Harmonicity Preservers of Projective Lines

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The projective line over a ring

- Let R be a ring with unity $1 \neq 0$.
- Let M be a free left R-module of rank 2, i. e., M has a basis with two elements.
- We say that $a \in M$ is *admissible* if there exists $b \in M$ such that (a, b) is a basis of M (with two elements). (We do not require that all bases of M have the same number of elements.)

Definition

The *projective line* over M is the set $\mathbb{P}(M)$ of all cyclic submodules Ra, where $a \in M$ is admissible. The elements of $\mathbb{P}(M)$ are called *points*.

Distant pairs and harmonic quadruples

Definition

A pair $(p_0, p_1) \in \mathbb{P}(M)^2$ are called *distant*, in symbols $p_0 \triangle p_1$, if there exists a basis (g_0, g_1) of M such that

$$p_0 = Rg_0, \quad p_1 = Rg_1.$$

Definition

A quadruple $(p_0, p_1, p_2, p_3) \in \mathbb{P}(M)^4$ is *harmonic* if there exists a basis (g_0, g_1) of M such that

$$p_0 = Rg_0, \quad p_1 = Rg_1, \quad p_2 = R(g_0 + g_1), \quad p_3 = R(g_0 - g_1).$$

Jordan homomorphisms of rings

Definition

A mapping $\alpha : R \to R'$ is a *Jordan homomorphism* if for all $x, y \in R$ the following conditions are satisfied:

1. $(x+y)^\alpha = x^\alpha + y^\alpha$,
2. $1^\alpha = 1'$,
3. $(xyx)^\alpha = x^\alpha y^\alpha x^\alpha$.

Examples

- Any homomorphism of rings.
- Any antihomomorphism of rings; e. g. the transposition $\mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$: $A \mapsto A^T$, where $\mathbb{F}$ is a commutative field.
- $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} : (A_1, A_2) \mapsto (A_1, A_2^T)$ which is neither homomorphic nor antihomomorphic for any $n \geq 2$.

Basic assumptions

Let $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ be a *harmonicity preserver*. Furthermore, we assume that R contains “sufficiently many” units; in particular $1 + 1 = 2$ has to be a unit in R.

We may choose bases (e_0, e_1) of M and (e'_0, e'_1) of M' such that

$$(Re_0)^\mu = R'e'_0, \quad (Re_1)^\mu = R'e'_1, \quad \big(R(e_0 \pm e_1)\big)^\mu = R'(e'_0 \pm e'_1).$$

Step 1: A local coordinate representation of μ

Then there exists a unique mapping $\beta : R \to R'$ with the property

$$\big(R(xe_0 + e_1)\big)^{\mu} = R'(x^{\beta}e'_0 + e'_1) \quad \text{for all} \quad x \in R.$$

This β is additive and satisfies $1^{\beta} = 1'$.

Step 2: Change of coordinates

We may repeat Step 1 for the new bases

$$(f_0, f_1) := (te_0 + e_1, -e_0) \quad \text{and} \quad (f'_0, f'_1) := (t^\beta e'_0 + e'_1, -e'_0),$$

where $t \in R$ is arbitrary. So the transition matrices are

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad E(t^\beta) := \begin{pmatrix} t^\beta & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the new local representation of μ yields the same mapping β as in Step 1.

Step 3: β is a Jordan homomorphism

By combining Step 1 and Step 2 (for $t = 0$) one obtains:

The mapping β from Step 1 is a Jordan homomorphism.

Part of the proof relies on previous work.

Step 4: Induction

Suppose that a point $p \in \mathbb{P}(M)$ can be written as

$$p = R(x_0 e_0 + x_1 e_1)$$

with

$$(x_0, x_1) = (1, 0) \cdot E(t_1) \cdot E(t_2) \cdots E(t_n) \quad \text{for some} \quad t_1, t_2, \ldots, t_n \in R,$$

where n is variable.

Then the image point of p under μ is

$$R'(x'_0 e'_0 + x'_1 e'_1)$$

with

$$(x'_0, x'_1) = (1', 0') \cdot E(t_1^\beta) \cdot E(t_2^\beta) \cdots E(t_n^\beta).$$

Concluding remarks

- For a wide class of rings in order to reach all points of $\mathbb{P}(M)$ it suffices to let $n \leq 2$ in Step 4.
- There are rings where the the description from Step 4 will not cover the entire line $\mathbb{P}(M)$, but only a connected component of the *distant graph* $(\mathbb{P}(M), \triangle)$. Here μ can be described in terms of several Jordan homomorphisms.
- Any Jordan homomorphism $R \to R'$ gives rise to a harmonicity preserver. This follows from work of C. Bartolone, A. Blunck, and others.
- For precise statements and further references see: H. H., Von Staudt's theorem revisited. *Aequationes Math.* **89** (2015), 459–472.