Tetrads of Lines Spanning $PG(7,2)$

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The non-zero decomposable tensors of $\bigotimes_{k=1}^{3} V_k$ determine the *Segre variety*

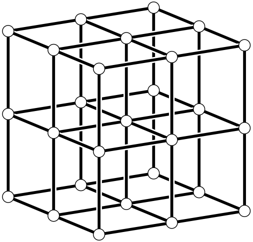
$$\mathcal{S}_{1,1,1}(2) = \left\{a_1 \otimes a_2 \otimes a_3 \mid a_k \in V_k \setminus \{0\}\right\}$$

with ambient projective space $\mathbb{P}\left(\bigotimes_{k=1}^{3} V_k\right) = \mathrm{PG}(7, 2)$.

(Over $\mathbb{F}_2$ we identify projective points and non-zero vectors.)

Orbits

The ambient $\mathrm{PG}(7,2)$ of the Segre $\mathcal{S}_{1,1,1}(2) =: \mathcal{S}$ has 255 points that fall into five orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_5$ under the subgroup $\mathcal{G}_{\mathcal{S}} < \mathrm{GL}(8,2)$ stabilising $\mathcal{S}$.



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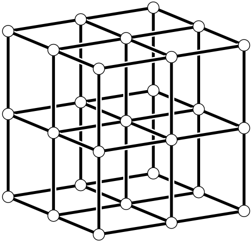

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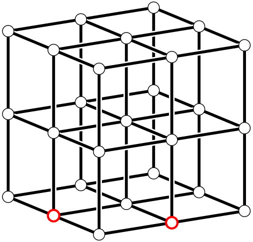

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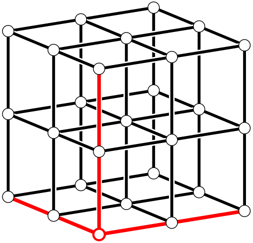

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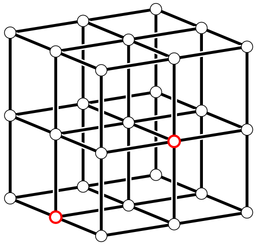

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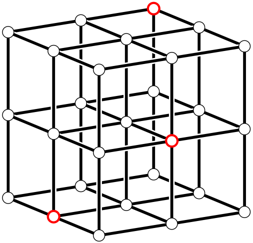

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- $\mathcal{O}_1$: 12 points (sums of triads of $\mathcal{S}$ at distance 3).

Orbits (cont.)

The results from the previous slide and generalisations thereof were established by various authors:

- D. Glynn, T. A. Gulliver, J. G. Maks, and M. K. Gupta (2006) [2].
- B. Odehnal, M. Saniga, and H. H. (2012) [3].
- R. Shaw, N. Gordon, and H. H. (2012) [5].
- M. R. Bremner and St. G. Stavrou (2013) [1].
- M. Lavrauw and J. Sheekey (2014) [4].

Orbits (cont.)

Two sets deserve special mention:

- The union $\mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$ (135 points) is a hyperbolic quadric $\mathcal{H}_7$ of PG$(7, 2)$.

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Two sets deserve special mention:

- The union $\mathcal{O}_2 \cup \mathcal{O}_4 \cup \mathcal{O}_5$ (135 points) is a hyperbolic quadric $\mathcal{H}_7$ of $\mathrm{PG}(7, 2)$.
- The orbit $\mathcal{O}_1$ (12 points) comprises a tetrad of lines spanning $\mathrm{PG}(7, 2)$.

Basic assumption

We start out from a(ny) direct sum decomposition

$$V_8 = V_a \oplus V_b \oplus V_c \oplus V_d$$

of $V_8 := V(8,2)$ into 2-dimensional spaces V_a, V_b, V_c, V_d.

So we obtain the tetrad of lines

$$\mathcal{L}_4 := \{L_a, L_b, L_c, L_d\},$$

where

$$L_h := \mathbb{P}(V_h), \quad h \in \{a,b,c,d\};$$

$\mathbb{P}(V_8) = \mathrm{PG}(7,2)$ is the span of this tetrad of lines.

The stabiliser group $\mathcal{G}(\mathcal{L}_4)$

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The group $\mathcal{G}(\mathcal{L}_4)$ has the semi-direct product structure

$$\mathcal{G}(\mathcal{L}_4) = \mathcal{N} \rtimes \mathrm{Sym}(4),$$

where

$$\mathcal{N} := \mathrm{GL}(V_a) \times \mathrm{GL}(V_b) \times \mathrm{GL}(V_c) \times \mathrm{GL}(V_d),$$

and where

$$\mathrm{Sym}(4) = \mathrm{Sym}(\{a, b, c, d\}).$$

Line weight

Let us define the *line-weight* $\mathrm{lw}(p)$ of a point $p \in \mathrm{PG}(7,2)$ as follows: Write

$$p = v_a + v_b + v_c + v_d \;\text{ with }\; v_h \in V_h, \quad h \in \{a, b, c, d\}.$$

Then

$$\mathrm{lw}(p) = r$$

whenever precisely r of the vectors v_a, v_b, v_c, v_d are non-zero.

Orbits

The 255 points of $\mathrm{PG}(7,2)$ fall into just four $\mathcal{G}(\mathcal{L}_4)$-orbits ω_1, ω_2, ω_3, ω_4, where

$$\omega_r = \{p \in \mathrm{PG}(7,2) \mid \mathrm{lw}(p) = r\}.$$

The lengths of these orbits are accordingly

$$|\omega_1| = 12, \qquad |\omega_2| = \binom{4}{2} \times 3^2 = 54,$$

$$|\omega_3| = \binom{4}{3} \times 3^3 = 108, \qquad |\omega_4| = 3^4 = 81.$$

The symplectic form of $\mathcal{L}_4$

There is a *unique* symplectic form B on V_8 such that the subspaces V_a, V_b, V_c, V_d are hyperbolic 2-dimensional spaces which are pairwise orthogonal.

The quadric of $\mathcal{L}_4$

The tetrad $\mathcal{L}_4$ also determines a particular non-degenerate quadric $\mathcal{Q}$ in $\mathrm{PG}(7,2)$. Such a quadric $\mathcal{Q}$ is uniquely determined by the two conditions

(i) it has equation $Q(x) = 0$ such that the quadratic form Q polarises to give the foregoing symplectic form B;

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(ii) the 12-set of points

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The quadric $\mathcal{Q}$ is seen to be $\omega_2 \cup \omega_4$ ($54 + 81 = 135$ points), and so it is hyperbolic.

The normal subgroup $\mathcal{G}_{81}$ of $\mathcal{G}(\mathcal{L}_4)$

For each $h \in \{a, b, c, d\}$ let us choose an element $\zeta_h \in \mathrm{GL}(V_h)$ of order 3 that effects a cyclic permutation of the points of L_h.

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We define

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Observe that ω_4 is a single $\mathcal{G}_{81}$-orbit.

A GF(3) view of $\mathcal{G}_{81}$

By viewing $0, 1, 2$ as the elements of $\mathbb{F}_3 = \mathrm{GF}(3)$ the map

$$(\mathbb{F}_3)^4 \to \mathcal{G}_{81} : ijkl \mapsto A_{ijkl}$$

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Example: The elements $I = A_{0000}$, A_{1000}, and $A_{1000}^2 = A_{2000}$ constitute that subgroup of $\mathcal{G}_{81}$ which fixes pointwise each of the three lines L_b, L_c, and L_d.

Z_3 subgroups of $\mathcal{G}_{81}$

Any Z_3 subgroup of $\mathcal{G}_{81}$ is of the form $\{I, A_\sigma, A_{2\sigma}\}$ for some non-zero $\sigma \in (\mathbb{F}_3)^4$ and vice versa. Thus:

The group $\mathcal{G}_{81}$ contains 40 subgroups $\cong Z_3$ which are in bijective correspondence with the 40 points of the projective space $\mathrm{PG}(3,3)$.

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Under the action by conjugacy of $\mathcal{G}(\mathcal{L}_4)$ on $\mathcal{G}_{81}$ the particular 4-set of Z_3 subgroups corresponding to

$$\mathcal{T} := \{\langle 1000\rangle, \langle 0100\rangle, \langle 0010\rangle, \langle 0001\rangle\}$$

is fixed. So $\mathcal{T}$ is a $\mathcal{G}(\mathcal{L}_4)$-distinguished tetrahedron of reference in $\mathrm{PG}(3,3)$.

Triplets of 27-sets

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$$\begin{aligned} \mathcal{R}_H &:= \{hu \mid h \in H\}, \\ \mathcal{R}'_H &:= \{h'u \mid h' \in H'\}, \\ \mathcal{R}''_H &:= \{h''u \mid h'' \in H''\}. \end{aligned} \tag{1}$$

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Each such subgroup $H < \mathcal{G}_{81}$ gives rise to a decomposition

$$\omega_4 = \mathcal{R}_H \cup \mathcal{R}'_H \cup \mathcal{R}''_H$$

of ω_4 into a triplet of 27-sets.

Classification of subgroups of $\mathcal{G}_{81}$

Theorem

The normal subgroup $\mathcal{G}_{81} < \mathcal{G}(\mathcal{L}_4)$ *contains precisely* 40 *subgroups* $H \cong Z_3 \times Z_3 \times Z_3$. *These fall into four conjugacy classes of* $\mathcal{G}(\mathcal{L}_4)$, *of respective sizes* $8, 16, 12, 4$.

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Proof. Any such H corresponds to one of the 40 projective planes in $\mathrm{PG}(3, 3)$.

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$$|\mathcal{P}_0| = 8, \quad |\mathcal{P}_1| = 16, \quad |\mathcal{P}_2| = 12, \quad |\mathcal{P}_3| = 4$$

the theorem now follows, since planes of the same kind are seen to correspond to conjugate $Z_3 \times Z_3 \times Z_3$ subgroups.

Segre varieties from $\mathcal{L}_4$

Theorem

A triplet of 27*-sets* $\{\mathcal{R}_H, \mathcal{R}'_H, \mathcal{R}''_H\}$ *in* (1) *which arises from a* $Z_3 \times Z_3 \times Z_3$ *subgroup H will consist of Segre varieties* $\mathcal{S}_{1,1,1}(2)$ *if, and only if, the corresponding projective plane in* $\mathrm{PG}(3,3)$ *is of kind* $\mathcal{P}_0$*.*

Our approach yields precisely 24 *copies of a Segre variety* $\mathcal{S}_{1,1,1}(2)$ *which are contained in* ω_4*.*

Final Remarks

- The five $\mathcal{G}_{\mathcal{S}}$-orbits are related to the four $\mathcal{G}(\mathcal{L}_4)$-orbits in the following simple manner:

$$\omega_1 = \mathcal{O}_1,\ \omega_2 = \mathcal{O}_2,\ \omega_3 = \mathcal{O}_3,\ \omega_4 = \mathcal{O}_4 \cup \mathcal{O}_5 = \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}''.$$

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- The article [6] contains a detailed description of the non-Segre-27-sets and their intersection properties.

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