A Three-Dimensional Laguerre Geometry

Hans Havlicek

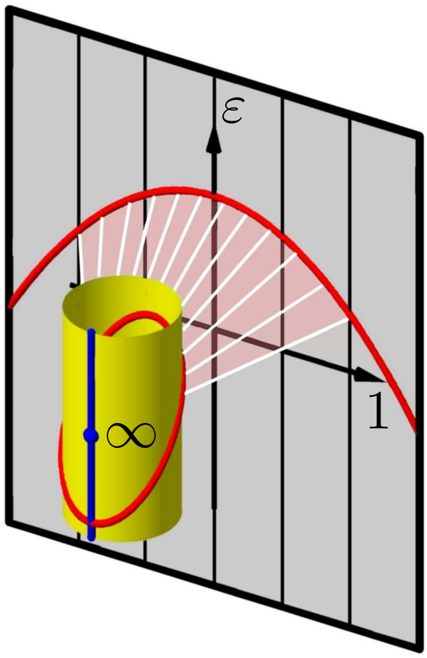
Institut für Geometrie
Technische Universität Wien
Vienna, Austria

H.-J. Samaga. Dreidimensionale Kettengeometrien über $\mathbb{R}$. *J. Geom.* **8** (1976), 61–73.

H. Havlicek and K. List. A Three-Dimensional Laguerre Geometry and Its Visualization. *Proceedings "Dresden Symposium Geometry: Constructive and Kinematic"*. Dresden February 2003 (in print).

Blaschke's Cylinder

A quadratic cylinder in the real affine 3-space is a point model for the *projective line* over the ring $\mathbb{R}[\varepsilon]$ of *real dual numbers*. Two points are called *parallel* exactly if they are on a common generator.



Under a stereographic projection (centre ∞) all points that are *distant*, i.e. non-parallel, to ∞ are mapped bijectively onto the affine plane of dual numbers (*isotropic plane*).

An Affine Description

The geometry of conics on Blaschke's cylinder is a model for the *2-dimensional Laguerre geometry*.

It may be interpreted as an *extension* of the isotropic plane by *improper points*. They are represented as follows:

- The distinguished point ∞: All non-isotropic lines.
- Any other improper point: All the translates of an isotropic circle.

The Laguerre Geometry $\Sigma(\mathbb{R}, \mathbb{L})$

Let $\mathbb{L}$ be the 3-dimensional real commutative algebra with an $\mathbb{R}$-basis $1_{\mathbb{L}}, \varepsilon, \varepsilon^2$ and the defining relation $\varepsilon^3 = 0$. We shall identify $x \in \mathbb{R}$ with $x \cdot 1_{\mathbb{L}} \in \mathbb{L}$.

$\mathbb{L}$ is a *local ring*: Its non-invertible elements form the only maximal ideal $N := \mathbb{R}\varepsilon + \mathbb{R}\varepsilon^2$.

Laguerre geometry $\Sigma(\mathbb{R}, \mathbb{L})$:

- The point set is the *projective line* over $\mathbb{L}$:
$$\mathbb{P}(\mathbb{L}) := \{\mathbb{L}(a, b) \subset \mathbb{L}^2 \mid a \text{ or } b \text{ is invertible}\}$$
- The *chains* are the images of $\mathbb{P}(\mathbb{R}) \subset \mathbb{P}(\mathbb{L})$ under the natural right action of $\mathrm{GL}_2(\mathbb{L})$ on $\mathbb{L}^2$.

If two distinct points of $\mathbb{P}(\mathbb{L})$ can be joined by a chain then they are called *distant* ($\triangle$) or *non-parallel* ($\nparallel$).

There is a unique chain through any three mutually distant points.

Splitting the Point Set

We fix the point $\mathbb{L}(1,0) =: \infty \in \mathbb{P}(\mathbb{L})$.

- *Proper points*: $\mathbb{L}(z,1) \leftrightarrow z$ with $z \in \mathbb{L}$.
- *Improper points*: $\mathbb{L}(1,z) \leftrightarrow z$ with $z \in N$.

We can regard $\mathbb{P}(\mathbb{L})$ as the real affine 3-space on $\mathbb{L}$ together with an extra "improper plane" which is just a copy of the maximal ideal N.

Problem: Geometric description of this extension.

The Absolute Flag

We shall also use the *projective extension* $\mathbb{P}_3(\mathbb{R})$ of the affine space on $\mathbb{L}$ as follows:

$$\underbrace{\mathbb{R}(1, x_1, x_2, x_3)}_{\in \mathbb{P}_3(\mathbb{R})} \leftrightarrow \underbrace{x_1 + x_2\varepsilon + x_3\varepsilon^2}_{\in \mathbb{L}}$$

There is an *absolute flag* (f, F, Φ):

$$f := \mathbb{R}(0, 0, 0, 1)$$

is the point at infinity of the affine line $\mathbb{R}\varepsilon^2$,

$$F := \mathbb{R}(0, 0, 0, 1) + \mathbb{R}(0, 0, 1, 0)$$

is the line at infinity of the affine plane N,

$$\Phi : x_0 = 0$$

is the plane at infinity.

Chains Through an Improper Point

For each chain $\mathcal{C}$ let $\mathcal{C}^\circ$ be its *proper part*. Each chain has a unique improper point.

The proper part of a chain $\mathcal{C}$ is an algebraic curve which can be extended projectively . . . $\mathcal{C}^+$.

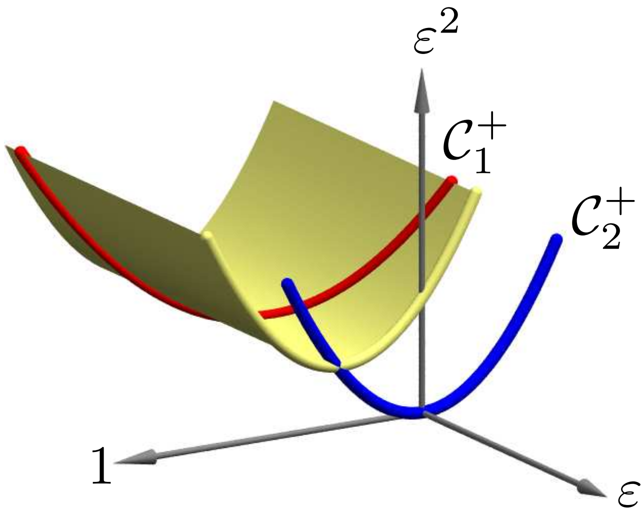
- $\mathbb{L}(1,0) = \infty \in \mathcal{C}$:
  $\mathcal{C}^+$ is a *line* with a point at infinity off the line F.
  All such lines arise from chains.

- $\mathbb{L}(1, x_3\varepsilon^2) \in \mathcal{C}$, $x_3 \neq 0$:
  $\mathcal{C}^+$ is a *parabola* through f with a tangent other than F.
  All such *admissible parabolas* arise from chains.

- $\mathbb{L}(1, x_2\varepsilon + x_3\varepsilon^2) \in \mathcal{C}$, $x_2 \neq 0$:
  $\mathcal{C}^+$ is a *cubic parabola* through f, with tangent F, and osculating plane Φ.
  Not all cubic parabolas of this form arise from chains.

Admissible Parabolas

Theorem. *Two admissible parabolas $\mathcal{C}_1^+$ and $\mathcal{C}_2^+$ represent the same improper point of $\mathbb{P}(\mathbb{L})$ if, and only if, the parallel projection of $\mathcal{C}_1^+$ to the plane of $\mathcal{C}_2^+$, in the direction of the ε-axis, is a translate of $\mathcal{C}_2^+$.*



Equivalent condition: The projection of $\mathcal{C}_1^+$ and the parabola $\mathcal{C}_2^+$ have second order contact at the point $f = \mathbb{R}(0, 0, 0, 1)$.

Admissible Cubic Parabolas

We say that a cubic parabola is *admissible* if it has the form $\mathcal{C}^+$ for a chain $\mathcal{C}$ of $\Sigma(\mathbb{L}, \mathbb{R})$.

Theorem. *A cubic parabola is admissible if, and only if, it has second order contact with the cubic parabola*

$$\{\mathbb{R}(1, t, t^2, t^3) \mid t \in \mathbb{R}\} \cup \{\mathbb{R}(0, 0, 0, 1)\}$$

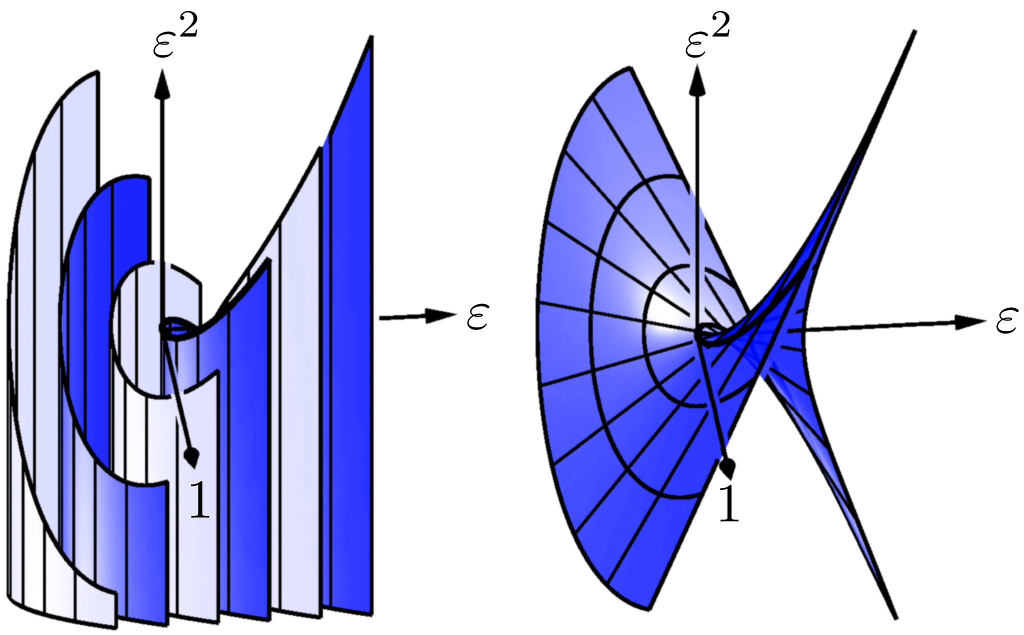
at the point $f = \mathbb{R}(0, 0, 0, 1)$.

Two admissible cubic parabolas represent the same improper point if, and only if, they have third order contact at $f = \mathbb{R}(0, 0, 0, 1)$.

Final Remarks

- Chains that *touch* each other at an improper point $\Leftrightarrow$ parallel lines or parabolas (cubic parabolas) with contact of order 3 (order 4) at the point f.

- A purely "affine" description of higher order contact of twisted cubics is possible.
  Example: Twisted cubics with contact of order 4 at f on *Cayley's ruled surface*:



- Similar results should hold for other local algebras of finite dimension. For more general algebras the problem seems to be intricate.