Adjacency Preservers vs. Diameter Preservers

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DIFFERENTIALGEOMETRIE UND
GEOMETRISCHE STRUKTUREN

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Part 1

Introduction

The first part deals with some basic notions and results from the geometry of matrices.

Rectangular Matrices

Let $M_{m,n}(\mathcal{D})$, $m, n \geq 2$, be the set of all $m \times n$ matrices over a division ring $\mathcal{D}$.

- Two matrices (linear operators) $A, B \in M_{m,n}(\mathcal{D})$ are *adjacent* if $A - B$ is of rank one. (Rank always means left row rank.)

- We consider $M_{m,n}(\mathcal{D})$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.

- Two matrices $A, B \in M_{m,n}(\mathcal{D})$ are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\operatorname{rank}(A - B) = k.$$

Grassmannians

Let $\mathcal{G}_{m+n,m}(\mathcal{D})$ be the Grassmannian of all m-dimensional subspaces of $\mathcal{D}^{m+n}$, where $m, n \geq 2$.

- Two subspaces $V, W \in \mathcal{G}_{m+n,m}(\mathcal{D})$ are *adjacent* if $\dim(V \cap W) = m - 1$.
- We consider $\mathcal{G}_{m+n,n}(\mathcal{D})$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent subspaces.
- Two subspaces $V, W \in \mathcal{G}_{m+n,m}(\mathcal{D})$ are at the graph-theoretical distance $k \geq 0$ if, and only if,

$$\dim(V \cap W) = m - k.$$

Connection

$M_{m,n}(\mathcal{D})$ can be embedded in $\mathcal{G}_{m+n,m}(\mathcal{D})$ as follows:

$$\begin{array}{ccccc} M_{m,n}(\mathcal{D}) & \longrightarrow & M_{m,m+n}(\mathcal{D}) & \longrightarrow & \mathcal{G}_{m+n,m}(\mathcal{D}) \\ A & \longmapsto & (A|I_m) & \longmapsto & \text{left rowspace of } (A|I_m) \end{array}$$

Note that $(X|Y)$ and $(TX|TY)$ have the same left row space for all $T \in \mathrm{GL}_m(\mathcal{D})$.

$\mathcal{G}_{m+n,m}(\mathcal{D})$ may be viewed as the *projective space* of $m \times n$ matrices over $\mathcal{D}$.

Many authors consider projective dimensions which are one less than dimensions of vector spaces.
E. g.: $\mathcal{G}_{4,2}(\mathcal{D})$ is the space of lines (1-subspaces) in the 3-dimensional projective space over $\mathcal{D}$.

Part 2

Adjacency Preservers

In the second part we present two classical results about adjacency preservers.

Hua's Theorem

Fundamental Theorem (1951). *Every bijective map* $\varphi : M_{m,n}(\mathcal{D}) \to M_{m,n}(\mathcal{D}) : A \mapsto A^\varphi$ *preserving adjacency in both directions is of the form*

$$A \mapsto TA^\sigma S + R,$$

where T *is an invertible* $m \times m$ *matrix,* S *is an invertible* $n \times n$ *matrix,* R *is an* $m \times n$ *matrix, and* σ *is an automorphism of the underlying division ring.*

If $m = n$*, then we have the additional possibility that*

$$A \mapsto T(A^\sigma)^t S + R$$

where T, S, R *are as above,* σ *is an anti-isomorphism of* $\mathcal{D}$*, and* A^t *denotes the transpose of* A*.*

The assumptions in Hua's fundamental theorem can be weakened.

W.-l. Huang and Z.-X. Wan (2004), P. Šemrl (2004).

Chow's Theorem

Fundamental Theorem (1947). *Every bijective map* $\varphi : \mathcal{G}_{m+n,n}(\mathcal{D}) \to \mathcal{G}_{m+n,n}(\mathcal{D}) : X \mapsto X^\varphi$ *preserving adjacency in both directions is induced by a semilinear mapping*

$$f : \mathcal{D}^{m+n} \to \mathcal{D}^{m+n} : x \mapsto x^\sigma T \text{ such that } X^\varphi = X^f,$$

where T *is an invertible* $(m+n) \times (m+n)$ *matrix and* σ *is an automorphism of the underlying division ring.*

If $m = n$*, then we have the additional possibility that* φ *is induced by a sesquilinear form*

$$g : \mathcal{D}^{m+n} \times \mathcal{D}^{m+n} \to \mathcal{D} : (x, y) \mapsto xL(y^\sigma)^t \text{ such that } U^\varphi = U^{\perp_g},$$

where T *is as above and* σ *is an anti-isomorphism of* $\mathcal{D}$*.*

The assumptions in Chow's fundamental theorem can be weakened.

W.-l. Huang (1998).

Geometries of Matrices

Similar fundamental theorems (subject to technical restrictions) hold for:

- Spaces of Hermitian matrices ($\mathcal{D}$ a division ring with involution $\bar{\ }$).
- Spaces of symmetric matrices ($\mathcal{D}$ commutative).
- Spaces of alternate matrices ($\mathcal{D}$ commutative)
  (with a different definition of adjacency: $\operatorname{rank} A - B = 2$).
- The associated projective matrix spaces (dual polar spaces).

In all cases the fundamental theorem is essentially a result on isomorphisms of graphs with finite diameter.

Part 3

Diameter Preservers

In the third part we exhibit diameter preservers in a purely graph-theoretic setting. Then we shall apply the results to several matrix spaces.

Diameter Preservers

Recent work focusses on diameter preservers between matrix spaces and related structures.

P. Abramenko, A. Blunck, D. Kobal, M. Pankov, P. Šemrl, H. Van Maldeghem, H. H.

In this lecture we aim at pointing out the common features.

Conditions (A1)–(A5)

We focus our attention on graphs Γ satisfying the following conditions:

(A1) Γ is connected and its diameter $\operatorname{diam}\Gamma$ is finite.

(A2) For any points $x, y \in \mathcal{P}$ there is a point $z \in \mathcal{P}$ with

$$d(x,z) = d(x,y) + d(y,z) = \operatorname{diam}\Gamma.$$

(A3) For any points $x, y, z \in \mathcal{P}$ with $d(x,z) = d(y,z) = 1$ and $d(x,y) = 2$ there is a point w satisfying

$$d(x,w) = d(y,w) = 1 \text{ and } d(z,w) = 2.$$

(A4) For any points $x, y, z \in \mathcal{P}$ with $x \neq y$ and $d(x,z) = d(y,z) = \operatorname{diam}\Gamma$ there is a point w with

$$d(z,w) = 1, \quad d(x,w) = \operatorname{diam}\Gamma - 1, \quad \text{and} \quad d(y,w) = \operatorname{diam}\Gamma.$$

(A5) For any adjacent points $a, b \in \mathcal{P}$ there exists a point $p \in \mathcal{P} \setminus \{a, b\}$ such that for all $x \in \mathcal{P}$ the following holds:

$$d(x,p) = \operatorname{diam}\Gamma \;\Rightarrow\; d(x,a) = \operatorname{diam}\Gamma \vee d(x,b) = \operatorname{diam}\Gamma.$$

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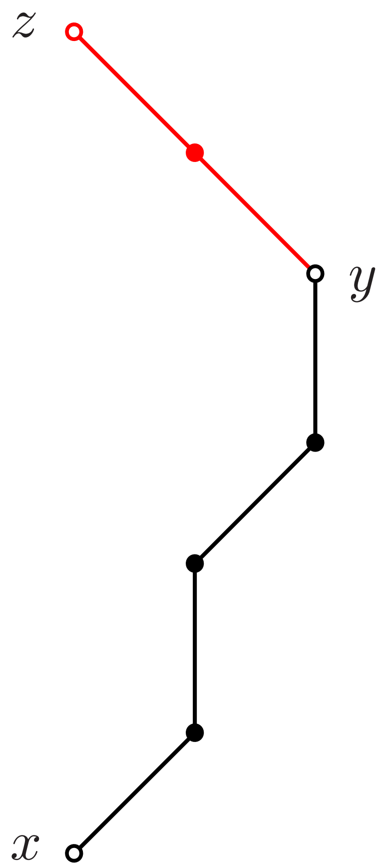

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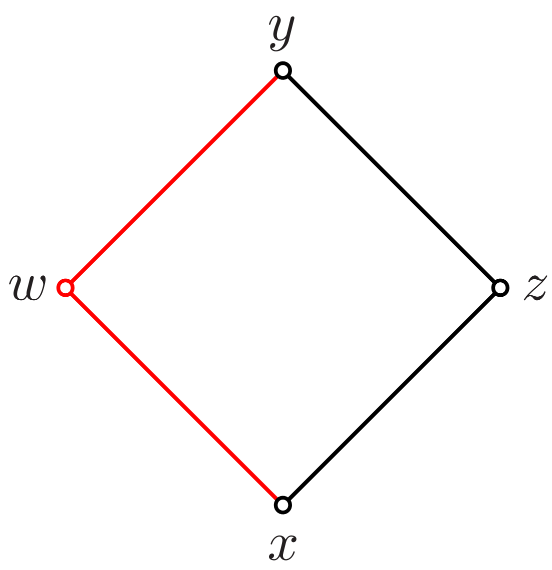

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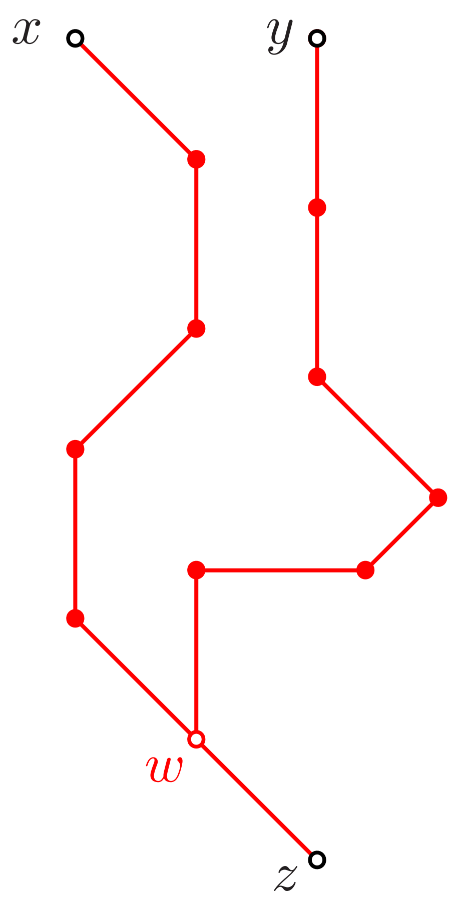

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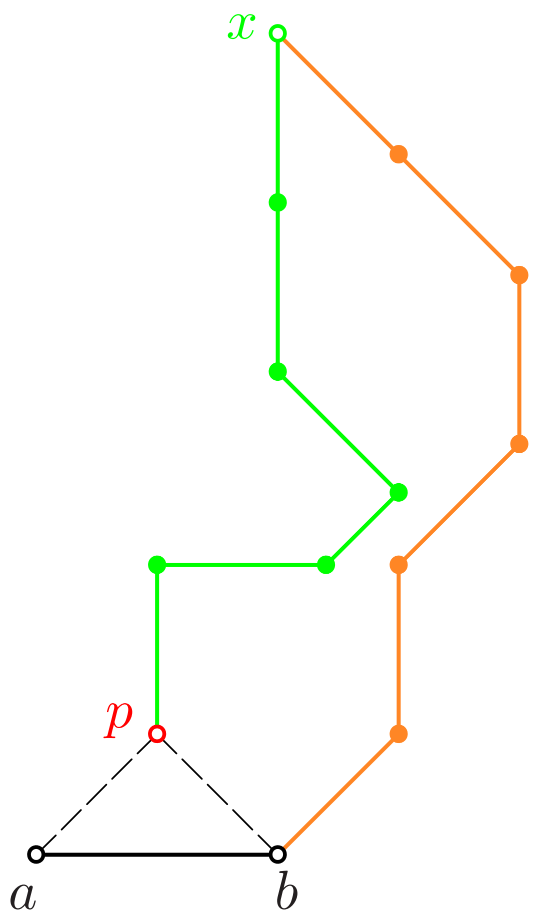

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A Lemma about Adjacency

Lemma. *Given a graph* Γ *which satisfies conditions* (A1)–(A4) *let*

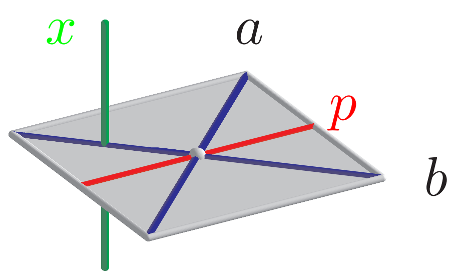
$$n := \operatorname{diam}\Gamma.$$

Suppose that $a, b \in \mathcal{P}$ *are distinct points with the following property:*

$$\exists\, p \in \mathcal{P} \setminus \{a, b\}\ \forall\, x \in \mathcal{P} \,:\, d(x,p) = n \ \Rightarrow\ d(x,a) = n \ \vee\ d(x,b) = n. \tag{1}$$

Then a *and* b *are adjacent.*

Geometric idea behind the proof for $m = n = 2$ from a projective point of view:



Condition (A5) just guarantees that (1) holds for any two adjacent points $a, b \in \mathcal{P}$.

Main Theorem

Theorem (W.-l. Huang and H. H., 2008). *Let* Γ *and* Γ' *be two graphs satisfying the above conditions* (A1)–(A5). *If* $\varphi : \mathcal{P} \to \mathcal{P}'$ *is a surjection which satisfies*

$$d(x, y) = \operatorname{diam}\Gamma \quad \Leftrightarrow \quad d(x^\varphi, y^\varphi) = \operatorname{diam}\Gamma' \text{ for all } x, y \in \mathcal{P},$$

then φ *is an isomorphism of graphs. Consequently,* $\operatorname{diam}\Gamma = \operatorname{diam}\Gamma'$*.*

Application

The graph on $M_{m\times n}(\mathcal{D})$ satisfies conditions (A1)–(A5) provided that $|\mathcal{D}| \neq 2$.

Theorem. *Let $\mathcal{D}$, $\mathcal{D}'$ be division rings with $|\mathcal{D}|, |\mathcal{D}'| \neq 2$. Let $m, n, p, q \geq 2$ be integers. If $\varphi : M_{m\times n}(\mathcal{D}) \to M_{p\times q}(\mathcal{D}')$ is a surjection which satisfies*

$$\operatorname{rank}(A - B) = \min\{m, n\} \quad \Leftrightarrow \quad \operatorname{rank}(A^{\varphi} - B^{\varphi}) = \min\{p, q\}$$

$$\text{for all} \quad A, B \in M_{m\times n}(\mathcal{D}),$$

then φ is bijective. Both φ and φ^{-1} preserve adjacency of matrices. Moreover, $\min\{m, n\} = \min\{p, q\}$.

The associated projective space of rectangular matrices (Grassmannian) satisfies conditions (A1)–(A5) for any $\mathcal{D}$.

Hermitian Matrices

Let $\mathcal{D}$ be a division ring which possesses an *involution*, i. e. an anti-automorphism of $\mathcal{D}$ whose square equals the identity map of $\mathcal{D}$. We fix one such involution of $\mathcal{D}$ and denote it by $\overline{}$. Also, we assume that the following restrictions are satisfied:

(R1) The set $\mathcal{F}$ of fixed elements of $\overline{}$ has more than three elements in common with the centre of $\mathcal{D}$.

(R2) When $\overline{}$ is the identity map, whence $\mathcal{D} = \mathcal{F}$ is a field, then assume that $\mathcal{F}$ does not have characteristic 2.

Let $\mathcal{H}_n(\mathcal{D})$ denote the space of Hermitian $n \times n$ matrices over $\mathcal{D}$ (with respect to $\overline{}$), where $n \geq 2$.

If $\overline{}$ is the identity map, then $\mathcal{H}_n(\mathcal{D}) =: \mathcal{S}_n(\mathcal{F})$ is the space of symmetric $n \times n$ matrices over $\mathcal{F}$.

Application

The graph on $\mathcal{H}_n(\mathcal{D})$ satisfies conditions (A1)–(A5) provided that the restrictions (R1) and (R2) are satisfied.

Theorem. *Let $\mathcal{D}, \mathcal{D}'$ be division rings which possess involutions $^-$ and $^{-'}$, respectively, subject to the restrictions* (R1) *and* (R2). *Let n, n' be integers ≥ 2. If $\varphi : \mathcal{H}_n(\mathcal{D}) \to \mathcal{H}_{n'}(\mathcal{D}')$ is a surjection which satisfies*

$$\operatorname{rank}(A - B) = n \quad \Leftrightarrow \quad \operatorname{rank}(A^\varphi - B^\varphi) = n' \ \text{ for all } \ A, B \in \mathcal{H}_n(\mathcal{D}),$$

then φ is bijective. Both φ and φ^{-1} preserve adjacency of Hermitian matrices. Moreover, $n = n'$.

Final remarks

Characterisations of geometric transformations under mild hypotheses.

W. Benz, *Geometrische Transformationen*, 1992.

Z.-X. Wan: Geometry of Matrices, 1996.

Preservation theorems can be seen as as consequences of first-order definability,

V. Pambuccian, 2000.

Generalisation from division rings to rings.

L. P. Huang: Geometry of Matrices over Ring, 2006.

M. Pankov, Grassmannians of Classical Buildings, 2008.