Preserver Problems in Geometry

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Introduction

Grassmannians

We consider a k-dimensional left vector space V over a (not necessarily commutative) field F, and denote by

$$\mathcal{G}_m(V)$$

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Since proper skew fields are included, we cannot use tools from exterior algebra.

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- **For** $k = 2m$ **only**: Semilinear bijections $f : V \to V^*$, where V^* denotes the dual of V:

  $$X \mapsto \text{annihilator of } X^f.$$

  Any such f is accompanied by a unique antiautomorphism of K. (There are skew fields admitting no antiautomorphism.)

Problem

Characterise the standard transformations of Grassmannians from the previous slide by as few geometric invariants as possible.

The Grassmann Graph

- Subspaces $X_1, X_2 \in \mathcal{G}_m(V)$ are called *adjacent* (in symbols: $X_1 \sim X_2$) if
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- The automorphisms of the Grassmann graph are precisely those bijections of $\mathcal{G}_m(V)$ that preserve adjacency in both directions.

- We shall often assume $2 \leq m \leq k - 2$ in order to avoid a complete graph.

Theorem (W. L. Chow (1949) [7])

Let $2 \leq m \leq k-2$.

A bijective mapping

$$\varphi : \mathcal{G}_m(V) \to \mathcal{G}_m(V) : X \mapsto X^\varphi$$

preserves adjacency in both directions, i. e.,

$$X_1 \sim X_2 \Leftrightarrow X_1^\varphi \sim X_2^\varphi \quad \text{for all} \quad X_1, X_2 \in \mathcal{G}_m(V),$$

if, and only if, φ is a standard transformation.

The Matrix Approach

Projective Matrix Spaces

Each element of the Grassmannian $\mathcal{G}_m(F^k)$ can be viewed as the left row space of a matrix $A|B$ with left row rank m, where $A \in F^{m\times(k-m)}$, $B \in F^{m\times m}$, and vice versa. We let $n := k - m$.

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- Let $\mathrm{rk}(A|B) = m$. Then $A|B$ and $A'|B'$ have the same left row space, if and only if, there is a $T \in \mathrm{GL}_m(F)$ with

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- One may consider a matrix pair

$$(A, B) \in F^{m\times n} \times F^{m\times m} \quad \text{with} \quad \mathrm{rk}(A|B) = m$$

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as left homogeneous coordinates of an element of $\mathcal{G}_m(F^k)$.

- $\mathcal{G}_m(F^k)$ is also called the point set of the *projective space of $m \times n$ matrices* over F.

An Embedding

We have an injective mapping:

$$\begin{array}{ccccc} F^{m\times n} & \rightarrow & F^{m\times k} & \rightarrow & \mathcal{G}_m(F^k) \\ A & \mapsto & A|I_m & \mapsto & \text{left rowspace of } A|I_m \end{array}$$

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- Matrices from $F^{m\times n}$ are adjacent precisely when their images in $\mathcal{G}_m(F^k)$ are adjacent.

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- **For $m = n$ only:**

$$A \mapsto P \cdot (A^{\sigma\mathsf{T}}) \cdot Q + R,$$

  where P, Q, R are as above, σ is an antiautomorphisms of F, and T denotes transposition.

Theorem (L. K. Hua (1951) [10])

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A link between the theorems of Chow and Hua is provided by the theory of *spine spaces*; see K. Prażmowski and M. Żynel [24].

Other Matrix Spaces

and Related Topics

Transformations on Symmetric Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any commutative field F:

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  This is also called the *projective space of $m \times m$ symmetric matrices* over F.

Transformations on σ-Hermitian Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any field F that admits an antiautomorphism σ of order two:

- The space of $m \times m$ σ-Hermitian matrices over F.

- The space of maximal totally isotropic subspaces of F^{2m} w.r.t. a particular skew σ-Hermitian sesquilinear form.

  This is also called the *projective space of $m \times m$ σ-Hermitian matrices* over F.

Transformations on Alternating Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any commutative field F:

- The space of $m \times m$ alternating matrices over F.

  Adjacency is not inherited from $F^{n \times n}$.

- The space of maximal totally singular subspaces of F^{2n} w.r.t. a particular quadratic form.

  This is also called the *projective space of $m \times m$ alternating matrices* over F.

Monographs and Surveys

- W. Benz: *Geometrische Transformationen* (1992) [1].
- W. Benz: *Real Geometries* (1994) [2].
- J. Lester: Distance preserving transformations (1995) [19].
- M. Pankov: *Grassmannians of Classical Buildings* (2010) [21].
- P. Šemrl: Maps on matrix and operator algebras (2006) [26].
- Z.-X. Wan: *Geometry of Matrices* (1996) [27].

Applications: light cone preservers, Jordan homomorphisms, ...

Chow's Theorem

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3. Why adjacency, why not . . . ?
4. Is there a unified theory?

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- $(k-2)$-flats are called *hyperplanes*.

Techniques: Maximal Cliques

For $2 \leq m \leq k-2$ the maximal cliques of the Grassmann graph $(\mathcal{G}_m(V), \sim)$ fall into two classes.

- A *star* is the set of all $(m-1)$-flats through a fixed $(m-2)$-flat, called the *centre* of the star.
- A *top* is the set of all $(m-1)$-flats within a fixed m-flat, called the *carrier* of the top.

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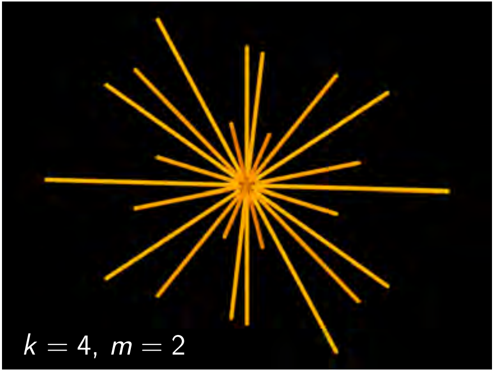

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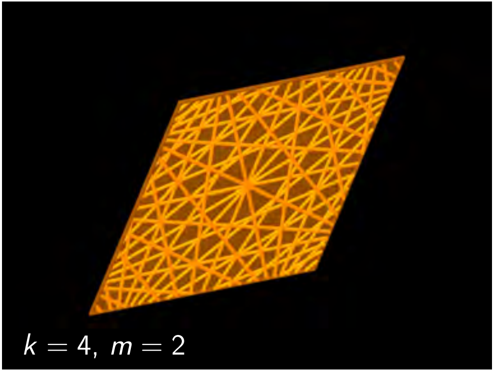

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- The intersection of two distinct stars (tops) is either empty or it contains a single $(m-1)$-flat.
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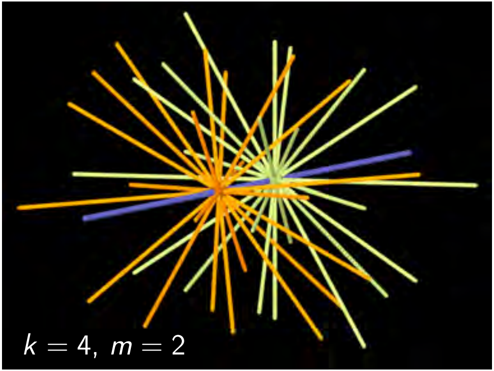

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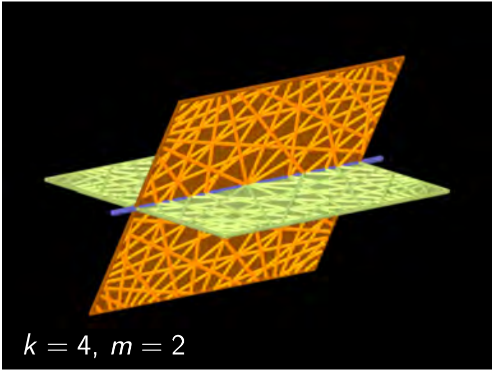

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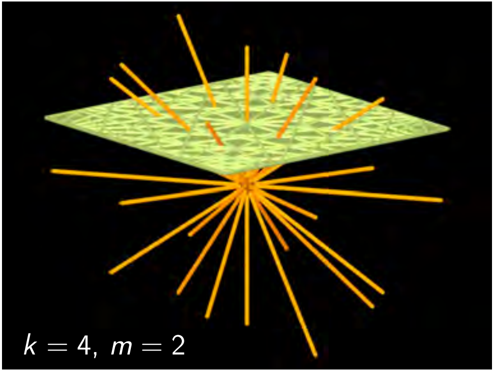

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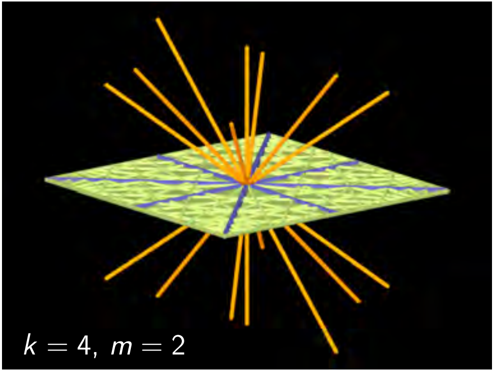

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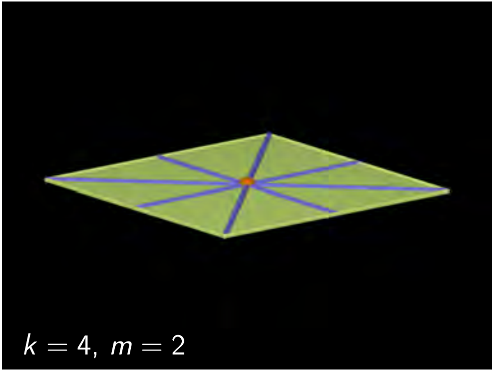

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Techniques: Collineations

Fundamental Theorem of Projective Geometry

All collineations between the point sets of projective spaces on vector spaces V, V' of dimension ≥ 3 stem from semilinear bijections $V \to V'$, and vice versa.

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A (rather intricate) example of an adjacency preserving bijection $\mathcal{G}_2(F^4) \to \mathcal{G}_2(F'^3)$ is due to A. Kreuzer [18].

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- $\operatorname{dist}(X, Y) = s \quad \Leftrightarrow \quad \dim(X \cap Y) = m - s.$
- The diameter of the Grassmann graph $\mathcal{G}_m(V)$ equals

$$\operatorname{diam} \mathcal{G}_m(V) = \min\{m, k - m\}.$$

Theorem (M.-H. Lim (2010) [20])

Let $2 \leq m \leq k-2$ and chose an integer s such that

$$1 \leq s < \operatorname{diam} \mathcal{G}_m(V).$$

A surjective mapping

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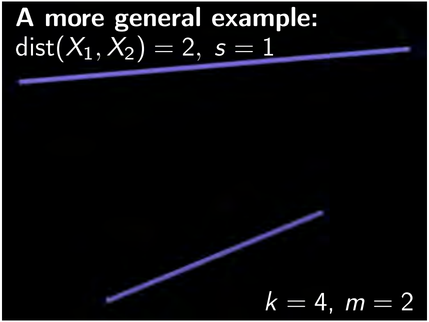

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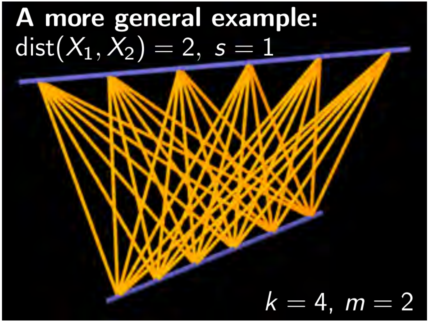

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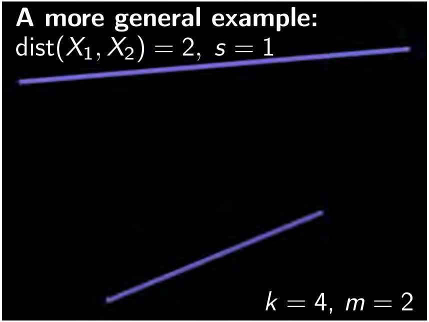

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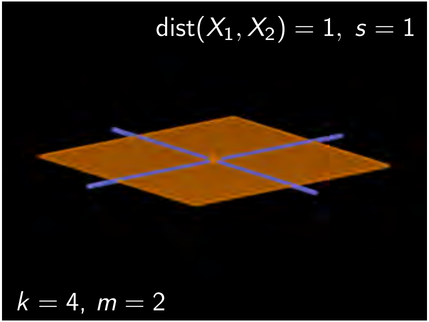

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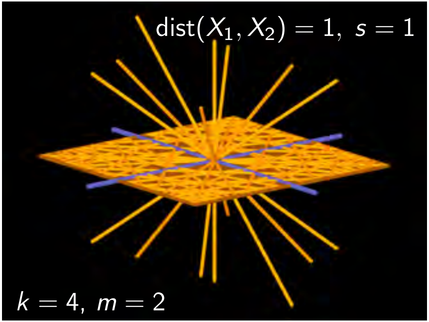

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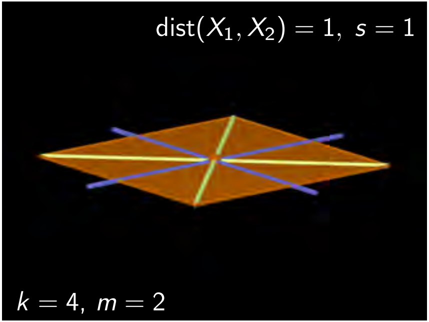

Corollary (Lim's theorem for $s = d - 1$)

Let $2 \leq m \leq k-2$ *and define* $d := \operatorname{diam} \mathcal{G}_m(V)$*. A surjective mapping*

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It overlaps with a characterisation of (not necessarily surjective) distance preserving mappings due to J. Kosiorek, A. Matraś, and M. Pankov [17], [22].

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- For Grassmannians over rings refer to L. P. Huang [11], [12].

Serdecznie dziękuję
za zaproszenie
i za Państwa uwagę!

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The bibliography focusses on preserver problems for Grassmannians, and includes only a few items of related work.

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