Diameter Preserving Surjections in the Geometry of Matrices

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DIFFERENTIALGEOMETRIE UND GEOMETRISCHE STRUKTUREN

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Introduction

Let $M_{m,n}(\mathcal{D})$, $m, n \geq 2$, be the set of all $m \times n$ matrices over a division ring $\mathcal{D}$.

Two matrices (linear operators) A and B are *adjacent* if $A - B$ is of rank one.

We may consider $M_{m,n}(\mathcal{D})$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.

Two matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if

$$\mathrm{rank}(A - B) = k.$$

Hua's Theorem

Fundamental Theorem (1951). *Every bijective map* $\phi : M_{m,n}(\mathcal{D}) \to M_{m,n}(\mathcal{D}) : A \mapsto A^\phi$ *preserving adjacency in both directions is of the form*

$$A \mapsto TA^\sigma S + R,$$

where T *is an invertible* $m \times m$ *matrix,* S *is an invertible* $n \times n$ *matrix,* R *is an* $m \times n$ *matrix, and* σ *is an automorphism of the underlying field.*

If $m = n$*, then we have the additional possibility that*

$$A \mapsto T(A^\sigma)^t S + R$$

where T, S, R *are as above,* σ *is an anti-isomorphisms of* $\mathcal{D}$*, and* A^t *denotes the transpose of* A*.*

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang and Z.-X. Wan: Beiträge Algebra Geom. **45** (2004), no. 2, 435–446.

P. Šemrl: J. Algebra **272** (2004), 801–837.

Geometries of Matrices

Similar fundamental theorems (subject to technical restrictions) hold for:

- spaces of *Hermitian matrices*,
- spaces of *symmetric matrices*,
- spaces of *alternate matrices*
  (with a different definition of adjacency: $\operatorname{rank} A - B = 2$)
- various *projective* matrix spaces, e. g. *Grassmannians*.

In all cases the fundamental theorem is essentially a result on isomorphisms of graphs with finite diameter.

Diameter Preservers

Recent work focusses on diameter preservers between matrix spaces and other structures.

P. Abramenko, A. Blunck, D. Kobal, M. Pankov, P. Šemrl, H. Van Maldeghem, H. H.

The aim of our present work is to exhibit diameter preservers in a purely graph-theoretic setting and to apply the results to several matrix spaces.

Conditions (A1)–(A5)

We focus our attention on graphs Γ satisfying the following conditions:

(A1) Γ is connected and its diameter $\operatorname{diam}\Gamma$ is finite.

(A2) For any points $x, y \in \mathcal{P}$ there is a point $z \in \mathcal{P}$ with

$$d(x,z) = d(x,y) + d(y,z) = \operatorname{diam}\Gamma.$$

(A3) For any points $x, y, z \in \mathcal{P}$ with $d(x,z) = d(y,z) = 1$ and $d(x,y) = 2$ there is a point w satisfying

$$d(x,w) = d(y,w) = 1 \text{ and } d(z,w) = 2.$$

(A4) For any points $x, y, z \in \mathcal{P}$ with $x \neq y$ and $d(x,z) = d(y,z) = \operatorname{diam}\Gamma$ there is a point w with

$$d(z,w) = 1, \quad d(x,w) = \operatorname{diam}\Gamma - 1, \quad \text{and} \quad d(y,w) = \operatorname{diam}\Gamma.$$

(A5) For any adjacent points $a, b \in \mathcal{P}$ there exists a point $p \in \mathcal{P} \setminus \{a, b\}$ such that for all $x \in \mathcal{P}$ the following holds:

$$d(x,p) = \operatorname{diam}\Gamma \quad \Rightarrow \quad d(x,a) = \operatorname{diam}\Gamma \ \vee \ d(x,b) = \operatorname{diam}\Gamma.$$

A Lemma about Adjacency

Lemma 1. *Given a graph* Γ *which satisfies conditions* (A1)–(A4) *let*

$$n := \operatorname{diam}\Gamma.$$

Suppose that $a, b \in \mathcal{P}$ *are distinct points with the following property:*

$$\exists\, p \in \mathcal{P} \setminus \{a, b\}\ \forall\, x \in \mathcal{P}\ :\ d(x, p) = n \ \Rightarrow\ d(x, a) = n \ \vee\ d(x, b) = n. \tag{1}$$

Then a *and* b *are adjacent.*

Condition (A5) just guarantees that (1) holds for any two adjacent points $a, b \in \mathcal{P}$.

Main Theorem

Theorem 1. *Let Γ and Γ' be two graphs satisfying the above conditions* (A1)–(A5)*. If $\phi : \mathcal{P} \to \mathcal{P}'$ is a surjection which satisfies*

$$d(x,y) = \operatorname{diam}\Gamma \quad \Leftrightarrow \quad d(x^{\phi}, y^{\phi}) = \operatorname{diam}\Gamma' \ \textit{ for all } \ x, y \in \mathcal{P},$$

then ϕ is an isomorphism of graphs. Consequently, $\operatorname{diam}\Gamma = \operatorname{diam}\Gamma'$.

Proof. Injectivity follows from condition (A2).

By Lemma 1 and (A5) the mapping ϕ is an isomorphism of graphs. □

Application

Lemma 2. *The graph on* $M_{m\times n}(\mathcal{D})$ *satisfies conditions* (A1)–(A5) *provided that* $|\mathcal{D}| \neq 2$.

Theorem 2. *Let* $\mathcal{D}$, $\mathcal{D}'$ *be division rings with* $|\mathcal{D}|$, $|\mathcal{D}'| \neq 2$. *Let* $m, n, p, q \geq 2$ *be integers. If* $\phi : M_{m\times n}(\mathcal{D}) \to M_{p\times q}(\mathcal{D}')$ *is a surjection which satisfies*

$$\operatorname{rank}(A - B) = \min\{m, n\} \quad \Leftrightarrow \quad \operatorname{rank}(A^\phi - B^\phi) = \min\{p, q\}$$

$$\text{for all } \ A, B \in M_{m\times n}(\mathcal{D}),$$

then ϕ *is bijective. Both* ϕ *and* ϕ^{-1} *preserve adjacency of matrices. Moreover,* $\min\{m, n\} = \min\{p, q\}$.

Hermitian Matrices

Let $\mathcal{D}$ be a division ring which possesses an *involution*, i. e. an anti-automorphism of $\mathcal{D}$ whose square equals the identity map of $\mathcal{D}$. We fix one such involution of $\mathcal{D}$ and denote it by $\overline{}$. Also, we assume that the following restrictions are satisfied:

(R1) The set $\mathcal{F}$ of fixed elements of $\overline{}$ has more than three elements in common with the centre of $\mathcal{D}$.

(R2) When $\overline{}$ is the identity map, whence $\mathcal{D} = \mathcal{F}$ is a field, then assume that $\mathcal{F}$ does not have characteristic 2.

Let $\mathcal{H}_n(\mathcal{D})$ denote the space of Hermitian $n \times n$ matrices over $\mathcal{D}$ (with respect to $\overline{}$), where $n \geq 2$.

If $\overline{}$ is the identity map, then $\mathcal{H}_n(\mathcal{D}) =: \mathcal{S}_n(\mathcal{F})$ is the space of symmetric $n \times n$ matrices over $\mathcal{F}$.

Application

Lemma 2. *The graph on* $\mathcal{H}_n(\mathcal{D})$ *satisfies conditions* (A1)–(A5) *provided that the restrictions* (R1) *and* (R2) *are satisfied.*

Theorem 2. *Let* $\mathcal{D}, \mathcal{D}'$ *be division rings which possess involutions* $^-$ *and* $^{-\prime}$, *respectively, subject to the restrictions* (R1) *and* (R2). *Let* n, n' *be integers* ≥ 2. *If* $\phi : \mathcal{H}_n(\mathcal{D}) \to \mathcal{H}_{n'}(\mathcal{D}')$ *is a surjection which satisfies*

$$\operatorname{rank}(A - B) = n \quad \Leftrightarrow \quad \operatorname{rank}(A^\phi - B^\phi) = n' \text{ for all } A, B \in \mathcal{H}_n(\mathcal{D}),$$

then ϕ *is bijective. Both* ϕ *and* ϕ^{-1} *preserve adjacency of Hermitian matrices. Moreover,* $n = n'$.

Final Remarks

The results can also be applied to Grassmannians.

Problem: Find more graphs which meet conditions (A1)–(A5).

Reference:

Wen-ling Huang and H. H.: Diameter preserving surjections in the geometry of matrices, *Linear Algebra Appl.* **429** (2008), 376–386.