Projective Ring Lines and Their Generalisations

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All our rings are associative, with a unit element $1 \neq 0$ which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of units (invertible elements) of a ring $R$ is denoted by $R^*$. 
Let $R$ be a ring. We consider the free left $R$-module $R^2$.

- A pair $(a, b) \in R^2$ is called *admissible* if $(a, b)$ is the first row of a matrix in $\text{GL}_2(R)$.
  This is equivalent to saying that there exists $(c, d) \in R^2$ such that $(a, b), (c, d)$ is a basis of $R^2$.

- **Projective line** over $R$ (X. Hubaut [30]):

  $$\mathbb{P}(R) := \{ R(a, b) \mid (a, b) \text{ admissible} \}$$

The elements of $\mathbb{P}(R)$ are called *points*.

- Two admissible pairs generate the same point if, and only if, they are left proportional by a unit in $R$.

- Note that $R^2$ need not have an *invariant basis number*: There may also be bases with cardinality $\neq 2$. 

The Distant Graph

- **Distant** points of $\mathbb{P}(R)$:

  \[ R(a, b) \triangle R(c, d) :\Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R) \]

- $(\mathbb{P}(R), \triangle)$ is called the *distant graph* of $\mathbb{P}(R)$.

- Non-distant points are also called *neighbouring*.

- The relation $\triangle$ is invariant under the action of $\text{GL}_2(R)$ on $\mathbb{P}(R)$.

- The group $\text{GL}_2(R)$ acts transitively on the triples of mutually distant points of $\mathbb{P}(R)$.

A. Blunck, A. Herzer: *Kettengeometrien* [12].

A. Herzer: *Chain Geometries* [25].
Examples: Rings with Four Elements

- $R = \text{GF}(4)$ (Galois field).
- $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $R = \mathbb{Z}_4$.
- $R = \mathbb{Z}_2[\varepsilon], \varepsilon^2 = 0$ (dual numbers over $\mathbb{Z}_2$).

$\#\mathcal{P}(R) = 5$
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$\#P(R) = 6$
The Elementary Linear Group $E_2(R)$

All elementary $2 \times 2$ matrices over $R$, i.e., matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \text{ with } t \in R,$$

generate the *elementary linear group* $E_2(R)$. The group $GE_2(R)$ is the subgroup of $GL_2(R)$ generated by $E_2(R)$ and all invertible diagonal matrices.

**Lemma (P. M. Cohn [17])**

A $2 \times 2$ matrix over $R$ is in $E_2(R)$ if, and only if, it can be written as a finite product of matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t \in R.$$
Connectedness

**Theorem (A. Blunck, H. H. [8])**

Let $R$ be any ring.

- $(\mathbb{P}(R), \bigtriangleup)$ is connected precisely when $GL_2(R) = GE_2(R)$.
- A point $p \in \mathbb{P}(R)$ is in the connected component of $R(1,0)$ if, and only if, it can be written as $R(a,b)$ with

$$
(a, b) = (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1).
$$

for some $n \in \mathbb{N}$ and some $t_1, t_2, \ldots, t_n \in R$.

See A. Blunck [6] and [7] for the orbit of $R(1,0)$ under certain subgroups of $GL_2(R)$. 
The formula
\[(a, b) = (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)\]
reads explicitly as follows:

\[
\begin{align*}
 n = 0 : & \quad (a, b) = (1, 0) \\
 n = 1 : & \quad (a, b) = (t_1, 1) \\
 n = 2 : & \quad (a, b) = (t_2 t_1 - 1, t_2) \quad (Cf. \ C. \ Bartolone \ [1]). \\
 n = 3 : & \quad (a, b) = (t_3 t_2 t_1 - t_3 - t_1, t_3 t_2 - 1) \\
 \vdots \\
\end{align*}
\]

Recursive formulas for the entries of \(E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)\) can be found in A. Blunck, H. H. [9].
A ring has *stable rank 2* (or: stable range 1) if for any unimodular pair \((a, b) \in R^2\), i.e., there exist \(u, v\) with \(au + bv \in R^*\), there is a \(c \in R\) with 
\[
ac + b \in R^*.
\]

Surveys by F. Veldkamp [40] and [41].
H. Chen: *Rings Related to Stable Range Conditions* [16].
Examples

Rings of stable rank 2 are ubiquitous:

- local rings;
- matrix rings over fields;
- finite-dimensional algebras over commutative fields;
- finite rings;
- direct products of rings of stable rank 2.

\( \mathbb{Z} \) is not of stable rank 2: Indeed, \((5, 7)\) is unimodular, but no number \(5c + 7\) is invertible in \( \mathbb{Z} \).
Examples

\( \mathbb{P}(R) \) is connected if . . .

- \( R \) is a ring of stable rank 2. Diameter \( \leq 2 \) (C. Bartolone [1]).
- \( R \) is the endomorphism ring of an infinite-dimensional vector space. Diameter 3 (A. Blunck, H. H. [8]).
- \( R \) is a polynomial ring \( F[X] \) over a field \( F \) in a central indeterminate \( X \). Diameter \( \infty \) (A. Blunck, H. H. [8]).

However, in \( R = F[X_1, X_2, \ldots, X_n] \) with \( n \geq 2 \) central indeterminates there holds

\[
\begin{pmatrix}
  1 + X_1 X_2 & X_1^2 \\
  -X_2^2 & 1 - X_1 X_2
\end{pmatrix} \in \text{GL}_2(R) \setminus \text{GE}_2(R)
\]

(J. R. Silvester [39]).
A chain space $\Sigma = (\mathbb{P}, C)$ is an incidence structure (consisting of points and chains) such that the following axioms hold:

1. Each point is on at least one chain. Each chain contains at least one point.

2. There is a unique chain through any three mutually distant points of $\mathbb{P}$. Here two points $p, q \in \mathbb{P}$ are called distant (in symbols: $p \triangle q$) if they are distinct and on at least one common chain.

3. For each point $p \in \mathbb{P}$ the residue $\Sigma_p := (\triangle(p), C_p)$, where

$$\triangle(p) := \{q \in \mathbb{P} \mid q \triangle p\} \quad \text{and} \quad C_p := \{C \setminus \{p\} \mid p \in C \in \mathcal{C}\},$$

is a partial affine space, i.e., an incidence structure resulting from an affine space by removing some (but not all) parallel classes of lines.
Example: The Chain Space on a Cylinder

An elliptic cylinder in the three-dimensional real affine space gives rise to a chain space $\Sigma = (\mathbb{P}, C)$ as follows:

The set $\mathbb{P}$ is the set of points of the cylinder. The set of chains $C$ is the set of ellipses on the cylinder.

- Two points are distant precisely when they are not on a common generator.
- The point set of any residue $\Sigma_p$ arises by removing the generator through $p$ from $\mathbb{P}$.
- All residues $\Sigma_p$ are real affine planes from which precisely one parallel class of lines is removed.

Any projective quadric (up to some degenerate cases) determines a chain space in a similar way.
The Chain Geometry of an Algebra

Let $R$ be an algebra over a commutative field $K$. By identifying $x \in K$ with $x \cdot 1_R \in R$ we may assume $K \subset R$.

- The injective mapping

$$\mathbb{P}(K) \rightarrow \mathbb{P}(R) : K(a, b) \mapsto R(a, b)$$

is used to identify $\mathbb{P}(K)$ with a subset of $\mathbb{P}(R)$.

- The $\text{GL}_2(R)$ orbit of $\mathbb{P}(K)$ is called the set of $K$-chains in $\mathbb{P}(R)$ and will be denoted by $C(K, R)$.

- For $K \neq R$ the incidence structure

$$\Sigma(K, R) := (\mathbb{P}(R), C(K, R))$$

is the chain geometry on $(K, R)$.
Properties of $\Sigma(K, R)$

**Proposition**

- *The chain geometry* $\Sigma(K, R)$ *is a chain space.*
- *The distant relation of the chain space* $\Sigma(K, R)$ *coincides with the distant relation of the projective line* $\mathbb{P}(R)$.
- *All residues of* $\Sigma(K, R)$ *are isomorphic to the partial affine space which arises from the vector space* $R$ *over* $K *by removing all lines with a non-invertible direction vector.*

A bijective correspondence between $R$ and the residue at $R(1,0)$ is given by $a \mapsto R(a,1)$.

W. Benz: *Vorlesungen über Geometrie der Algebren* [2].
A. Herzer: *Chain Geometries* [25].
A. Blunck, A. Herzer: *Kettengeometrien* [12].
Example: The Blaschke Cylinder

The chain space on the cylinder which we exhibited before is actually a model for the chain geometry

$$\Sigma(\mathbb{R}, \mathbb{R}[\varepsilon]),$$

where $\mathbb{R}[\varepsilon]$ denotes the real dual numbers (W. Blaschke [3]).
Example

Let $R = K^{n \times n}$ be the $K$-algebra of $n \times n$ matrices over a commutative field $K$. There is the a bijective correspondence:

<table>
<thead>
<tr>
<th>Chain geometry $\Sigma(K, R)$</th>
<th>Vector space $K^{2n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>Subspace with dimension $n$</td>
</tr>
<tr>
<td>Chain</td>
<td>Regulus</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Complementarity relation</td>
</tr>
</tbody>
</table>

**Theorem (A. Blunck and H. H. [11])**

The $K$-chains of $\Sigma(K, K^{n \times n})$ are definable in terms of the distant relation of $\mathbb{P}(K^{n \times n})$.

Actually, in [11] a more general result is shown.

Cf. also M. Pankov [38] and Z.-X. Wan [42] for relations with Grassmann spaces and the geometry of matrices.
Let \((P, C)\) be a chain space. Given any subset \(S\) of \(P\) we denote by \(C(S)\) the set of all chains which are entirely contained in \(S\).

The set \(S\) is called a *subspace* of the chain space \((P, C)\) if it satisfies the following conditions:

1. \(S\) has at least three mutually distant points.
2. For any three mutually distant points of \(S\) the unique chain through them belongs to \(C(S)\).
3. \((S, C(S))\) is a chain space.
Subspaces of $\Sigma(K, R)$

Examples:

- Any connected component of the distant graph on $\mathbb{P}(R)$ is a subspace.
- Let $S$ is a $K$-subalgebra of $R$ which is inversion invariant, i.e., for all $x \in S \cap R^*$ holds $x^{-1} \in S$.
  Then $\mathbb{P}(S)$ (embedded in $\mathbb{P}(R)$) is a subspace.
- There are various “sporadic” examples of subspaces.

Problem

Find all subspaces of a chain geometry $\Sigma(K, R)$ containing $R(1, 0)$, $R(0, 1)$, and $R(1, 1)$ with a neat algebraic description.
A Jordan system $J$ of $(K, R)$ is $K$-subspace of $R$ satisfying the following conditions:

1. $1 \in J$.
2. For all $x \in J \cap R^*$ holds $x^{-1} \in J$.

A Jordan system $J$ is called strong provided that the following extra condition holds:

3. For all $x \in J$ we have

$$\#(k \in K | x + k \notin R^*) < \#(k \in K | x + k \in R^*).$$

A. Herzer [24], H. J. Kroll [31].

See O. Loos [35] for relations with Jordan algebras and Jordan pairs.
Examples

- Let $R$ be the algebra of $n \times n$ matrices over a commutative field $K$. Then the subset of symmetric matrices is a Jordan system. It is strong if $\#K > 2n$.
  This may be generalised to Hermitian matrices.

- For commutative algebras $(K, R)$ with $\text{Char } K \neq 2$, any strong Jordan system is necessarily a subalgebra (H. J. Kroll [32], [33]).

- Many examples, even for commutative algebras, can be found in A. Blunck, A. Herzer [12], A. Herzer [26].

- All inversion invariant additive subgroups of a field $R$ were determined by D. Goldstein et al. [19] and A. Mattarei ($R$ commutative) [36].
Properties

An essential tool in the investigation of strong Jordan systems is **Hua’s identity**: Let $a, b$ and $a - b$ be invertible elements of a ring $R$. Then $a^{-1} - b^{-1}$ is invertible too, and there holds

$$(a^{-1} - b^{-1})^{-1} = a - a(a - b)^{-1}a.$$ 

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**Theorem (A. Herzer [24])**

Any strong Jordan-System $J$ is closed under the Jordan triple product:

$$xyx \in J \text{ for all } x, y \in J.$$ 

Easy consequences:

- $x^n \in J$ for all $x \in J$ and all $n \in \mathbb{N}$.
- $xy + yx \in J$ for all $x, y \in J$. 

The Projective Line over a Strong Jordan System

Let $J$ be a strong Jordan system in $R$. The projective line over $J$ is defined as

$$\mathbb{P}(J) = \{ R(t_2 t_1 - 1, t_2) \mid t_1, t_2 \in J \}.$$ 

**Theorem (A. Herzer [24])**

The projective line over any strong Jordan-System $J$ in $R$ is a connected subspace of $\Sigma(K, R)$.

Under certain technical conditions the theorem describes all connected subspaces containing $R(1, 0)$, $R(0, 1)$, and $R(1, 1)$ (A. Herzer [24]).

See also A. Blunck [4], H.-J. Kroll [31], [32], [33].
Final Remarks

- Strong Jordan systems of the matrix algebra $R = K^{n \times n}$ ($K$ commutative) yield subsets of Grassmannians which are closed under reguli (A. Herzer [24]).

- Chain spaces on quadrics (with quadratic form $Q$) can be described algebraically via strong Jordan systems of the Clifford algebra of $Q$ (A. Blunck [5]).

Question

Is it possible to replace the strongness condition for Jordan systems by closedness under triple multiplication without affecting the known results about projective lines?

Cf. [10] for an affirmative answer concerning Hermitian matrices, using results about dual polar spaces (see P. J. Cameron [15]) and matrix spaces (see Z.-X. Wan [42]) rather than ring geometry.
The bibliography focusses on the presented material and recent related work. The books and surveys [2], [12], [20], [25], [29], [41], [42] contain a wealth of further references.

Refer to [13], [14], [18], [21], [22], [23], [34] for deviating definitions of projective lines which we could not present in our lecture.
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