On bijections that preserve complementary subspaces

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The distant graph

V ... vector space over a (skew) field K.

$\mathcal{G}$ ... all subspaces $X \leq V$ such that $X \cong V/X$.

We assume that $\mathcal{G} \neq \emptyset$.

The *distant relation*

$$X \bigtriangleup Y :\Leftrightarrow X \oplus Y = V,$$

where $X, Y \in \mathcal{G}$, yields the *distant graph* on $\mathcal{G}$.

The distant graph is connected. Its diameter is as follows:

$\dim V$	0	2	$4, 6, \ldots$	∞
diameter	0	1	2	3

A. Blunck and H. H. The connected components of the projective line over a ring. *Adv. Geom.* **1** (2001), 107–117

The Grassmann graph

The *adjacency relation*

$$\begin{array}{rcl} X \sim Y \quad :\Leftrightarrow \quad 1 & = & \dim((X+Y)/X) \\ & = & \dim((X+Y)/Y), \end{array}$$

where $X, Y \in \mathcal{G}$ yields the *Grassmann graph* on $\mathcal{G}$.

Distance of $X, Y \in \mathcal{G}$ in the Grassmann graph:

$$d = \dim((X+Y)/X) = \dim((X+Y)/Y)$$

which is equivalent to

$$d = \dim(X/(X \cap Y)) = \dim(Y/(X \cap Y))$$

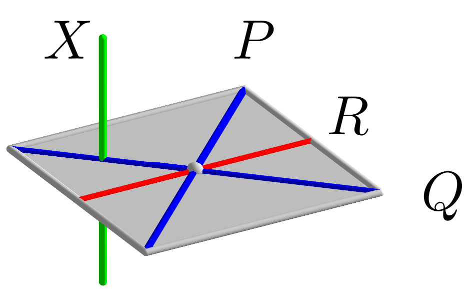
The Grassmann graph is connected iff $\dim V < \infty$.

A characterization of adjacency

Theorem. *For all $P, Q \in \mathcal{G}$ the following statements are equivalent:*

- *P and Q are adjacent.*
- *There is an element $R \in \mathcal{G}$ satisfying the following conditions:*

$$\begin{aligned} R &\neq P, Q, \\ \forall X \in \mathcal{G} : X \vartriangle R &\Rightarrow X \vartriangle P \text{ or } X \vartriangle Q. \end{aligned}$$



Comparing Automorphisms

Theorem. *The following statements hold:*

- *Every automorphism of the distant graph is an automorphism of the Grassmann graph.*
- *If* $\dim V < \infty$ *then every automorphism of the Grassmann graph is an automorphism of the distant graph.*

Automorphisms of the distant graph

Theorem. *Let $4 \le \dim V < \infty$ and let $\varphi : \mathcal{G} \to \mathcal{G}$ be an automorphism of the distant graph. Then there is either a semilinear bijection*

$$f : V \to V \ \textit{such that}\ X^{\varphi} = X^{f}$$

or a semilinear bijection

$$f : V^{*} \to V \ \textit{such that}\ X^{\varphi} = (X^{\perp})^{f};$$

here V^{*} denotes the dual of V.

W.-L. Chow. On the geometry of algebraic homogeneous spaces. *Ann. of Math.* **50** (1949), 32–67.

W.-l. Huang. Adjacency preserving transformations of Grassmann spaces. *Abh. Math. Sem. Univ. Hamburg* **68** (1998), 65–77.