Jordan homomorphisms of rings and their action on projective lines

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We present an affirmative answer to a long standing question:

Does a Jordan homomorphism $\alpha : R \to R'$ *of rings determine a mapping* $\mathbb{P}(R) \to \mathbb{P}(R')$ *of the corresponding projective lines?*

If α is a homomorphism of rings then, of course, the image of a point $R(a, b)$ may be defined as $R'(a^\alpha, b^\alpha)$, but in the general setting such a 'definition' does not make sense anymore.
Contributions to this problem are due to A. Herzer (1987) and C. Bartolone (1989), who treated the particular cases where R is a local and a ring of stable rank 2, respectively.
Our approach is based on the fact that the projective line over a ring can be viewed as a graph. In general, this so-called *distant graph* is not connected. We show that each Jordan homomorphism $\alpha : R \to R'$ gives rise to a mapping $\overline{\alpha}$ of a *subset* of $\mathbb{P}(R)$ into the projective line over R'. This subset is a connected component of the distant graph on $\mathbb{P}(R)$.
The mapping $\overline{\alpha}$ is *harmonic*, i.e., it preserves harmonic pairs. If there is more than one connected component of $\mathbb{P}(R)$ then $\overline{\alpha}$ can be extended in various ways to a harmonic mapping $\mathbb{P}(R) \to \mathbb{P}(R')$.

This is joint work with Andrea Blunck, Fachbereich Mathematik, Universität Hamburg, Germany.