Three-dimensional Chain Geometries and their Visualization

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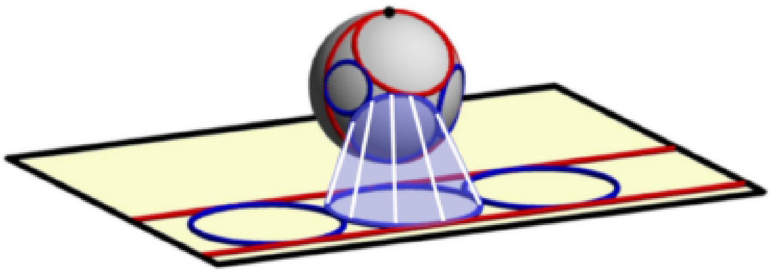
The Real Möbius Plane

Algebraic definition

Points: $\mathbb{C} \cup \{\infty\}$ (complex projective line)
Circles: Images of $\mathbb{R} \cup \{\infty\}$ under $\mathrm{PGL}_2(\mathbb{C})$

Other models

Elliptic quadric / conics
Euclidean plane + one point / circles and lines



The Projective Line over a Ring

All our rings are associative, with unit element 1 which is inherited by subrings and acts unitally on modules.

Let $\mathrm{GL}_2(A)$ be the group of invertible (2×2)-matrices with entries in a ring A.

A pair $(a, b) \in A^2$ is called *admissible* if (a, b) is the first row of a matrix in $\mathrm{GL}_2(A)$.

Projective line over A: $\mathbb{P}(A) := \{A(a, b) \mid (a, b) \text{ admissible}\}$

Real Chain Geometries

Assume that A is a real algebra. We identify $a \in \mathbb{R}$ and $a \cdot 1_A$. There is the natural embedding

$$\mathbb{P}(\mathbb{R}) \to \mathbb{P}(A) : \mathbb{R}(a, b) \mapsto A(a, b).$$

The images of $\mathbb{P}(\mathbb{R})$ under $\mathrm{PGL}_2(A)$ are the *chains* of the *chain geometry* $\Sigma(\mathbb{R}, A)$.

$\mathrm{PGL}_2(A)$ is a group of automorphisms of $\Sigma(\mathbb{R}, A)$.

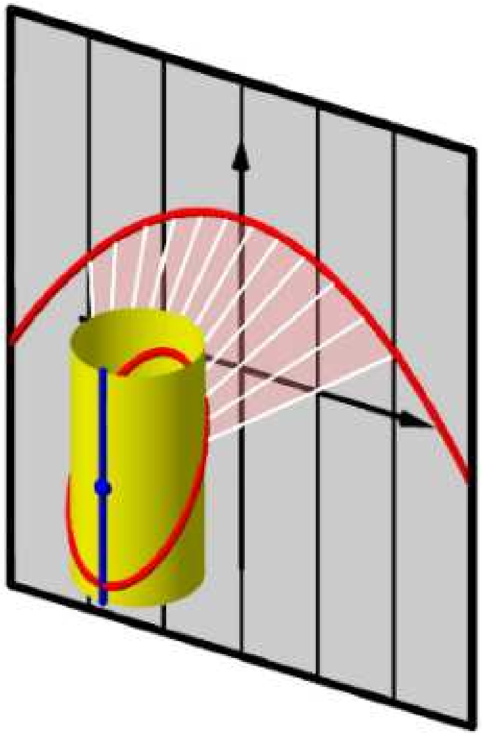
Two-dimensional Real Chain Geometries

$A = \mathbb{C} = \mathbb{R}[i]$, $i^2 = -1$	complex numbers	Möbius
$A = \mathbb{R} \times \mathbb{R}$	double numbers	Minkowski
$A = \mathbb{D} = \mathbb{R}[\varepsilon]$, $\varepsilon^2 = 0$	dual numbers	Laguerre

Blaschke's Cone

A quadratic cone (without its vertex) in the real projective 3-space is a point model for the projective line over $\mathbb{R}[\varepsilon]$. Two points are *parallel* (*non-distant*) if they are on a common generator.

Under a stereographic projection all points that are distant to the centre of projection are mapped bijectively onto the affine plane of dual numbers (*isotropic plane*).



Three-dimensional Real Chain Geometries

$$A_1 = \mathbb{C} \times \mathbb{R}$$

$$A_2 = \mathbb{D} \times \mathbb{R}$$

$$A_3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

$$A_4 = \mathbb{R}[\varepsilon],\ \varepsilon^3 = 0$$

$$A_5 = \mathbb{R}[\varepsilon_1, \varepsilon_2],\ \varepsilon_i \varepsilon_j = 0$$

$$A_6 = \mathbb{R}[j, \varepsilon],\ j^2 = 1,\ \varepsilon^2 = 0,\ j\varepsilon = -\varepsilon j = \varepsilon$$

The Chain Geometry on A_4

$A := A_4 = \mathbb{R}[\varepsilon]$, $\varepsilon^3 = 0$. $z = z_0 + z_1\varepsilon + z_2\varepsilon^2$, $z_i \in \mathbb{R}$.

$N := \mathbb{R}\varepsilon + \mathbb{R}\varepsilon^2$ is the only maximal ideal of A.

$$\mathbb{P}(A) = \{\underbrace{A(z,1)}_{=z\in A} \mid a \in A\} \cup \{\underbrace{A(1,u)}_{=u\in N} \mid u \in N\}$$

proper and *improper* points; $\infty := A(1,0)$.

Projectivities fixing the Improper Plane

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$. Then γ fixes the improper plane if, and only if,

$$b \in N = \mathbb{R}\varepsilon + \mathbb{R}\varepsilon^2.$$

In this case, γ yields the bijections

$$A \to A : z \mapsto \frac{za + c}{zb + d} \text{ and } N \to N : u \mapsto \frac{b + ud}{a + uc},$$

since the denominators are invertible for all $z \in A$ and all $u \in N$.

Dilative Rotations

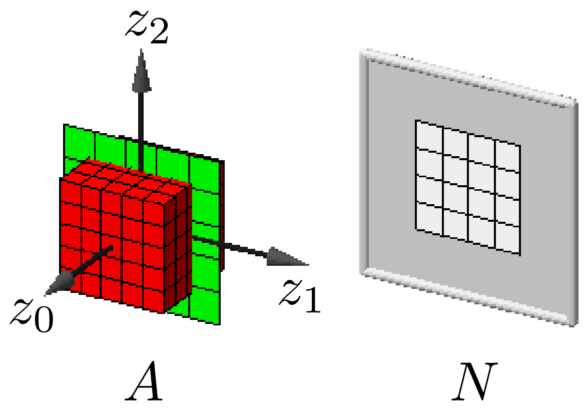
$$\gamma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(A):$$

We get the *dilative rotation*

$$A \to A : z \mapsto za$$

and the shear

$$N \to N : u \mapsto \frac{u}{a}.$$



Translations

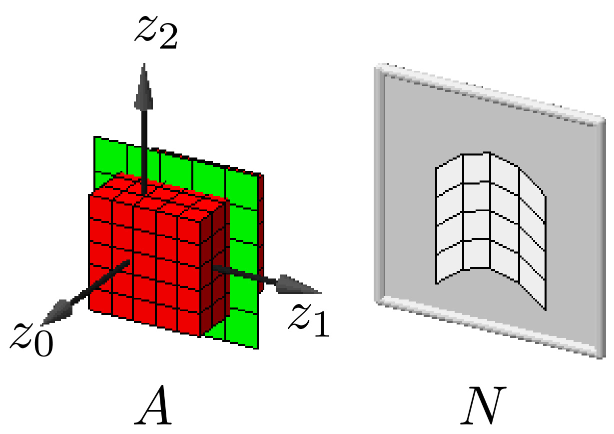
$$\gamma = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in \mathrm{GL}_2(A):$$

We get the translation

$$A \to A : z \mapsto z + c$$

and the quadratic Cremona transformation

$$N \to N : u \mapsto \frac{u}{1+uc}.$$



Vertical shears and translations

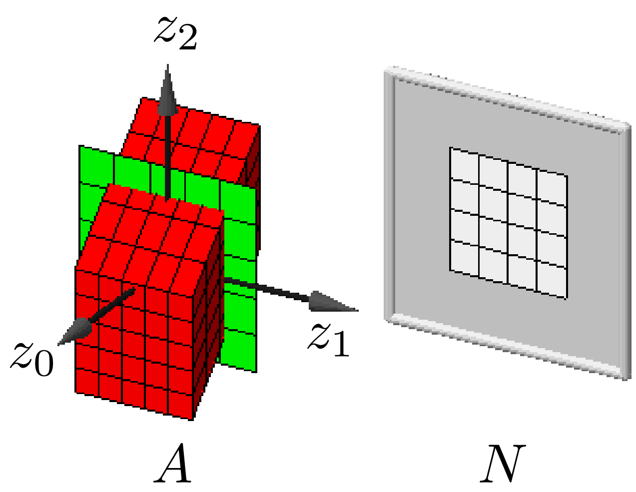
$$\gamma = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \in \mathrm{GL}_2(A)$$

with $a = 1 + a_2\varepsilon$ and $c = c_2\varepsilon^2$: We get a shear or a translation

$$A \to A : z \mapsto za + c$$

and the identity

$$N \to N : u \mapsto u\,.$$



Quadratic and Cubic Cremona Transformations

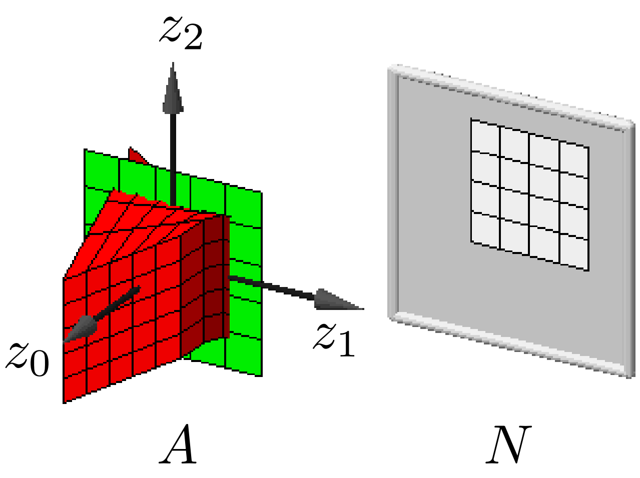
$$\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(A) :$$

We get the Cremona transformation

$$A \to A : z \mapsto \frac{z}{zb+1}$$

and the translation

$$N \to N : u \mapsto b + u\,.$$



The Flag Space

We consider the *projective closure* of the affine space on A. Its plane at infinity, the line at infinity of $N = \mathbb{R}\varepsilon + \mathbb{R}\varepsilon^2$ and the point of infinity of $\mathbb{R}\varepsilon^2$ comprise the *absolute flag*.

All dilative rotations $z \mapsto az$ (when extended projectively) fix the absolute flag, so they are *similarities* of the flag space (*"zweifach isotroper Raum"*).

In particular, we obtain a *motion* of the flag space if, and only if,

$$a = 1 + a_1\varepsilon + a_2 \text{ with } a_i \in \mathbb{R}.$$

However, neither all motions nor all similarities arise in this way.

The Chains

A chain is either:

- a line of A together with $\infty = A(1,0)$;
- a parabola of A together with a point $A(1, b_2\varepsilon^2)$, $b_2 \neq 0$.
- a cubic parabola of A together with a point $A(1, b_1\varepsilon + b_2\varepsilon^2)$, $b_1 \neq 0$.

More precisely, ...

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