, *mutatis mutandis*, an alternative approach to projective lines over the Jordan system comprising the fixed elements of the given anti-automorphism. More precisely, one would obtain a defining equation similar to (5) rather than a parametric representation.

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